ON CHARACTERISTIC CLASSES OF RIEMANNIAN FOLIATIONS

SHIGEYUKI MORITA

(Received February 17, 1978)

0. Introduction

In [6], Lazarov and Pasternack defined characteristic classes for Riemannian foliations and investigated their properties very closely. Their theory is a special case of the theory of characteristic classes for foliated bundles due to Kamber and Tondeur [4]. From this point of view, the characteristic classes are defined by looking at the unique Riemannian connection on the orthonormal frame bundle of the foliation, whose structure group is the orthogonal group $O(n)$ ($n$ is the codimension of the foliation). However, if we enlarge the structure group to $E(n)$, the group of Euclidean motions on $\mathbb{R}^n$, and if we look at a system of differential forms defined by considering the Cartan connection, then we obtain more characteristic classes than those defined by Lazarov and Pasternack. The purpose of this note is to clarify this point. Thus this note could be considered as an addendum to [6].

In §1 we give the main construction of the characteristic classes and in §2, the concept of “p-th scalar curvature” is defined for every Riemannian manifold. In §3 the cohomology of a truncated Weil algebra of e(n), the Lie algebra of $E(n)$, is determined and §4 is devoted to the study of continuous variation of the new characteristic classes.

1. Construction of the characteristic classes

Let $F$ be a Riemannian foliation on a smooth manifold $M$ defined by a maximal family of submersions

$$f_\alpha: U_\alpha \to (\mathbb{R}^n_\alpha, g_\alpha)$$

from open sets $U_\alpha$ in $M$ to a Riemannian manifold $(\mathbb{R}^n_\alpha, g_\alpha)$ ($g_\alpha$ is a Riemannian metric on $\mathbb{R}^n$) such that for every $x \in U_\alpha \cap U_\beta$ there exists a local isometry $\gamma_{\beta\alpha}: f_\alpha(x) \to$ neighborhood of $f_\beta(x)$ with $f_\beta = \gamma_{\beta\alpha} f_\alpha$ near $x$. Now let $O(\mathbb{R}^n_\alpha)$ be the orthonormal frame bundle of $\mathbb{R}^n_\alpha$. Since $O(\mathbb{R}^n_\beta)|_{\gamma_{\beta\alpha}(U)} =$

1) Partially supported by the Sakkokai Foundation
\[ 162 \quad S. \quad M. \quad MORITA \]

where \( U \) is a small neighborhood of \( f_a(x) \), we can define a principal bundle \( O(F) \) over \( M \) such that \( O(F)|_{U_a} = f_a^*(O(R^n_a)|_{f_a(U_a)}) \). We call \( O(F) \) the orthonormal frame bundle of the foliation \( F \). Now since the canonical form and the Riemannian connection form of Riemannian manifolds are preserved by isometries, we can define \( R^n \) and \( \mathfrak{so}(n) \) valued one forms \( \theta_0 \) and \( \theta_1 \) on \( O(F) \) such that \( \theta_0|_{U_a} = f_a^*(\theta_0^a) \) and \( \theta_1|_{U_a} = f_a^*(\theta_1^a) \), where \( \theta_0^a \) and \( \theta_1^a \) are the canonical form and the Riemannian connection form of \( R^n_a \), respectively. We call \( \theta_0 \) and \( \theta_1 \) the canonical form and the Riemannian connection form of \( F \).

