UNIFORM APPROXIMATION IN SEVERAL COMPLEX VARIABLES

AKIRA SAKAI

(Received June 10, 1977)

Introduction

Let $K$ be a compact subset of the $n$-dimensional complex Euclidean space $\mathbb{C}^n$. Let $C(K)$ be the Banach algebra of all complex valued continuous functions defined on $K$, equipped with the sup-norm. There are several important closed subalgebras of $C(K)$. The problem of uniform approximation is to find the conditions that some of these subalgebras coincide with each other. Among these, we shall mainly deal with the problem for the subalgebra $H(K)$, the closure in $C(K)$ of the class of functions each of which is the restriction of a function holomorphic in a neighborhood of $K$. When $n=1$, this is the problem of rational approximation. When $n>1$, known results for $H(K)$, for the most part, were concerned with the case when $K$ is the closure of a bounded domain with smooth boundary or a compact subset of a smooth real submanifold of $\mathbb{C}^n$.

The problem of finding the conditions under which $H(K)$ coincides with $C(K)$, when $K$ is a compact subset of a smooth real submanifold of $\mathbb{C}^n$, originated with Wermer [13] and has been studied by several authors (Hörmander-Wermer [5], Nirenberg-Wells [7], Čirka [1], and Harvey-Wells [2]). The result of Hörmander-Wermer is the following:

Let $M$ be a smooth real submanifold of $\mathbb{C}^n$ without complex tangent. Then, for every compact subset $K$ of $M$, $H(K)=C(K)$ holds.

In this paper, we shall deal with the case when $K$ is a subset of the zero set $T$ of a nonnegative strictly plurisubharmonic function. Such a set $T$ will be called a totally real set. (This use of terminology is supported by the fact that a smooth real submanifold $M$ of $\mathbb{C}^n$ is a totally real set if and only if $M$ has no complex tangents (Corollary of Proposition 6).) It is known that a totally real set is locally a subset of a totally real submanifold (Harvey-Wells [3]). Therefore, the local approximation theorem for totally real sets follows at once from the theorem of Hörmander-Wermer cited above. The main purpose of this paper is to establish the following global approximation theorem:

Let $T$ be a totally real set. Then, for every compact subset $K$ of $T$, $H(K)$

*) Recent address: Himeji Institute of Technology.
590 A. Sakai

\(=C(K)\) holds.

The main tool that we shall make use of is the \(L^2\)-estimate for solutions of \(\bar{\partial}\)-problem due to Hörmander [4].

In order to establish the uniform approximation theorem for \(H(K)\), we shall consider, in Section 2, some convexity condition on \(K\) that is a variant of the uniform \(H\)-convexity condition introduced by Cirka [1]. It will be proved in Section 5 that a compact totally real set satisfies a certain convexity condition of this kind.

In Section 3, we shall give a local representation of \(CR\)-submanifolds (Theorem 1). In Section 4, we shall define a totally real set and study some of its properties. Section 5 will be devoted to the proof of the main theorem (Theorem 2). The essential part of the proof consists of Lemma 4 and 6. In Section 6, we shall give an example of a smooth real submanifold \(M\) containing a set of points at which \(M\) has non zero complex tangents, while \(H(K)=C(K)\) holds for every compact subset of \(M\). To derive this example, we need to generalize a theorem due to Mergeljan. Section 7 is concerned with the problem of the (peak) interpolation for a nondegenerate analytic polyhedron or a strictly pseudoconvex domain (not necessarily with smooth boundary), as an application of the main theorem. This problem has been extensively discussed for a polydisk (cf. Stout [11]). In the last section, we shall prove an approximation theorem for \(CR\)-functions in some globally presented case. It seems to the author that the main difficulty in proving the approximation theorem for \(CR\)-functions in general form consists in the proof of the extension lemma corresponding to Lemma 6.

1. Notations and preliminaries

We denote by \(C^n\) the complex \(n\)-dimensional Euclidean space. When we must emphasize the complex coordinates \(z=(z_1, \ldots, z_n)\), it will be denoted by \(C^n_z\). Similarly, \(C^n_{t_1, \ldots, t_k}\) or \(R^n_{u_1, \ldots, u_k}\) denotes the subspace with the coordinates \((z_1, \ldots, z_k)\) or \((u_1, \ldots, u_k)\) respectively. For any point \(z\) of \(C^n\), \(|z|\) denotes the Euclidean norm of \(z\). For a subset \(S\) of \(C^n\), we define the distance function \(d_S(z)=\inf \{|\xi-z|: \xi \in S\}\) and the \(\varepsilon\)-neighborhood \(U_\varepsilon(S)=\{z: d_S(z)<\varepsilon\}\) of \(S\). \(B_n(a, r)\) denotes the \(n\)-dimensional ball \(\{z \in C^n: |z-a|<r\}\). If \(f\) is a continuous function defined on \(S\), the sup- and \(L^2\)-norms are denoted by \(\|f\|_S\) and \(\|f\|_{L^2(S)}\) respectively.

Let \(U\) be an open subset of \(C^n\). For any real valued function \(\rho \in C^0(U)\) and for any vector \(\xi\) of \(C^n\), we write

\[H[\rho; \xi](z) = \sum \frac{\partial^2 \rho(z)}{\partial z_i \partial z_k} \xi_i \xi_k, \quad z \in U.\]

If \(H[\rho; \xi](z)>0\) holds for every nonzero \(\xi\) and for every point \(z\) of \(U\), then \(\rho\) is
called a \textit{strictly plurisubharmonic} function in $U$. Let $\mathcal{P}(U)$ be the class of $C^\infty$ functions nonnegative and strictly plurisubharmonic in $U$.

**Lemma 1.** Let $U$ be an open subset of $\mathbb{C}^n$. Let $\rho$ and $\sigma$ be strictly plurisubharmonic functions defined in $U$. Suppose that there exists a real number $c$ such that

$$G_0 = \{z \in U: \rho(z) < c\}$$

is relatively compact in $U$. Then, for any real number $\varepsilon$,

$$G_\varepsilon = \{z \in G_0: \sigma(z) < \varepsilon\}$$

is holomorphically convex. In particular, $G_0$ is holomorphically convex.

**Proof.** Set

$$u(z) = \frac{1}{c - \rho(z)} + \frac{1}{\varepsilon - \sigma(z)}.$$  

Then, we have, for any vector $\xi \in \mathbb{C}^n$ and for any $z \in G_0$,

$$H[u; \xi](z) = \frac{1}{[c - \rho(z)]^2} H[\rho; \xi](z) + \frac{2}{[c - \rho(z)]^3} \sum_k \frac{\partial \rho(z)}{\partial \bar{z}_k} \xi_k^2$$

$$+ \frac{1}{[\varepsilon - \sigma(z)]^2} H[\sigma; \xi](z) + \frac{2}{[\varepsilon - \sigma(z)]^3} \sum_k \frac{\partial \sigma(z)}{\partial \bar{z}_k} \xi_k^2.$$  

The right member is positive for nonzero vector $\xi$. If $z$ is a point of $\partial G_\varepsilon$, then $\rho(z) = c$ or $\sigma(z) = \varepsilon$ holds. From this it follows that, for any real number $a$, the set $\{z \in G_0: u(z) \leq a\}$ is compact. This proves the lemma.

If $\omega = \sum_k \alpha_k(z)dz_k$ is a $C^\infty$ form of type $(0, 1)$ defined in an open set $U$, then we write

$$|\omega(z)| = \sum_k |\alpha_k(z)|.$$  

If $V$ is any relatively compact open subset of $U$, then the sup- and $L^2$-norms of $\omega$ on $V$ are denoted by $||\omega||_V$ and $||\omega||_{L^2(V)}$ respectively. The main tool which we shall make use of is the following theorem. (This is a special form of the theorem proved by Hörmander [4].)

**Hörmander’s theorem.** Let $K$ be a compact subset of a bounded open set $U$. Suppose that $\omega$ is a $C^\infty$ form of type $(0, 1)$ satisfying $\overline{\partial}\omega = 0$ in $U$. Then, for every holomorphically convex open set $G$ such that $K \subset G \subset U$, there exists a $C^\infty$ function $u$ such that

$$\overline{\partial} u = \omega \quad \text{and} \quad ||u||_{L^2(G)} \leq \gamma ||\omega||_{L^2(G)}.$$
where \( \gamma \) is a constant depending only on \( K \).

We shall also make use of the following lemma due to Hörmander-Wermer [5].

**Lemma 2.** Let \( u \) be a \( C^\infty \) function defined in a neighborhood of the closed ball \( B = B_\alpha(a, \varepsilon) \). Then
\[
|u(a)| \leq \gamma_0 \left\{ \varepsilon^{-n}|u||\varphi| \right\} + \varepsilon \left\| \partial u \right\|
\]
holds, where \( \gamma_0 \) is an absolute constant.