We may also consider the pair \((\theta_0, \theta_1)\) as an \( e(n) \)-valued one form on \( O(F) \) whose restriction to \( U_a \) is the pull back under \( f_a^* \) of the \( e(n) \)-valued one form \((\theta_0^a, \theta_1^a)\) on \( O(R^n_a) \), which may be considered as the unique torsionfree Cartan connection form of \( R^n_a \). With respect to the usual basis of \( e(n) = R^n \oplus \mathfrak{so}(n) \), we can represent \( \theta_0 \) and \( \theta_1 \) by \( n \) forms \( \theta_1, \theta_2, \ldots, \theta_n \) and a skew symmetric matrix of differential forms \( \theta^j \). Now if we denote \( W(e(n)) \) for the Weil algebra of \( e(n) \), then \( \theta_0 \) and \( \theta_1 \) define a d.g.a. map

\[ \varphi: W(e(n)) \rightarrow \Omega^*(O(F)) \]

where \( \Omega^*(O(F)) \) is the de Rham complex of \( O(F) \). Let \( \omega^j, \Omega^j, \Omega^i \in W(e(n)) \) be the universal connection and curvature forms corresponding to the usual basis of \( e(n) \). Then \( \varphi \) satisfies \( \varphi(\omega^j) = \theta^j \) and \( \varphi(\Omega^j) = \theta^j \). Now we know the following conditions (cf. [5]).

\[
\begin{align*}
(i) \quad d\theta^i &= -\sum_j \theta^j \wedge \theta^i \quad \text{(torsionfree-ness)}, \\
(ii) \quad d\theta^j &= -\sum_k \theta^k \wedge \theta^i + \Theta^j, \\
&\quad \text{where} \quad \Theta^j = \frac{1}{2} \sum_{k,l} R^j_{kl} \theta^k \wedge \theta^l, \\
(iii) \quad \sum_j \Theta^j \wedge \theta^j &= 0 \quad \text{(the first Bianchi's identity)}. 
\end{align*}
\]

In view of these conditions, we define an ideal \( I \) of \( W(e(n)) \) as the one generated by the following elements.

\[
\begin{align*}
(i)' \quad \Omega^i, \\
(ii)' \quad \text{elements whose "length" \( l \) is greater than \( n \), where \( l \) is defined by the conditions: \( l(\omega^j) = l(\Omega^j) \neq 0 \), \( l(\omega^i) = 1 \) and \( l(\Omega^i) = 2 \)}, \\
(iii)' \quad \sum_j \Omega^j \wedge \omega^j. 
\end{align*}
\]

Then it is easy to see that \( I \) is a subcomplex of \( W(e(n)) \). The condition (1.1) shows that \( \varphi(I) = 0 \). Therefore, if we denote \( \tilde{W}(e(n)) = W(e(n))/I \), then \( \varphi \) induces a d.g.a. map

\[ \varphi: W(e(n)) \rightarrow \Omega^*(O(F)) \]
Now suppose that the normal bundle of $F$ is trivialized by a cross section $s: M \to O(F)$, then we obtain

$$H^*(\mathcal{W}(e(n))) \to H^*_{DB}(O(F)) \to H^*_{DB}(M).$$

Since this construction is functorial, we finally obtain

$$H^*(\mathcal{W}(e(n))) \to H^*(BR\Gamma_n; R)$$

where $BR\Gamma_n$ is the classifying space for codimension $n$ Riemannian Haefliger structures with trivial normal bundles (cf. [6]). In the general case we have

$$H^*(\mathcal{W}(e(n))_{O(n)} \to H^*(BR\Gamma_n^+; R)$$

where the left hand sides are the cohomology of subcomplexes $\mathcal{W}(e(n))_{O(n)}$ (resp. $\mathcal{W}(e(n))_{SO(n)}$) of $O(n)$ (resp. $SO(n)$) basic elements of $\mathcal{W}(e(n))$ and $BR\Gamma_n^+$ (resp. $BR\Gamma_n^+$) are the classifying space for the Riemannian (resp. oriented Riemannian) Haefliger structures. This is our construction of the characteristic classes for Riemannian foliations. Now if we ignore the canonical form $\theta_\sigma$, then we obtain

$$H^*(\mathcal{W}(\mathfrak{so}(n))) \to H^*(BR\Gamma_n; R)$$

where $\mathcal{W}(\mathfrak{so}(n)) = W(\mathfrak{so}(n))$ modulo the ideal $I \cap W(\mathfrak{so}(n))$. This is nothing but the characteristic classes defined by Kamber and Tondeur [4] and is the same as those defined by Lazarov and Pasternack [6].