Let \( K \) be a compact subset of \( C^n \). A subalgebra \( \mathcal{A} \) of \( C(K) \) is called a uniform algebra on \( K \), if it is closed in \( C(K) \), contains the constants, and separates the points of \( K \). If \( \mathcal{A} \) is a uniform algebra on \( K \), then \( K \) is naturally imbedded in the maximal ideal space \( M(\mathcal{A}) \) of \( \mathcal{A} \), and the Šilov boundary \( \Gamma(\mathcal{A}) \) of \( \mathcal{A} \) is contained in \( K \). We shall consider some uniform algebras on \( K \). \( A(K) \) is the algebra of functions in \( C(K) \) which is holomorphic in the interior of \( K \). \( H(K) \) is the algebra of functions approximated uniformly on \( K \) by functions each holomorphic in a neighborhood of \( K \). If \( K \subset U, H(K, U) \) is the algebra of uniform limits on \( K \) of functions holomorphic in \( U \). If \( \{f_1, \ldots, f_m\} \) is a set of functions in \( C(K) \) separating the points of \( K \), \( [f_1, \ldots, f_m; K] \) is the algebra of uniform limits of polynomials of \( f_1, \ldots, f_m \). In particular, \( [z_1, \ldots, z_n; K] \) is denoted by \( P(K) \). Evidently, we have
\[
P(K) = H(K, C^n) \subset H(K, U) \subset H(K) \subset A(K) \subset C(K).
\]

2. Holomorphically convexity

Let \( K \) be a compact subset of \( C^n \). \( K \) is called a \( H \)-convex set, if the maximal ideal space \( M(H(K)) \) of \( H(K) \) coincides with \( K \). It is known that, if \( K \) is the intersection of holomorphically convex open sets containing \( K \), then \( K \) is \( H \)-convex (cf. Rossi [9]). To establish the approximation theorem for \( H(K) \), we need to impose a stronger convexity condition on \( K \).

Let \( \delta(z) \) be a nonnegative continuous function defined in an open subset \( U \) of \( C^n \). A compact subset \( K \) of \( U \) is said to be in the class \( \mathfrak{K}(\delta) \), if we can find constants \( \eta \) and \( \varepsilon_0 \) so that, for every \( \varepsilon, 0 < \varepsilon < \varepsilon_0 \), there exists a holomorphically convex open set \( G_\varepsilon \subseteq U \) satisfying
\[
U_\varepsilon(K) \subseteq G_\varepsilon \subseteq \{z \in U: \delta(z) < \eta \varepsilon\}.
\]

(1) \( K \) is called a \( \delta \)-convex set, if, in addition to (1), the condition
\[
K = \bigcap_{\varepsilon > 0} G_\varepsilon
\]
is satisfied. A \( \delta \)-convex set is \( H \)-convex by definition. When \( \delta(z) \) is the
distance function $d_{\kappa}(z)$, (1) implies (2). The $d_{\kappa}$-convexity is nothing but the uniform $H$-convexity introduced by Čirka [1]. We shall give some examples of $\delta$-convex sets.

**Example 1.** Let $G$ be a bounded strictly pseudoconvex domain defined by a strictly plurisubharmonic function $\sigma$ in an open set $U$ containing $\partial G := \{z \in U: \sigma(z) < 0\}$. Suppose that $d\sigma$ does not vanish on $\partial G$. Let $V$ be a relatively compact open subset of $U$ containing $\partial G$. Then, there exist positive constants $c_1$ and $c_2$ such that

$$c_1 d_{\kappa}(z) < \sigma(z) < c_2 d_{\kappa}(z), \quad z \in V \setminus \partial G.$$  

We choose $\varepsilon_0$ so that the open set $\{z \in V: \sigma(z) < c_2 \varepsilon_0\}$ is relatively compact in $V$. It follows from Lemma 1 that, for any $\varepsilon$, $0 < \varepsilon < \varepsilon_0$, the open set

$$G_\varepsilon = \{z \in V: \sigma(z) < c_2 \varepsilon\}$$

is holomorphically convex. Setting $\eta = \varepsilon_0^{-1} c_2$, we have

$$U_\eta(G) \subset G_\varepsilon \subset \{z \in V: d_{\kappa}(z) < \eta \varepsilon\}.$$  

Therefore, $G$ is $d_{\kappa}$-convex. Moreover, if we set $\delta(z) = \max \{\sigma(z), 0\}$, then $G$ is also a $\delta$-convex set.

**Example 2.** Let $U$ be an open subset of $\mathbb{C}^n$. Let $f_\nu$, $1 \leq \nu \leq s$, $s \leq n$, be functions in $C^\infty(U)$. Suppose that $f_\nu$ are holomorphic in $z_{s+1}, \ldots, z_n$ in $U$ and that

$$\det \left[ \frac{\partial f_\nu}{\partial z_k}; \quad \nu = 1, \ldots, s; \quad k = 1, \ldots, s \right] = 0 \quad \text{on} \quad U.$$  

Let $K$ be a $d_{\kappa}$-convex compact subset of $U$, and let $K^*$ be the graph of $(f_1, \ldots, f_s)$ on $K$:

$$K^* = \{(z, w) \in C^{s+s}: w_\nu = f_\nu(z_1, \ldots, z_n), \quad 1 \leq \nu \leq s\}.$$  

If $G_\varepsilon$ is a holomorphically convex open subset of $U$ in the definition of the $d_{\kappa}$-convexity of $K$, then the open set

$$V_\varepsilon = \{(z, w) \in C^{s+s}: \sum_{\nu=1}^s |w_\nu - f_\nu(z)|^2 < \varepsilon^2, \quad z \in G_\varepsilon\}$$

is holomorphically convex (cf. Sakai [10]). We can choose a positive constant $\varepsilon$ so that, for every sufficiently small $\varepsilon > 0$,

$$U_\varepsilon(K^*) \subset V_\varepsilon \subset \{(z, w) \in C^{s+s}: d_{\kappa}(z, w) < c \varepsilon\}.$$  

holds. Therefore, $K^*$ is $d_{\kappa^*}$-convex.
Example 3. A real $C^\infty$ submanifold $M$ of $C^n$ is said to be finite, if $M$ is a manifold with boundary and if $\partial M$ is a real $C^\infty$ submanifold of $C^n$. If $M$ is a compact or finite $C^\infty$ submanifold of $C^n$ and has no complex tangents, then $M$ is $d_M$ convex. This is derived from the fact that $d_M(z)^2$ is strictly plurisubharmonic in a neighborhood of $M$ (cf. Hörmander-Wermer [5]). More generally, if $\rho$ is a function in $\mathcal{P}(U)$ and if the zero set $K$ of $\rho$ is compact, then $K$ is $\delta$-convex, where $\delta$ is the function defined by

$$\delta(z)^2 = \sum_1^n |\partial \rho(z)|^2$$

(see Lemma 5, Section 5).

Let $\delta$ be a nonnegative continuous function in $U$ and $m$ a positive integer. A function $F \in C^\infty(U)$ is said to be in the class $\mathfrak{M}_m(U, \delta)$, if, for any relatively compact open subset $V$ of $U$, there exists a constant $c$ such that

$$|\bar{\partial} F(z)| \leq c \delta(z)^m, \quad z \in V.$$

Proposition 1. Let $K$ be a compact subset of $U$ in the class $\mathfrak{M}(\delta)$. If $F \in \mathcal{M}_{n+1}(U, \delta)$, then $F|_K$ belongs to $H(K)$.

Proof. Set $\omega = \bar{\partial} F$ in $U$. Let $\varepsilon_0$ and $\eta$ be the constants in (1), and let $\varepsilon$ be an arbitrary number with $0 < \varepsilon < \varepsilon_0$. We use the notation $\gamma$ for unspecific constants that are independent of $\varepsilon$. By Hörmander’s theorem, we can find a function $u \in C^\infty(G_\varepsilon)$ such that

$$\bar{\partial} u_\varepsilon = \omega \quad \text{and} \quad ||u_\varepsilon||_{L^2(G_\varepsilon)} \leq \gamma ||\omega||_{L^2(G_\varepsilon)}.$$

Let $z$ be an arbitrary point of $K$. The ball $B=B_n(z, \varepsilon)$ is contained in $G_\varepsilon$. It follows from Lemma 2 that

$$|u_\varepsilon(z)| \leq \gamma \varepsilon \{ \varepsilon^{-n} ||u_\varepsilon||_{L^2(G_\varepsilon)} + \varepsilon ||\omega||_{G_\varepsilon} \}.$$

Since $F \in \mathcal{M}_{n+1}(U, \delta)$, (1) yields $||\omega||_{G_\varepsilon} < \gamma \varepsilon^{n+1}$. Therefore, we obtain $|u_\varepsilon(z)| < \varepsilon \varepsilon$. We set $F_\varepsilon = F - u_\varepsilon$ in $G_\varepsilon$. Then $F_\varepsilon$ is holomorphic in $G_\varepsilon$. For every $z$ of $K$, we have

$$|F(z) - F_\varepsilon(z)| = |u_\varepsilon(z)| < \varepsilon \varepsilon,$$

which proves the proposition.

Let $G$ be a bounded domain in $C^n$. Let $A^\infty(G)$ denote the closure in $C(G)$ of the class of functions of $A(G)$ each of which can be extended as a $C^\infty$ function in a neighborhood of $G$.