2. Scalar curvatures

In this section, we define the notion of "$p$-th scalar curvature" for every Riemannian manifold $M$ of dimension $n$, where $p$ is an even integer $\leq n$. First of all we recall the concept of $p$-th sectional curvature $\gamma_p$ defined by Allendoerfer and studied by Thorpe [8]. Let $G_p(M)$ be the Grassman bundle of tangent $p$-planes of $M$. For every $p$-plane $(x, P) \in G_p(M)$, $\gamma_p(x, P)$ is defined to be the Lipschitz-Killing curvature at $x \in M$ of the $p$-dimensional submanifold of $M$ geodesic at $x$ and tangent to $P$ at $x$. Thus $\gamma_p$ is a smooth function on $G_p(M)$. In terms of the curvature tensor $R$ of $M$, $\gamma_p$ is expressed by

$$\gamma_p(x, P) = \frac{(-1)^{p/2}}{2^{p/2}p!} \sum_{\sigma \tau} \text{sgn}(\sigma) \text{sgn}(\tau) g(R(u_{\sigma(1)}^\tau, u_{\sigma(2)}^\tau)u_{\tau(1)}^\sigma, u_{\tau(2)}^\sigma)$$

$$\cdots g(R(u_{\sigma(p-1)}^\tau, u_{\sigma(p)}^\tau)u_{\tau(p-1)}^\sigma, u_{\tau(p)}^\sigma)$$

where $g$ is the metric tensor of $M$, $u_1, \ldots, u_p$ is an orthonormal basis for $P$ and $\sigma, \tau$ range over the $p$-th symmetric group $S_p$. Now if we average this $\gamma_p$ over
each fibre of \( G_p(M) \to M \), then we obtain a real valued smooth function \( R_p \) on \( M \). Let us call this function the "\( p \)-th scalar curvature" of \( M \). In terms of the curvature tensor \( R \), \( R_p \) is expressed by the formula,

\[
R_p(x) = \frac{(-1)^{n/2}}{2^{n/2}p! \binom{n}{p} \sum_i \gamma_p(x, P_i)}.
\]

Here the sum ranges over all \( p \)-tuples \( i = (i(1), \ldots, i(p)) \) with \( 1 \leq i(1) < \cdots < i(p) \leq n \) and \( P_i \) is the \( p \)-plane at \( x \in M \) spanned by \( u_{i(1)}, \ldots, u_{i(p)} \), where \( u_1, \ldots, u_n \) is any orthonormal frame at \( x \). \( R_2 \) is the usual scalar curvature of \( M \) (up to a non-zero constant) and \( R_n \) is the Lipschitz-Killing curvature if \( n \) is even. Now as in \( \S \) 1, let \( \theta^i \), \( i = 1, \ldots, n \) and \( \theta_j \) be the canonical form and the Riemannian connection form of \( M \). Thus they are one forms defined on \( O(M) \), the orthonormal frame bundle of \( M \). Let us define a smooth function \( \bar{R}_{jkl} \) on \( O(M) \) by

\[
d\theta_j = -\sum_k \theta^i \wedge \theta^k + \Theta_j^i,
\]

\[
\Theta_j^i = \frac{1}{2} \sum_{k,l} \bar{R}_{jkl} \theta^k \wedge \theta^l.
\]

For every even integer \( p \leq n \), we define a smooth function \( \bar{R}_p \) on \( O(M) \) as follows. We consider the \( n \)-form:

\[
det((n-p)\theta, (p/2)\Theta) = \sum_{\sigma} \text{sgn}(\sigma) \theta^\sigma(1) \wedge \cdots \wedge \theta^\sigma(n-p) \wedge \Theta^\sigma(n-p+1) \wedge \cdots \wedge \Theta^\sigma(n-1),
\]

where \( \sigma \) ranges over the \( n \)-th symmetric group \( S_n \). Then \( \bar{R}_p \) is defined to be a function satisfying the equality.

\[
det((n-p)\theta, (p/2)\Theta) = \bar{R}_p \theta^1 \wedge \cdots \wedge \theta^n.
\]

Then it is easy to see that the function \( \bar{R}_p \) is constant on each fibre of the bundle \( \pi: O(M) \to M \). In fact we have

**Proposition 2.1.** \( \bar{R}_p = (-1)^{n/2} n! \pi^* R_p \).

Proof. Let \( x \in M \) and let \( u_1, \ldots, u_n \) be an orthonormal frame at \( x \). We choose a coordinate around \( x \) such that \( \frac{\partial}{\partial x_i} = u_i \) for \( i = 1, \ldots, n \). If \( R_{jkl} \) is the component of \( R \) with respect to this coordinate, then from (2.2) we have

\[
R_p(x) = \frac{(-1)^{n/2}}{2^{n/2}p! \binom{n}{p} \sum_{\sigma, i}} \text{sgn}(\sigma) \text{sgn}(i) R_{jkl}^{(\sigma(1))} \cdots
\]

where \( i = (i(1), \ldots, i(p)) \) ranges over all \( p \)-tuples with \( 1 \leq i(1) < \cdots < i(p) \leq n \). On the other hand we have
where j = (j(1), ..., j(p)) ranges over every permutation of \((n-p+1, \ldots, n)\). We have

\[
(2.7) \quad \bar{R}_p(x, u) = \frac{1}{2^p} \sum_{\sigma, \tau} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \begin{pmatrix} \sigma(1) & \ldots & \sigma(n-p) & j(1) & \ldots & j(p) \
\end{pmatrix}
\]

where \(\sigma = (\sigma(1), \ldots, \sigma(n))\) ranges over every permutation of \((n-p+1, \ldots, n)\). Therefore by comparing (2.5) and (2.7) we obtain

\[
\bar{R}_p(x, u) = (-1)^{p/2} n! R_p(x, u).
\]

Since it is easy to see that \(\bar{R}_p(x, u)\) does not depend on the choice of the frame \(u\), this completes the proof. q.e.d.

Now let \(f : M \to N\) be an isometry of Riemannian manifolds \(M, N\). Then from the definition of \(R_p\), it is clear that \(R_p(M) = f^* R_p(N)\). Next we investigate how the \(p\)-th scalar curvature behaves under the scale change \(g \to k^2 g\). Let \(\bar{R}_p\) be the \(p\)-th scalar curvature of the Riemannian manifold \((M, k^2 g)\). Then we have

**Proposition 2.2.** \(\bar{R}_p = k^{-p} R_p\).

**Proof.** This follows from an elementary calculation.

**3. Cohomology of \(\bar{W}(e(n))\)**

In this section we compute the cohomology of the truncated Weil algebra \(\bar{W}(e(n))\). For each even integer \(p \leq n\), let \(r_p\) be an element of \(\bar{W}(e(n))\) defined by

\[
r_p = \det((n-p)\omega_s, (p/2)\Omega) = \sum_{\sigma} \operatorname{sgn}(\sigma) \omega^{\sigma(1)} \cdots \omega^{\sigma(n-p)} \Omega_{\sigma(n-p+1)} \cdots \Omega_{\sigma(n)}.
\]