Corollary. If $G$ is a bounded domain such that $G$ is $d_G$-convex, then $A^\infty(G) = H(G)$ holds.
Proof. Let $F$ be a $C^\infty$ function defined in a neighborhood $U$ of $\bar{G}$ and holomorphic in $G$. Since $\partial F = 0$ in $G$, we have $F \in \mathcal{M}_{n+1}(U, d\sigma)$. Therefore, $F|_\partial$ belongs to $H(\bar{G})$.

A $C^\infty$ map $\phi = (\phi_1, \cdots, \phi_n)$ of an open subset $U$ of $C^n$ into $C^n$ is said to be in the class $\mathcal{M}_n(U, \delta)$, if every $\phi_k$ is in $\mathcal{M}_n(U, \delta)$.

**Proposition 2.** Let $\delta$ be a nonnegative continuous function defined in an open set $U$ and $K$ a compact subset of $U$ in the class $\mathcal{M}(\delta)$. Let $\phi$ be a map in $\mathcal{M}^+_{n+1}(U, \delta)$. If $f'$ is a function in $H(\phi(K))$, then $f = f' \circ \phi$ is in $H(K)$.

Proof. There exists a sequence of functions $\{g'_v\}$ each holomorphic in a neighborhood $U_v$ of $K' = \phi(K)$ such that $f'$ is the uniform limit of $\{g'_v\}$ on $K'$. Set $g^v = g'_v \circ \phi$ and $U_v = \phi^{-1}(U_v)$. Since $\phi \in \mathcal{M}^+_{n+1}(U, \delta)$ we have $g^v \in \mathcal{M}^+_{n+1}(U_v, \delta)$. Therefore, by Proposition 1, we have $g^v \in H(K)$. Since $f$ is the uniform limit of $\{g^v\}$ on $K$, we have $f \in H(K)$, as required.

**Corollary.** Let $\phi$ be a diffeomorphism in $\mathcal{M}^+_{n+1}(U, \delta)$. Set $\delta' = \delta \circ \phi^{-1}$. If $K$ is in $\mathcal{M}(\delta)$ and if $K' = \phi(K)$ is in $\mathcal{M}(\delta')$, then $H(K)$ and $H(K')$ are isomorphic as uniform algebras.

Proof. The inverse map $\phi^{-1}$ is in $\mathcal{M}^+_{n+1}(U', \delta')$, where $U' = \phi(U)$. Hence, we have the Corollary.

3. **CR-submanifolds**

Let $M$ be a real $C^\infty$ submanifold of $C^n$. We denote by $T_z(M)$ the real tangent space of $M$ at $z$. We say that $M$ has the complex rank $r$ at $z$, if the complex tangent space

$$CT_z(M) = T_z(M) \cap iT_z(M)$$

has the complex dimension $r$. $M$ is called a **CR-submanifold** of complex rank $r$, if it has a constant complex rank $r$ at every point, which will be denoted by $r(M)$.

The following lemma gives an example of a CR-submanifold of $C^n$.

**Lemma 3.** Let $f = (f_1, \cdots, f_n)$ be a $C^\infty$ imbedding of an subset $N$ of $R^d$ into $C^n$. Then, the image $M = f(N)$ is a CR-submanifold of $C^n$ of complex rank $r$, if and only if

$$\text{rank} \left[ \frac{\partial f_k}{\partial x_v}, \ k=1, \cdots, n, \ v=1, \cdots, d \right] = d - r$$

holds at every point of $N$.

Proof. Let $x^0$ be any point of $N$ and set $\alpha = f(x^0)$. $M$ has a nonzero complex tangent at $\alpha$, if and only if there exist two nonzero vectors $t, s \in R^d$ such that
(1) \[ \sum \frac{\partial f_k}{\partial x_v}(x^v) t_v = i \sum \frac{\partial f_k}{\partial x_v}(x^v) s_v, \quad 1 \leq k \leq n. \]

If \( t=0 \), then (1) implies that

\[ \sum \frac{\partial u_k}{\partial x_v}(x^v) s_v = 0, \quad \sum \frac{\partial v_k}{\partial x_v}(x^v) s_v = 0, \quad 1 \leq k \leq n, \]

where \( f_k = u_k + iv_k \). Since \[ rank \begin{bmatrix} \frac{\partial u_k}{\partial x_v} & k=1, \ldots, n \\ \frac{\partial v_k}{\partial x_v} & v=1, \ldots, d \end{bmatrix} = d, \]
we have \( s=0 \). Set \( \xi = t-i s \). Then \( \xi \) is nonzero solution of

(2) \[ \sum \frac{\partial f_k}{\partial x_v}(x^v) \xi_v = 0, \quad 1 \leq k \leq n, \]

if and only if \( t \) and \( s \) are nonzero and satisfy (1). Thus, the complex dimension of \( CT_a(M) \) coincides with the number of linearly independent complex vectors \( \xi \) satisfying (2). This proves the lemma.

Let \( M \) be a CR-submanifold of complex rank \( r \). We say that \( M \) is holomorphic if, for every point \( z \) of \( M \), there exists a neighborhood \( U_z \) of \( z \) in \( C^n \) such that \( M \cap U_z \) is represented as a real \( C^\infty \) parametric family of complex submanifolds of \( C^n \) of complex dimension \( r \). If \( r(M)=0 \), then \( M \) is trivially holomorphic. A CR-submanifold of positive rank is not necessarily holomorphic. For example, the hypersphere \( S^{2n-1} \) in \( C^n, n>1 \), is a CR-submanifold of complex rank \( n-1 \). However, \( S^{2n-1} \) can contain no complex submanifolds of \( C^n \) of positive dimensions. To see this, we suppose that \( S^{2n-1} \) contains a complex submanifold \( X \) of \( C^n \). We may assume that \( X \) contains the point \( z^0=(1,0,\ldots,0) \).

Then the function \( f(x)=\frac{i}{2}(1+z_i) \) induces a holomorphic function \( F \) on \( X \). \( F \) attains its maximal value 1 in \( X \) only at \( z^0 \). It follows from the maximal modulus principle that \( X \) reduces to \( \{z^0\} \).

We shall now give a local representation of holomorphic CR-submanifolds. For simplicity, we use the abbreviations

\[ u' = (u_1, \ldots, u_t), \quad v' = (v_1, \ldots, v_t), \]
\[ w' = (w_{t+1}, \ldots, w_{t+r}), \quad \text{and} \quad w'' = (w_{t+r+1}, \ldots, w_n), \]

where \( w_v=u_v+iv_v \) and \( 0 \leq t \leq t+r \leq n \).

Let \( V \) be an open subset of \( C^n \). Suppose that \( N=V \cap (R^t_x \times C^{n-r}) \) is not empty. If \( \phi \) is a diffeomorphism of \( V \) into \( C^n \) which is in \( \tilde{M}(V, d_N) \), then \( \phi|_N \) is holomorphic in \( w'' \) on \( N \). Therefore, \( M=\phi(N) \) is a holomorphic CR-submanifold of \( C^n \).

Conversely, we have the following theorem.

**Theorem 1.** Let \( M \) be a holomorphic CR-submanifold of \( C^n \). For any
positive integer \( m \) and for any point \( z \) of \( M \), there exist a neighborhood \( U_z \) of \( z \) in \( C^\infty \), a neighborhood \( V \) of the origin of \( C^\infty \), and a \( C^\infty \) diffeomorphism \( \phi \) of \( V \) onto \( U_z \) satisfying the following conditions:

(i) \( N = \phi^{-1}(M \cap U_z) \) is an open subset of \( \mathbb{R}^r \times C^\infty \), where \( r = r(M) \) and \( t = \text{dim}_\mathbb{R} M - 2r \);

(ii) every component \( \phi_k \) of \( \phi \) is holomorphic in \( w_{t+1}, \ldots, w_n \);

(iii) \( \phi \in \mathcal{O}_m(V, d_N) \).

Proof. Since \( M \) is holomorphic, we can choose a neighborhood \( U'_z \) of \( z \), a neighborhood \( N' \) of the origin of \( \mathbb{R}^r \times C^\infty \), and a \( C^\infty \) map \( \psi \) of \( N' \) into \( C^\infty \) with \( \psi(0) = z \), satisfying the following conditions:

(a) \( \psi(N') = M \cap U'_z \);

(b) every component \( \psi_k \) of \( \psi \) is holomorphic in \( w'' \) on \( N' \).

By Lemma 3, we can assume that

(c) \( \det \left[ \frac{\partial \psi_k}{\partial u_\nu} ; \ k = 1, \ldots, t+r \right] \neq 0 \quad \text{on} \quad N' \).

We define the function \( \tilde{\psi} \in C^\infty(\Omega) \), \( \Omega = N' \times \mathbb{R}^r \times C^\infty \), by

\[
\tilde{\psi}(w_1, \ldots, w_n) = \psi(u', w') + i \sum_{\nu=1}^{t+r} \frac{\partial \psi_k}{\partial u_\nu} (u', w') v_\nu
\]

It follows at once from (b) that \( \frac{\partial \tilde{\psi}_k}{\partial \overline{w}_\mu} = 0 \) for \( \mu = t+1, \ldots, n \). For \( \mu = 1, \ldots, t \), we have

\[
\frac{i^p}{p!} \sum_{\nu_1, \ldots, \nu_p} \frac{\partial^{p+1} \psi_k}{\partial u_{\nu_1} \cdots \partial u_{\nu_p} \partial \overline{w}_\mu} v_{\nu_1} \cdots v_{\nu_p}
\]
Therefore, we have

$$\frac{\partial \tilde{\gamma}_k}{\partial w_\mu} = i^m \sum_{\nu_1, \ldots, \nu_m} \frac{\partial^{m+\nu} \tilde{\gamma}_k}{\partial w_\mu \partial u_{\nu_1} \cdots \partial u_{\nu_m}} (w', w'') v_{\nu_1} \cdots v_{\nu_m}.$$ 

From this it follows that $\tilde{\gamma} \in \mathcal{M}_m(\Omega, d_N)$.