Then it is easy to see that \(r_p\) is closed so that it defines a cohomology class in \(H^*(\bar{W}(e(n)))\). Next we define \(h_i \in \bar{W}(e(n)) \quad (i = 2, 4, \ldots, n-1)\) for \(n\) odd and \(h_i, h_x \in \bar{W}(e(n)) \quad (i = 2, 4, \ldots, n-2)\) for \(n\) even by
where $P_i$, $X$ are the $i$-th Pontrjagin form and the Euler form respectively, $T$ denotes Chern-Simons’ transgression form [2] (at the Weil algebra level) and $\sim$ is the projection modulo the ideal $I$. Let

$$E_n = E(h_2, h_4, \ldots, h_{n-1}) \quad n \text{ odd},$$

$$= E(h_2, h_4, \ldots, h_{n-2}, h_n) \quad n \text{ even},$$

be the exterior algebra generated by $h_2$, $\ldots$, and let $r_pE_n$ be the vector space over $R$ with basis $\{r_px_i\}$ where $\{x_i\}$ is a basis of $E_n$. Then by the truncation, every element $r_px_i$ is closed. We have the following.

**Theorem 3.1.** $H^*(W(e(n))) = H^*(W(so(n))) \oplus \sum_{p \text{ even}} r_pE_n$.

Here $H^*(W(so(n)))$ has been determined by Kamber and Tondeur and is isomorphic to $H^*(RW_n)$, where $RW_n$ is the differential complex defined by Lazarov and Pasternack [6]. Before proving the Theorem we describe the geometric meaning of the second term of Theorem 3.1. Thus let $F$ be a codimension $n$ Riemannian foliation on a smooth manifold $M$ defined by sumbersons $f_a: U_a \to R^*_a$ (see § 1). We define a smooth function $R_p$ on $M$, for every even integer $p$ with $0 \leq p < n$, as follows. $R_0$ is the identity function of $M$ and for $p \geq 2$, $R_p| U_a = f_a^*(R^*_p)$, where $R^*_p$ is the $p$-th scalar curvature of $R^*_a$ defined in § 2. This is well-defined because $R_p$ is invariant under isometries. Also we have an $n$-form $v$ on $M$ such that $v| U_a = f_a^*(\text{volume form of } R^*_a)$. With these understood, $r_p$ of the foliation $F$ is represented by the $n$-form $(-1)^p/n!R_pv$ (cf. (2.4) and Proposition 2.1.). Now assume that we have a cross section $s: M \to O(F)$ and let $TP_i(F), TX(F)$ be the Chern-Simons’ transgression forms corresponding to the Riemannian connection on $O(F)$. Then $h_i$ (resp. $h_o$) of the foliation is represented by the form $s*TP_i(F)$ (resp. $s*TX(F)$). Now we prove our Theorem 3.1.

**Proof of Theorem 3.1.** Let us define the “weight” function $w$ on the elements of $W(e(n))$ by $w(o^j) = 1$, $w(o^j) = w(\Omega^j) = 0$, and define $J_0 = \{x \in W(e(n)); w(x) = 0\}$, $J_+ = \{x \in W(e(n)); w(x) > 0\}$. Then it is easy to see that both $J_0$ and $J_+$ are subcomplexes of $W(e(n))$. Moreover we have $W(e(n)) = J_0 \oplus J_+$. Therefore

$$H^*(W(e(n))) = H^*(J_0) \oplus H^*(J_+).$$

(3.1)

Now let us define a decreasing filtration $F^p$ on $J_+$ by

$$F^p = \{x \in J_+; l(x) \geq p\}$$

where $l$ is the “length” on $W(e(n))$ induced from that on $W(e(n))$. Let
\{E_r^q, d_r\} be the spectral sequence associated with this filtration. Define \(M_p\) be a sub-vector space of \(J_+\) spanned by \(\omega^{i(1)} \cdots \omega^{(p-2k)}\Omega^{j(1)} \cdots \Omega^{j(2k-1)}\) for all \(i, j, k\). Then \(\mathfrak{so}(n)\) acts on \(M_p\) by the Lie derivation. Thus \(M_p\) is an \(\mathfrak{so}(n)\)-module.