Let us now define the $C^m$ map $\phi=(\phi_1, \ldots, \phi_n)$ of $\Omega$ into $C^n$ by

$$\phi_k(w_1, \ldots, w_n) = \begin{cases} \tilde{\gamma}_k(w) & (1 \leq k \leq t+r) \\ \tilde{\gamma}_k(w)+w_k & (t+r+1 \leq k \leq n). \end{cases}$$

Then $\phi$ is clearly in the class $\mathcal{M}_m(\Omega, d_N)$, and hence, for any point $w$ of $N'$ and for any index $\nu$, $1 \leq \nu \leq t+r$, we have $\frac{\partial \tilde{\gamma}_k}{\partial w_\nu}(w) = \frac{\partial \tilde{\gamma}_k}{\partial u_\nu}(w)$. Therefore, the Jacobian of $\phi$ at $w$ is

$$J(w) = \left| \det \left[ \frac{\partial \phi_k}{\partial w_\nu} ; \begin{array}{c} k=1, \ldots, n \\ \nu=1, \ldots, n \end{array} \right] \right|^2$$

$$= \left| \det \left[ \frac{\partial \tilde{\gamma}_k}{\partial u_\nu} ; \begin{array}{c} k=1, \ldots, t+r \\ \nu=1, \ldots, t+r \end{array} \right] \right|^2$$

$$= \left| \det \left[ \frac{\partial \tilde{\gamma}_k}{\partial u_\nu} ; \begin{array}{c} k=1, \ldots, t+r \\ \nu=1, \ldots, t+r \end{array} \right] \right|^2.$$ 

It follows from (c) that the last member does not vanish on $N'$ and hence in a neighborhood of $N'$ in $C^n$. Therefore, we can find a neighborhood $V$ of the origin in $C^n$ such that $\phi$ is diffeomorphic in $V$. Setting $N=N' \cap V$ and $U_z=\phi(V)$, we have the theorem.

### 4. Totally real sets

A subset $T$ of $C^n$ is called a totally real set, if there exist an open subset $U$ of $C^n$ containing $T$ and a function $\rho$ in $\mathcal{D}(U)$ such that

$$T = \{ z \in U : \rho(z) = 0 \}.$$ 

$\rho$ is then called a defining function of $T$. We note that if $T$ is totally real, then $\rho(z)=d\rho(z)=0$ holds for $z \in T$.

**Proposition 3.** If every point $z$ of $T$ has a neighborhood $U_z$ in $C^n$ such that $T \cap U_z$ is totally real, then $T$ is totally real.

**Proof.** We can find a locally finite open covering $\{ U_z \}$ of $T$ such that, in each $U_z$, there exists a function $\rho_z \in \mathcal{D}(U_z)$ satisfying $T \cap U_z = \{ z \in U_z : \rho_z(z) = 0 \}$. Let $\{ \lambda_z \}$ be a partition of unity subordinate to $\{ U_z \}$. We set $\rho(z) = \sum \lambda_z(z) \rho_z(z)$, $z \in U = \cup U_z$. 

For any vector $\xi$ of $\mathbb{C}^n$, we have

$$H[\rho; \xi] = \sum_{\nu} \lambda_{\nu} H[\rho_{\nu}; \xi] + \sum_{\nu} \rho_{\nu} H[\lambda_{\nu}; \xi] + \sum_{\nu, i, k} \left( \frac{\partial \lambda_{\nu}}{\partial z_i} \frac{\partial \rho_{\nu}}{\partial z_k} + \frac{\partial \lambda_{\nu}}{\partial z_k} \frac{\partial \rho_{\nu}}{\partial z_i} \right) \xi_i \bar{z}_k.$$  

If $\xi \neq 0$, the first sum is positive. The second and the third sums vanish on $T$, since $\rho_{\nu}(z) = d\rho_{\nu}(z) = 0$ for $z \in T \cap U_{\nu}$. Therefore, we can find a neighborhood $V$ of $T$ so that $\rho$ is a defining function of $T$ in $\mathcal{P}(U)$.

**Proposition 4.** Let $T$ be a totally real set defined by $\rho \in \mathcal{P}(U)$. If $f_{\nu}$ $(1 \leq \nu \leq t)$ are functions in $C^\infty(U)$, then

$$T_1 = \{ z \in T: f_{\nu}(z) = 0, 1 \leq \nu \leq t \}$$

is a totally real set.

**Proof.** Set $\rho_1(z) = \rho(z) + \sum_{\nu} |f_{\nu}(z)|^2$, $z \in U$. Then we have

$$H[\rho; \xi] = H[\rho; \xi] + \sum_{\nu} \sum_{i} \frac{\partial f_{\nu}}{\partial z_i} \xi_i |^2 + \sum_{\nu} \sum_{i} \frac{\partial f_{\nu}}{\partial z_i} \xi_i |^2
+ \sum_{\nu} f_{\nu} \sum_{i, k} \frac{\partial^2 f_{\nu}}{\partial z_i \partial z_k} \xi_i \bar{z}_k
+ \sum_{\nu} f_{\nu} \sum_{i, k} \frac{\partial^2 f_{\nu}}{\partial z_i \partial z_k} \xi_i \bar{z}_k.$$  

For any nonzero vector $\xi$, the right member is positive at every point of $T_1$. Hence there exists a neighborhood $V$ of $T_1$ such that $\rho_1$ is in $\mathcal{P}(V)$.

**Corollary.** Let $T$ be a totally real set defined by $\rho \in \mathcal{P}(U)$. If $V$ is a relatively compact open subset of $U$ with the smooth boundary, then $T \cap V$ is a totally real set.

**Proof.** We can choose a $C^\infty$ function $\lambda(z)$ such that $\lambda(z) = 0$ for $z \in V$ and $\lambda(z) > 0$ for $z \in U \cap \overline{V}$. Since $T \cap V = \{ z \in T: \lambda(z) = 0 \}$, $T \cap V$ is totally real.

**Proposition 5.** If $T_1$ and $T_2$ are totally real sets in $C^\infty$ and $C^\infty$ respectively, then $T = T_1 \times T_2$ is totally real in $C^\infty \times C^\infty$.

**Proof.** Let $\rho_1$ be defining functions of $T_1$ respectively. Then, $\rho(z, w) = \rho_1(z) + \rho_2(w)$ is a defining function of $T$.

Let $f_{\nu}(z_{\nu})$ be holomorphic functions in open subsets $U_{\nu}$ of $C^1_{1,\nu}(1 \leq \nu \leq N)$ respectively. Set $T_{\nu} = \{ z_{\nu} \in U_{\nu}: |f_{\nu}(z_{\nu})| = 1 \}$, $1 \leq \nu \leq N$. Suppose that every $f_{\nu}'(z_{\nu})$ does not vanish on $T$. Then every $T_{\nu}$ is a totally real set in $C^1_{1,\nu}$ defined by $\rho_{\nu}(z_{\nu}) = (|f_{\nu}(z_{\nu})|^2 - 1)^2$. Therefore, $T = T_1 \times \cdots \times T_N$ is a totally real set in $C^N$.

**Proposition 6.** Let $\delta$ be a nonnegative continuous function defined in an open set $U$ and $\phi$ a diffeomorphism in $\mathcal{M}(U, \delta)$. Let $T$ be a subset of the zero set
of $\delta$ (assumed to be nonempty). Then $T$ is totally real if and only if $T' = \phi(T)$ is totally real.

Proof. Set $U' = \phi(U)$ and $\delta' = \delta \circ \phi^{-1}$. Since $\phi^{-1}$ is in $\tilde{M}(U', \delta')$, it suffices to prove that if $T'$ is totally real then so is $T$. There exists an open set $V'$ such that $T' \subset V' \subset U'$ and a function $\rho' \in \mathcal{P}'(V')$ such that $T' = \{w \in V': \rho'(w) = 0\}$. We write $\phi$ as $w = \phi(z), 1 \leq \nu \leq n$. Then, for any point $z \in V = \phi^{-1}(V')$ and for any vector $\xi \in \mathbb{C}^n$, we have

$$H[\rho; \xi](z) = H[\rho'; \eta_{\nu}(w)] + \sum_{i,k} \epsilon_{ik}(z) \xi_i \bar{\xi}_k,$$

where $\eta_{\nu}$ is the vector $(\eta_{\nu}(z), \ldots, \eta_{\nu}(z))$ defined by

$$\eta_{\nu}(z) = \sum_{k} \frac{\partial \phi_{\nu}}{\partial z_k}(z) \xi_k, \quad 1 \leq \nu \leq n,$$

and

$$\epsilon_{ik} = \sum_{r,s} \frac{\partial^{2} \rho'}{\partial w_{\mu} \partial w_{\nu}} \frac{\partial \phi_{\nu}}{\partial z_k} \frac{\partial \phi_{\mu}}{\partial z_s} + \sum_{r,s} \frac{2^{2} \rho'}{\partial \phi_{\nu} \partial \phi_{\mu}} \frac{\partial \phi_{\nu}}{\partial z_k} \frac{\partial \phi_{\mu}}{\partial z_s} + \sum_{r,s} \frac{2^{2} \rho'}{\partial \phi_{\nu} \partial \phi_{\mu}} \frac{\partial \phi_{\nu}}{\partial z_k}.$$

Since $\phi \in \tilde{M}(V, \delta)$, the first three sums of the last expression vanish on $T$. Since $d\rho' = 0$ on $T$, the other terms vanish on $T$. Therefore, we have $\epsilon_{ik}(z) = 0$ for $\xi \in T$. If $\xi \neq 0$, then we have $\eta_{\nu}(z) = 0$ on $T$. Thus we can find a neighborhood $V$ of $T$ such that $\rho \in \mathcal{P}(V)$.

**Corollary.** Let $M$ be a real $C^\infty$ submanifold of $\mathbb{C}^n$. Then $M$ is totally real if and only if $r(M) = 0$.

Proof. Suppose that $r(M) = 0$. Then, for any point $z^0$ of $M$, there exists a neighborhood $U$ of $z^0$ such that $M \cap U$ is mapped by a diffeomorphism $\phi \in \tilde{M}(U, d_{M \cap U})$ onto an open subset $N$ of the real subspace $\mathbb{R}^{u'}$ of $\mathbb{C}^u$, where $d = \dim_{\mathbb{R}} M$ and $u' = (u_1, \ldots, u_d)$. Since $\mathbb{R}^{u'}$, is clearly a totally real set in $\mathbb{C}^u$, we can assume that $N$ is totally real by Corollary of Proposition 4. It follows from Proposition 3 and 6 that $M$ is totally real.

Conversely, we suppose that $M$ is totally real. For every point $z$ of $M$ there exist a neighborhood $U_z$ and a real submanifold $M_z$ with $r(M_z) = 0$ such that $M \cap U_z \subset M_z$ (Harvey-Wells [3]). Since $T_z(M \cap U_z) \subset T_z(M_z)$ and $iT_z(M \cap U_z) \subset iT_z(M_z)$, we have $CT_z(M \cap U_z) = \{0\}$ as required.

It should be noted that some of the properties of totally real sets has been studied in Harvey-Wells [2], [3]. We note also that the necessary part of Proposition 6 is due to Hörmander-Wermer [5]. Our proof is a different one.
5. Uniform approximation on totally real sets

The purpose of this section is to prove the following theorem.

**Theorem 2.** Let $T$ be a totally real set in $\mathbb{C}^n$. Then, for every compact subset $K$ of $T$, $H(K)=C(K)$ holds.

We begin by proving the following lemmas. We write

$$|\text{grad } \rho(z)| = \left( \sum_{k} \left| \frac{\partial \rho}{\partial z_k} (z) \right|^2 \right)^{1/2}.$$

**Lemma 4.** Let $T$ be a totally real set defined by $\rho \in \mathcal{P}(U)$. Then, there exists an open set $V$ such that $T \subset U \subset V$ and that $u(z) = |\text{grad } \rho(z)|^2$ is in $\mathcal{P}(V)$. ($T$ is contained in the zero set of $u$.)

**Proof.** For any vector $\xi$ of $\mathbb{C}^n$, we have

$$H[u; \xi] = \sum_k \left( \sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial z_j} \xi_i \xi_j \right) \frac{\partial \rho}{\partial z_k} + \sum_k \left( \sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial z_j} \xi_i \xi_j \right) \frac{\partial \rho}{\partial z_k}$$

$$+ \sum_k \left| \sum_i \frac{\partial^2 \rho}{\partial z_i \partial z_k} \xi_i \right|^2 + \sum_k \left| \sum_i \frac{\partial^2 \rho}{\partial z_i \partial z_k} \xi_i \right|^2.$$

The first and the second sums vanish on $T$, since $d \rho(z) = 0$ for $z \in T$. The fourth sum is positive for any nonzero $\xi$, since the matrix $\left[ \frac{\partial^2 \rho}{\partial z_i \partial z_k} \right]$ is nonsingular. Therefore, we can find a neighborhood $V$ of $T$ such that $H[u; \xi] > 0$ for any $z \in V$ and for any nonzero $\xi$.

**Lemma 5.** Let $K$ be a compact totally real set defined by $\rho \in \mathcal{P}(U)$. Set $\delta(z) = |\text{grad } \rho(z)|$. Then, $K$ is $\delta$-convex.

**Proof.** By Lemma 4, there exists an open set $V$ such that $K \subset V \subset U$ and that $\delta^2$ is in $\mathcal{P}(V)$. Since $K$ is the zero set of $\rho$, we can choose a positive constant $c$ so that the open set

$$G_0 = \{z \in V : \rho(z) < c\}$$

is relatively compact in $V$. Since $d \rho(z) = 0$ for $z \in K$, there exists a positive constant $\eta$ such that

$$\delta(z) < \eta d_k(z), \quad z \in G_0.$$

It follows from Lemma 1 that, for any positive number $\varepsilon$, the open set

$$G_\varepsilon = \{z \in G_0 : \delta(z) < \varepsilon \}$$

is holomorphically convex. Setting $\varepsilon_0 = \text{dist}(K, G_0)$, we have, for any $\varepsilon$, $0 < \varepsilon < \varepsilon_0$, ...
Since $c$ can be chosen arbitrarily small, we have $K = \prod_{i>0} G_i$ which completes the lemma.

**Lemma 6.** Let $T$ be a totally real set defined by $\rho \in \mathcal{P}(U)$. For any positive integer $m$ and for any function $f \in C^m(U)$, there exists a function $F \in \mathcal{M}_m(U, \{\text{grad} \rho\})$ such that $F(z) = f(z)$ for $z \in T$.

**Proof.** The system of equations

\[
\frac{\partial f}{\partial z_i} + \sum_{a} g_a(z) \frac{\partial^2 \rho}{\partial z_a \partial z_i}(z) = 0, \quad z \in U, \quad 1 \leq i \leq n
\]

has a unique solution $(g_a), g_a \in C^m(U)$, since the matrix $\left[ \frac{\partial^2 \rho}{\partial z_a \partial z_i} \right]$ is nonsingular.

Differentiating (1) by $z_j$, we have

\[
\frac{\partial g_a}{\partial z_j} \frac{\partial^2 \rho}{\partial z_a \partial z_j} = - \frac{\partial^2 f}{\partial z_i \partial z_j} - \sum_{a} g_a \frac{\partial^2 \rho}{\partial z_a \partial z_i \partial z_j}.
\]

Since the right member is symmetric with respect to $i$ and $j$, we have

\[
\sum_{a} \frac{\partial g_a}{\partial z_j} \frac{\partial^2 \rho}{\partial z_a \partial z_i} = \sum_{a} \frac{\partial g_a}{\partial z_i} \frac{\partial^2 \rho}{\partial z_a \partial z_j}.
\]

For every $\alpha$, the system of equations

\[
\frac{\partial g_{a\beta}}{\partial z_j} + \sum_{\beta} g_{a\beta} \frac{\partial^2 \rho}{\partial z_\beta \partial z_j} = 0, \quad 1 \leq j \leq n
\]

has a unique solution $(g_{a\beta}), g_a \in C^m(U)$. Substituting (3) to (2), we have

\[
\sum_{a,\beta} g_{a\beta} \frac{\partial^2 \rho}{\partial z_\beta \partial z_i} \frac{\partial^2 \rho}{\partial z_a \partial z_j} = \sum_{a,\beta} g_{a\beta} \frac{\partial^2 \rho}{\partial z_a \partial z_j} \frac{\partial^2 \rho}{\partial z_\beta \partial z_i}, \quad 1 \leq i, j \leq n,
\]

or equivalently,

\[
\sum_{a,\beta} (g_{a\beta} - g_{\beta a}) \frac{\partial^2 \rho}{\partial z_\beta \partial z_i} \frac{\partial^2 \rho}{\partial z_a \partial z_j} = 0, \quad 1 \leq i, j \leq n,
\]

which implies that $g_{a\beta} = g_{\beta a}$.