Let \(C^r(\mathfrak{so}(n); M_p)\) be the set of \(r\)-cochains on \(\mathfrak{so}(n)\) with coefficient in \(M_p\). Then it is easy to see that

\[(3.2) \quad E^0_0 \cong C^0(\mathfrak{so}(n); M_p) \]
of \(W(e(n))\). Let \(\tilde{I}(SO(n)) = I(SO(n))/I \cap I(SO(n))\) and \(\tilde{I}(O(n)) = I(O(n))/I \cap I(O(n))\). Then, by similar arguments we obtain

**Theorem 3.2.** \(H^*(W(e(n)))_{SO(n)} = \tilde{I}(SO(n)) \oplus \sum_{\rho \text{ even}} r_{\rho} R\).

\[H^*(W(e(n)))_{O(n)} = \tilde{I}(O(n)).\]

**Remark 3.3.** One may hope that one can obtain more characteristic classes for smooth foliations than those defined by Bott and Haefliger [1] by considering the Cartan connection. However this is false because we have an isomorphism \(H^*(\tilde{W}(\mathfrak{a}(n; R))) = H^*(\tilde{W}(\mathfrak{a}(n; R)))\) where \(\mathfrak{a}(n; R)\) is the Lie algebra of the \(n\)-th affine group and \(\tilde{W}\) denotes Weil algebras modulo certain ideals which are constructed by a similar argument as in the Riemannian case.

4. Continuous variation

In this section we prove that the new characteristic classes \(\sum_{\rho} r_{\rho} E_{\rho}\) defined in §3 vary continuously and independently under deformations of Riemannian foliations. Precisely we prove

**Theorem 4.1.** Let \(\dim H^*(SO(n)) = d\). Then there is a surjective homomorphism

\[H_{n+k}(BR\Gamma_n; Z) \to R^{k[(n+1)/2]} \to 0.\]

As before, let \(M\) be an oriented Riemannian manifold and let \(\pi: SO(M) \to M\) be the oriented orthonormal frame bundle of \(M\). We consider the codimension \(n\) Riemannian foliation \(F\) on \(SO(M)\) induced from the given Riemannian structure on \(M\) by the projection \(\pi\). The oriented orthonormal frame bundle of this foliation, \(SO(F)\), is the pull back of the principal bundle \(\pi: SO(M) \to M\) by the map \(\pi\). Thus we have \(SO(F) = \{(x; u, v); x \in M, u, v \in \pi^{-1}(x)\}\) and there is a commutative diagram

\[
\begin{array}{ccc}
SO(F) & \xrightarrow{f} & SO(M) \\
\pi \downarrow & & \pi \downarrow \\
SO(M) & \xrightarrow{\pi} & M
\end{array}
\]

where \(f(x; u, v) = (x, v)\) and \(\pi(x; u, v) = (x, u)\). Now we define a cross section \(s: SO(M) \to SO(F)\) of the bundle \(\pi\) by \(s(x, u) = (x, u, u)\). Then clearly the composition map \(f \circ s: SO(M) \to SO(F) \to SO(M)\) is the identity. Henceforth we denote \(F(M)\) for the foliation on \(SO(M)\) described above with the trivialization \(s\) of the normal bundle. Now assume that \(M\) satisfies the following conditions.
(i) \( P_t(M), X(M) = 0 \) where \( P_t(M) \) and \( X(M) \) are the Pontrjagin and the Euler forms of \( M \), respectively.

(ii) \( M \) is parallelizable so that there is given a bundle isomorphism \( i: M \times SO(n) \sim SO(M) \).