Suppose that, for a multi-index $J = (j_1, \ldots, j_p), g_J$ is already defined. Let $J_\alpha$ denotes the multi-index $(j_1, \ldots, j_p, \alpha)$. Then $(g_{J_\alpha})$ will be defined as a unique solution of the system of equations

\[
\frac{\partial g_{J_\alpha}}{\partial z_j} + \sum_{a} g_{J_\alpha} \frac{\partial^2 \rho}{\partial z_a \partial z_j} = 0, \quad 1 \leq j \leq n.
\]
Thus we define $g_J$ for all multi-indices $J$ inductively. We shall now prove the symmetry of $g_J$ with respect to $J$ by induction. Suppose that $g_J$ are symmetric for all multi-indices $J$ of length $p$. Fix any positive number $\nu$, $1 \leq \nu \leq p$, and for any indices $\alpha, \beta$, we write

$$I = (j, \ldots, j^\nu, \ldots, j_p), \quad J' = (j, \ldots, \alpha, \ldots, j_p)$$

and

$$J'' = (j, \ldots, \beta, \ldots, j_p),$$

where $j^\nu$ means that $j_\nu$ shall be omitted and $\alpha$ (or $\beta$) in $J'$ (or $J''$ resp.) means that $\alpha$ (or $\beta$ resp.) shall be posed at $\nu$-th position. By the assumption of induction, we have

$$\frac{\partial g_J}{\partial z_i} + \sum_\alpha g_{J'} \frac{\partial^2 \rho}{\partial z_\alpha \partial z_i} = 0, \quad 1 \leq i \leq n. \tag{5}$$

Differentiating (5) by $\bar{z}_j$ and using the argument analogous to one in the case of $p=1$, we have

$$\sum_{\alpha, \beta} (g_{\nu' \beta} - g_{\nu'' \alpha}) \frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_j} \frac{\partial^2 \rho}{\partial z_\beta \partial \bar{z}_j} = 0, \quad 1 \leq i, j \leq n.$$ 

Therefore, we obtain $g_{\nu' \beta} = g_{\nu'' \alpha}$, which implies the symmetry of $g_J$ with respect to all multi-indices $J$ of length $p+1$.

Now, we define the function $F \in C^\infty(U)$ by

$$F(z) = f(z) + \sum_{p=1}^n \frac{1}{p!} \sum_{j_1, \ldots, j_p} g_{j_1, \ldots, j_p}(z) \frac{\partial \rho(z)}{\partial z_{j_1}} \ldots \frac{\partial \rho(z)}{\partial z_{j_p}}. \tag{6}$$

Since $\partial \rho(z)=0$ for $z \in T$, we have $F(z)=f(z)$ for $z \in T$. Differentiating (6) by $z_k$, we have

$$\frac{\partial F}{\partial z_k} = \frac{\partial f}{\partial z_k} + \sum_{j_1} g_{j_1} \frac{\partial^2 \rho}{\partial z_k \partial z_{j_1}} + \sum_{p=1}^{n-1} \frac{1}{p!} \sum_{j_1, \ldots, j_p} g_{j_1, \ldots, j_p} \frac{\partial \rho}{\partial z_{j_1}} \ldots \frac{\partial \rho}{\partial z_{j_p}}$$

$$+ \sum_{p=1}^{n-1} \frac{1}{(p+1)!} \sum_{j_1, \ldots, j_{p+1}} g_{j_1, \ldots, j_{p+1}} \frac{\partial \rho}{\partial z_{j_1}} \ldots \frac{\partial \rho}{\partial z_{j_{p+1}}}$$

$$+ \frac{1}{m!} \sum_{j_1, \ldots, j_m} g_{j_1, \ldots, j_m} \frac{\partial \rho}{\partial z_{j_1}} \ldots \frac{\partial \rho}{\partial z_{j_m}}.$$

By the way of construction of $g_{j_1, \ldots, j_p}$, we have

$$\frac{1}{(p+1)!} \sum_{j_1, \ldots, j_{p+1}} g_{j_1, \ldots, j_{p+1}} \frac{\partial \rho}{\partial z_{j_1}} \ldots \frac{\partial \rho}{\partial z_{j_{p+1}}}$$

$$= \frac{1}{p!} \sum_{j_1, \ldots, j_p} g_{j_1, \ldots, j_p} \frac{\partial^2 \rho}{\partial z_\alpha \partial z_k} \frac{\partial \rho}{\partial z_{j_1}} \ldots \frac{\partial \rho}{\partial z_{j_p}} \frac{\partial \rho}{\partial z_{j_1}} \ldots \frac{\partial \rho}{\partial z_{j_p}}.$$
Thus, we obtain
\[
\frac{\partial F}{\partial \bar{z}_k} = \frac{1}{m!} \sum_{j_1, \ldots, j_m} \frac{\partial g_{j_1 \cdots j_m}}{\partial \bar{z}_k} \partial \bar{p} \cdots \partial \bar{p},
\]
which implies that $F$ belongs to $\mathcal{M}_m(U, |\text{grad } \rho|)$. The lemma is proved.

Proof of Theorem 2. If $K$ is a compact subset of $T$, then there exists an open set $V, K \subset V \subset U$, with the smooth boundary. It follows from Corollary of Proposition 2 that $T \subset V$ is totally real. Since $H(T \cap V)|_K \subset H(K) \subset C(K) = C(T \cap V)|_K$, it is sufficient to prove the theorem in the case when $K$ is totally real. Let $\rho \in P(U)$ be a defining function of $K$, and set $\delta = |\text{grad } \rho|$. Then, by Lemma 5, $K$ is $\delta$-convex. Let $f$ be an arbitrary function in $C^\infty(U)$. Then, by Lemma 6, $f|_K$ has a $C^\infty$ extension $F$ in $\mathcal{M}_{n+1}(U, \delta)$. It follows from Proposition 1 that $f|_K = F|_K \in H(K)$, which proves the theorem.

Theorem 2, when $T$ is a $C^k$ totally real submanifold $M$ of $C^n$, was proved by Hörmander-Wermer [5] for $2k \geq \dim_x M + 2$, and by Harvey-Wells [2] for $k = 1$, Nirenberg-Wells [7] proved a corresponding result when $M$ is a $C^\infty$ totally real submanifold of a complex manifold.

Theorem 2 implies that any compact subset $K$ of a totally real set $T$ is $H$-convex. This fact is due to a strong convexity of $T$. ($K$ is contained in a compact totally real set $K_0$. $K_0$ is $\delta$-convex, $\delta = |\text{grad } \rho|$, (Lemma 5), and $K$ is $\mathcal{O}_{K_0}$-convex (Harvey-Wells [2]).)

6. A theorem of Mergeljan

Let $f$ be a real valued continuous function defined on the closed unit disk $D$ in $C^1$. For every real number $u$, the level set $\{z \in D : f(z) = u\}$ will be denoted by $L_u$. We consider a uniform algebra

\[ B = \{g \in C(D) : |g|_{L_u} \in A(L_u) \text{ for every } u \in f(D)\}. \]

Mergeljan [6] proved the following theorem.

**Theorem of Mergeljan.** If, for every $u \in f(D)$, $L_u$ does not divide the plane, then $[z, f; D] = B$. In particular, if every $L_u$ has no interior points in addition, then $[z, f; D] = C(D)$ holds.

We shall generalize this theorem to the higher dimensional case. Let $K$ be a compact subset of $C^*_n$ and $T$ a totally real subset of $C^m$. Let $f = (f_1, \ldots, f_m)$ be a continuous map of $K$ into $T$. For any point $w$ of $f(K)$, we set $L_w = \{z \in K : f(z) = w\}$. We consider a uniform algebra
$B = \{ g \in C(K) : g\vert_{L_w} \in A(L_w) \text{ for every } w \in f(K) \}$.

If $g$ is in $A = [z_1, \ldots, z_n, f_1, \ldots, f_m, K]$, then, for every $w \in f(K)$, $g\vert_{L_w} \in P(L_w)$, and therefore we have $A \subseteq B$. If $A = B$ holds, then we have $P(L_w) = A(L_w)$ for every $w \in f(K)$.

**Theorem 3.** Suppose that there exists a polynomially convex compact set $T_0$ such that $f(K) \subseteq T_0 \subseteq T$. If $P(L_w) = A(L_w)$ holds for every $w \in f(K)$, then we have $A = B$. In particular, if $P(L_w) = C(L_w)$ holds for every $w \in f(K)$, then $A = C(K)$.

Proof. Fix an arbitrary function $g$ in $B$ and an arbitrary positive number $\varepsilon$. Let $f$ be any point of $f(K)$. Since $P(L_w) = A(L_w)$, we can find a polynomial $P_a(z)$ such that $\| g - P_a \|_{L_a} < \varepsilon /2$. By the continuity of $g - P_a$, there exists a positive number $\delta$ such that $\| g - P_a \|_{L_w} < \varepsilon$ for every $L_w$ contained in the $\delta$-neighborhood of $L_a$ in $C_z^n$. By the continuity of $f$, we can find a positive number $\eta$ such that $L_w$ is contained in the $\delta$-neighborhood of $L_a$ for every $w \in f(K)$ with $|w - f_a| < \eta$. Thus we have $\| g - P_a \|_{L_w} < \varepsilon$ for every $w \in f(K)$ with $|w - f| < \eta$. By the compactness of $f(K)$, we can choose a finite open covering $\{ V_\nu \}_{\nu=1}^N$ of $f(K)$ in $C_z^n$ and a set of polynomials $\{ P_\nu(z) \}_{\nu=1}^N$ so that, for every $\nu$ and for every $w \in f(K) \cap V_\nu$, $\| g - P_\nu \|_{L_w} < \varepsilon$ holds. Let $\{ \lambda_\nu(w) \}$ be a partition of unity subordinate to $\{ V_\nu \}$. Since $T$ is totally real and $T_0$ is polynomially convex, we have $P(T_0) = C(T_0)$ by Theorem 2 and Oka's approximation theorem. Therefore, for every $\nu$, there exists a polynomial $Q_\nu(w)$ in $w$ such that

$$\| \lambda_\nu - Q_\nu \|_{T_0} < \varepsilon [\sum P_\nu]^{-1}.$$ 

We set $h(z) = \sum \nu Q_\nu(f(z)) P_\nu(z)$. Then $h$ belongs to $A$.

Let $x$ be an arbitrary point of $K$. We denote by $\Lambda$ the set of indices $\nu$ for which the point $f(x)$ belongs to $V_\nu$. Since $\lambda_\nu(f(x)) = 0$ for any $\nu \in \Lambda$, we have

$$|g(z) - h(z)| \leq \sum_{\nu \in \Lambda} \lambda_\nu(f(z)) |g(z) - P_\nu(z)| + \sum_{\nu = 1}^{N} \lambda_\nu(f(z))|Q_\nu(f(z))| |P_\nu(z)| < \varepsilon [\sum \lambda_\nu(f(z)) + \varepsilon [\sum P_\nu]^{-1} \sum P_\nu(z)] < 2\varepsilon .$$

From this it follows that $g$ belongs to $A$.

**Corollary.** Suppose that $f_1$ are real valued continuous functions defined on $K$, $1 \leq k \leq m$. If $P(L_w) = A(L_w)$ holds for every $u \in f(K)$, then we have $A = B$. In particular, if $P(L_w) = C(L_w)$ holds for every $u \in f(K)$, then we have $A = C(K)$.

Proof. We can consider $f = (f_1, \ldots, f_m)$ as a continuous map of $K$ into a real subspace $R^n_u$ of $C_z^n$. $R^n_u$ is a totally real set in $C_z^n$. For a sufficiently large
polydisk \(D_0, T_0 = D_0 \cap \mathbb{R}^n\) is a polynomially convex set containing \(f(K)\). Thus, all the conditions of Theorem 5 are satisfied.

As an application of this corollary, we give an example to show that a compact set \(K\) satisfying \(H(K) = C(K)\) is not necessarily a subset of a totally real set.

Let \(K = \prod_{\nu=1}^n K_\nu\) be a compact subset of \(C^n\) and let \(f_\nu\) be real valued continuous functions defined on \(K_\nu\) respectively. Suppose that, for every \(u_\nu \in f_\nu(K_\nu)\), the level set \(L_{\nu u} = \{z_\nu \in K_\nu : f_\nu(z_\nu) = u_\nu\}\) has no interior points and does not divide \(C_{1,\nu}\). Then, we have \(P(L_{\nu u}) = C(L_{\nu u})\). Set \(f(z) = (f_1(z), \ldots, f_n(z))\), \(z \in K\). Then for every vector \(u \in f(K)\), we have \(L_u = \prod_y L_{u y}\), and therefore \(P(L_u) = C(L_u)\).

This follows from Stone-Weierstrass's theorem, since the totality of polynomials that are real valued on \(K\) separates the points of \(K\). It follows from Corollary of Theorem 3 that \(\mathcal{A} = [z_1, \ldots, z_n, f_1, \ldots, f_n; K]\) coincides with \(C(K)\). Set

\[
K^* = \{ (z, f(z)) \in C^{2n}_{(z, w)} : w_\nu = f_\nu(z_\nu), \ 1 \leq \nu \leq n \}.
\]

The projection of \(C^{2n}_{(z, w)}\) onto \(C^n\) induces isomorphisms of \(P(K^*)\) onto \(\mathcal{A}\) and of \(C(K^*)\) onto \(C(K)\). Thus, we have \(P(K^*) = C(K^*) = H(K^*)\).

However, \(K^*\) is not necessarily totally real. We consider a simple case when \(f_\nu(z_\nu) = q_\nu(x_\nu)\), where \(q_\nu(x)\) are \(C^\infty\) functions defined on an open interval \(I = (-2, 2)\) of a real variable \(x\). Suppose that, for every \(\nu\), and for every \(s \in q_\nu(I)\), the level set \(\{x \in I : q_\nu(x) = s\}\) is a discrete set. Set \(K_\nu = \{z_\nu \in C_{1,\nu} : |z_\nu| \leq 1\}\). Then, we have \(H(K^*) = C(K^*)\) by the argument above. Set

\[
M = \{ (z, f(z)) \in C^{2n}_{(z, w)} : w_\nu = f_\nu(z_\nu), \ x_\nu \in I, \ 1 \leq \nu \leq n \}.
\]

Then, \(M\) is a \(C^\infty\) real submanifold of \(C^{2n}_{(z, w)}\). It follows from Lemma 3 that the complex rank of \(M\) at the point \((z^0, f(z^0))\) of \(M\) is given by \(n - \text{rank} \left[ \frac{\partial f_\nu}{\partial z_\nu}(z^0) \right]\).

We impose an additional assumption that every \(q_\nu(x)\) has an isolated zero at \(x = 0\). Set

\[
E_0 = \{ z \in C^n : x_\nu = 0, \ 1 \leq \nu \leq n \},
\]

and

\[
E = \{ (z, f(z)) \in C^{2n}_{(z, w)} : z \in E_0 \}.
\]

Since

\[
\text{rank} \left[ \frac{\partial f_\nu}{\partial z_\nu}(z) \right] = \text{rank} \begin{bmatrix} q_1'(0) & \cdots & 0 \\ 0 & \cdots & q_n'(0) \end{bmatrix} = 0
\]

at every point \(z \in E_0\), \(M\) has the complex rank \(n\) at every point of \(E\). Thus, \(K^*\)
can not be a subset of a totally real set in $C^2_{\mathbb{R}}$. We remark that $E$ is an $n$-dimensional totally real subspace of $C^2_{\mathbb{R}}$.

7. Interpolation sets

Let $G$ be a bounded domain in $C^n$. A closed subset $K$ of the Silov boundary of $A(G)$ is called an interpolation set for $A(G)$, if $A(G)|_K=C(K)$. If, for every function $f$ in $A(G)$, there exists a function $F$ in $A(G)$ such that $F(z)=f(z)$ for $z\in K$ and $|F(z)|<\|f\|_K$ for $z\in \overline{G}\setminus K$, then $K$ is called a peak interpolation set for $A(G)$. It is known that an interpolation set $K$ is a peak interpolation set if and only if $K$ is a peak set for $A(G)$, that is, there exists a function $f$ in $A(G)$ such that $f(z)=1$ for $z\in K$ and $|f(z)|<1$ for $z\in \overline{G}\setminus K$.

When $G$ is a compact polydisk $D^n$, the Silov boundary of $A(D^n)$ is the n-dimensional torus $T^n$. In [11], Stout proved that a closed subset $K$ of $T^n$ is a peak interpolation set if and only if $K$ is the zero set of a function in $A(D^n)$. We shall consider the case when $G$ is an analytic polyhedron or a strictly pseudo-convex domain.

We use the following lemma known in the theory of uniform algebras (cf. Stout [12], Chap. 4).

**Lemma 7.** Suppose that $K$ satisfies the following conditions:

(i) $\overline{G}\setminus K$ is simply connected;

(ii) there exists a function $h\in A(G)$ such that $h(z)=0$ for $z\in \overline{G}\setminus K$.

Then, $K$ is a peak set for $A(G)$, and $A(G)|_K$ is a closed subalgebra of $C(K)$.

**Theorem 4.** Let $G$ be a bounded analytic polyhedron in $C^n$ defined by

$$G = \{z\in U: |f_\nu(z)|<1, 1\leq \nu \leq n\},$$

where $U$ is an open set containing $G$, and $f_\nu$ are functions holomorphic in $U$. Suppose that $\det \begin{bmatrix} \frac{\partial f_\nu}{\partial z_k} \end{bmatrix}$ has no zeros on the set

$$\Gamma = \{z\in U: |f_\nu(z)|=1, 1\leq \nu \leq n\}.$$ 

If $K$ is a closed subset of $\Gamma$ satisfying the condition (i) and (ii) of Lemma 7, then $K$ is a peak interpolation set for $A(G)$.

Proof. It suffices to prove that $A(G)|_K$ is dense in $C(K)$, by Lemma 7. Choose a positive number $r>1$ so that the open set

$$V = \{z\in U: |f_\nu(z)|<r, 1\leq \nu \leq n\}$$
SAKAI is relatively compact in $U$. We first prove that $K$ is $O_Y$-convex. If $z_0$ is a point of $V \setminus \bar{G}$, then we can find a function $g$ holomorphic in $V$ such that $|g(z_0)| > |g|_G$, since $\bar{G}$ is $O_Y$-convex. If $z_0$ is a point of $\bar{G} \setminus K$, we have $|h(z_0)| > 0$ by the condition (ii) of Lemma 7. There exists a function $\bar{h}$ holomorphic in a neighborhood of $\bar{G}$ such that $||h-\bar{h}||_G < |h(z_0)|/4$ (cf. Petrosjan [8]). Since $\bar{G}$ is $O_Y$-convex, we can find a function $g$ holomorphic in $V$ such that $||h-g||_G < |h(z_0)|/4$. Therefore, we have $|g(z_0)| > |g|_G$. Thus, $K$ is $O_Y$-convex. From this it follows that $H(K, V) = H(K)$. Since $H(K, V)$ is contained in the closure of $A(\bar{G})|_K$ in $C(K)$ and since $A(\bar{G})|_K$ is closed in $C(K)$, it remains to prove that $H(K) = C(K)$.

We consider the function $$\rho(z) = \sum_{\nu}(|f_{\nu}(z)|^2 - 1)^2$$ in $U$. Then we have $$H(\rho; \xi)(z) = 2 \sum_{\nu}(|f_{\nu}(z)|^2 - 1)\sum_{k} \frac{\partial f_{\nu}(z)}{\partial \xi_k} \xi_k|^2 + 2 \sum_{\nu}|f_{\nu}(z)|^2 \sum_{k} \frac{\partial f_{\nu}(z)}{\partial \xi_k} \xi_k|^2.$$ Since $\text{det} \left[ \frac{\partial f_{\nu}}{\partial \xi_k} \right] \neq 0$ and $|f_{\nu}(z)| = 1$ on $\Gamma$, $\rho$ is in $\mathcal{D}(U_i)$ for a neighborhood $U_i$ of $\Gamma$, and therefore $\Gamma$ is totally real. Thus the theorem follows from Theorem 2.

In the next place, we consider the case of strictly pseudoconvex domains (not necessarily with smooth boundaries).

**Theorem 5.** Let $G$ be a bounded strictly pseudoconvex domain in $C^n$ defined by $$G = \{z \in U: \sigma(z) < 0\},$$ where $U$ is an open set containing $\bar{G}$ and $\sigma$ is strictly plurisubharmonic in $U$. Let $K$ be a closed subset of $\partial G$ satisfying the condition (i) of Lemma 7 and the following condition:

(ii)' there exists a function $h$ holomorphic in $U$ such that $K = \{z \in G: h(z) = 0\}$ and $dh$ has no zero on $K$.

Then $K$ is a peak interpolation set for $A(G)$.

**Proof.** Since the condition (ii)' is stronger than (ii) of Lemma 7, it suffices to prove that $A(\bar{G})|_K$ is dense in $C(K)$. We can assume that $U$ is holomorphically convex and the zero set $X$ of $h$ in $U$ is a closed submanifold of $U$.

Let $z$ be any point of $K$. Then there exist a neighborhood $U_z$ of $z$ with $U_z \subset U$ and a local complex coordinate $\xi = (\xi_1, \ldots, \xi_n)$ in $U_z$ such that $X \cap U_z = \{\xi \in U_z: \xi_n = 0\}$. Let $\sigma(\xi_1, \ldots, \xi_{n-1})$ be the restriction of $\sigma$ to $X \cap U_z$ and set $$\rho(\xi) = \sigma(\xi_1, \ldots, \xi_{n-1}) + |\xi_n|^2, \quad \xi \in U_z.$$
Then $p$ is a function in $\mathcal{P}(U)$ and satisfies
\[ K \cap U = \{ \zeta \in U : p(\zeta) = 0 \} . \]

It follows from Proposition 1 that $K$ is totally real. Thus we have $H(K) = C(K)$.

Since $\sigma$ is a strictly plurisubharmonic function on $X$ and since $K = \{ \zeta' \in X : \sigma(\zeta') \leq 0 \}$, $K$ is $\mathcal{O}_X$-convex. Since $X$ is the zero set of $h$ in $U$, $K$ is $\mathcal{O}_U$-convex. Thus we can conclude that $A(\overline{G})|_K = C(K)$, by using an argument similar to one used in the proof of Theorem 4.

We remark that, when $n=1$, the condition (ii) reduces our problem to a very simple one, since then $K$ is a finite point set. When $n>1$, it is not trivial. We consider, for example, the case when $G$ is the unit ball in $C^n$ and $h(z) = \sum x_i^2 - 1$. Then, all the conditions of Theorem 5 are satisfied for $K = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sum x_i^2 = 1 \}$. In this case, $K$ is an $(n-1)$-dimensional totally real submanifold of $C^n$.

8. CR-functions

Let $T$ be a totally real set in $C^n$ and $G$ a holomorphically convex open subset of $C^n$. The projections of $C^n = C^n \times C^n$ onto $C^n$ and $C^n$ are denoted by $\pi_1$ and $\pi_2$ respectively. Let $K$ be a compact set of $C^n$. For any $z$ of $C^n$, we set $K_z = \pi_1^{-1}(z) \cap K$ and $K'_z = \pi_2(K_z)$.

**Theorem 6.** Let $K$ be a compact subset of $C^n$ satisfying the following conditions:

(i) $\pi_1(K) = T$;

(ii) for every $z \in T$, there exists a complex analytic subvariety $X_z$ of $G$ such that $K'_z$ is a $\mathcal{O}_{X_z}$-convex compact subset of $X_z$. If $f$ is a $C^\infty$ function defined in a neighborhood $U$ of $K$ in $C^n$ which is holomorphic in $w$ on $X_z \cap U$ for every $z \in T$, then $f$ belongs to $H(K)$.

**Proof.** Let $\alpha$ be any point of $T$. By the compactness of $K_\alpha$, for every number $\delta > 0$, there exists a $\eta > 0$ such that $K_{\alpha}$ is contained in the $\delta$-neighborhood of $K_\alpha$ in $C^n$, for every $z \in T$ with $|z - \alpha| < \eta$. Since $K_\alpha$ is $\mathcal{O}_{X_\alpha}$-convex, and since $X_\alpha$ is a closed subvariety of $G$, there exists a function $g_\alpha(w)$ holomorphic in $G$ such that
\[ |f(\alpha, w) - g_\alpha(w)| < \varepsilon/2 , \quad w \in K'_\alpha . \]

By the continuity of $f - g_\alpha$, there exists $\eta > 0$ such that
\[ |f(z, w) - g_\alpha(w)| < \varepsilon , \quad w \in K'_z . \]
holds for every \( z \in T \) with \( |z - \alpha| < \eta \).

By the compactness of \( T \), we can find a finite open covering \( \{ V_\nu \}_{\nu=1}^N \) of \( T \) in \( C_r^2 \) and a set of functions \( \{ g_\nu(w) \}_{\nu=1}^N \) each holomorphic in \( G \) and satisfying

\[
|f(z, w) - g_\nu(w)| < \varepsilon, \quad z \in T \cap V_\nu, \quad w \in K'_\nu.
\]

Choose a partition of unity \( \{ \lambda_\nu(z) \} \) subordinate to \( \{ V_\nu \} \). Since \( T \) is totally real, there exists a set of functions \( \{ h_\nu(z) \} \) each holomorphic in a neighborhood \( V \) of \( T \) in \( C_r^2 \) satisfying

\[
|h_\nu(z) - \lambda_\nu(z)| < \varepsilon \sum_\nu \| g_\nu \|_{\varphi_2(K)}^{-1}, \quad z \in T.
\]

We set \( f_\nu(z, w) = \sum_\nu h_\nu(z) g_\nu(w) \).

Let \((z, w)\) be any point of \( K \). We denote by \( \Lambda \) the set of indices \( \nu \) for which \( z \) belongs to \( V_\nu \). Then we have

\[
|f(z, w) - f_\nu(z, w)| \leq \sum_{\nu \in \Lambda} \lambda_\nu(z) |f(z, w) - g_\nu(w)|
\]

\[
+ \sum_{\nu \in \Lambda} \lambda_\nu(z) - h_\nu(z) |g_\nu(w)|
\]

\[
< \varepsilon \sum_\nu \nu_\nu(z) + \varepsilon \sum_\nu \left[ \sum_\nu \| g_\nu \|_{\varphi_2(K)}^{-1} \right] |g_\nu(w)| < 2 \varepsilon.
\]

Since \( f_\nu(z, w) \) is holomorphic in the open set \( V \times G \), we have \( f|_K \in H(K) \), as required.

Let \( M \) be a holomorphic \( CR \)-submanifold of \( C^N \). A \( C^\infty \) function \( f \) defined on \( M \) is called a \( CR \)-function, if \( f \) is holomorphic with respect to the complex coordinates in \( M \). Let \( K \) be a compact subset of \( M \). We denote by \( CR(K) \) the closure in \( C(K) \) of the class of functions each of which is the restriction of a \( CR \)-function defined on a neighborhood of \( K \) in \( M \). If \( r(M) = 0 \), then \( CR(K) \) trivially coincides with \( C(K) \). Let \( T \) be a totally real submanifold of \( C_r^2 \) and \( G \) a holomorphically convex compact subset of \( C^2_\nu \). If \( M \) is a closed real submanifold of \( T \times G \) such that, for every \( z \in T \), \( M'_z = \pi_2(M) \) is an \( r \)-dimensional closed complex analytic submanifold of \( G \), then \( M \) is a \( CR \)-submanifold of \( C^N \) of complex rank \( r \). In this case, we have the following corollary.

**Corollary.** Let \( K \) be a compact subset of \( M \) such that every \( K'_z \) is \( C^r \)-convex. Then we have \( H(K) = CR(K) \).

OSAKA UNIVERSITY
References