Let \( \tau(P_t), \tau(X) \in H^*(SO(n)) \) be the transgression images of the Pontrjagin class \( P_t \) and the Euler class \( X \), and let \( TP_t(M), TX(M) \in \Omega^*(SO(M)) \) be the Chern-Simons’ transgression forms of \( M \) corresponding to the Riemannian connection on \( SO(M) \). Then \( h_i \) and \( h_x \) of the foliation \( F(M) \) are represented by \( s*f*TP_t(M) \) and \( s*f*TX(M) \). But since \( f \circ s = id \), we obtain

\[
\begin{align*}
h_i(F(M)) &= TP_t(M) \\
h_x(F(M)) &= TX(M)
\end{align*}
\]

By the assumption (4.2)-(i), both \( TP_t(M) \) and \( TX(M) \) are closed forms and define cohomology classes in \( H^*(SO(M)) \) which is isomorphic to \( H^*(M) \otimes H^*(SO(n)) \) under the homomorphism \( i^* \). (Hereafter we identify \( H^*(SO(M)) \) with \( H^*(M) \otimes H^*(SO(n)) \) by \( i^* \).) Since the forms \( TP_t(M), TX(M) \) restricted to each fibre are closed and represent the cohomology classes \( \tau(P_t) \) and \( \tau(X) \) (cf. [2]), we have

\[
\begin{align*}
[h_i(F(M))] &= 1 \times \tau(P_t) \mod I, \\
[h_x(F(M))] &= 1 \times \tau(X) \mod I,
\end{align*}
\]

where \([\ ]\) denotes the cohomology class and \( I \) is the ideal \( H^*(M) \otimes H^*(SO(n)) \) of \( H^*(M) \otimes H^*(SO(n)) \sim H^*(SO(M)) \). Now for each even integer \( p \) with \( 0 \leq p < n \), we have the \( p \)-th scalar curvature \( R_p(M) \) of \( M \). (\( R_0 \) is defined to be the identity function of \( M \).) Then clearly \( R_p(F(M)) = \pi^* R_p(M) \) and the characteristic class \( r_p \) of \( F(M) \) is represented by

\[
r_p(F(M)) = (-1)^{p/2} n! \int_M R_p(M) \nu(M) \cdot \pi^*[M],
\]

where \( \nu(M) \) is the volume form of \( M \) and \([M]\) is the fundamental cohomology class. From (4.4) and (4.5) we obtain

**Proposition 4.2.** Let \( F(M) \) be as above and assume that \( M \) satisfies the condition (4.2). Moreover assume \( \int_M R_p(M) \nu(M) \neq 0 \) for an even integer \( p \). Then the cohomology classes \( r_p h_{i_1} \cdots h_{i_1}, r_p h_{i_1} \cdots h_{i_1} h_x \) of \( F(M) \) are represented by

\[
\begin{align*}
r_p h_{i_1} \cdots h_{i_1}(F(M)) &= (-1)^{p/2} n! \int_M R_p(M) \nu(M) \cdot [M] \times \tau(P_{i_1}) \cdots \tau(P_{i_1}), \\
r_p h_{i_1} \cdots h_{i_1} h_x(F(M)) &= (-1)^{p/2} n! \int_M R_p(M) \nu(M) \cdot [M] \times \tau(P_{i_1}) \cdots \tau(P_{i_1}) \tau(X).
\end{align*}
\]
Now we consider the special case when $M$ is the Riemannian product $S^1 \times S^{n-1}$ of unit spheres. For a unit sphere $S^1$, clearly we have $R_p(S^1) = 1$ for every $p$ (cf. [8]), and from the definition of $R_p$, it is easy to see that

$$R_p(S^1 \times S^{n-1}) = \frac{n-1}{\binom{n}{p}}$$

$$= \frac{n}{n-p}.$$  

Since $S^1 \times S^{n-1}$ satisfies the condition (4.2), from Proposition 4.2 we obtain

**Proposition 4.3.** The characteristic classes of $F(S^1 \times S^{n-1})$ are given by

$$r_p h_i \cdots h_n(F(S^1 \times S^{n-1})) = (-1)^{p/2} n! \frac{n}{n-p} v_1 v_{n-1} [S^1 \times S^{n-1}] \times$$

$$\tau(P_i) \cdots \tau(P_{i_0}),$$

$$r_p h_i \cdots h_n(F(S^1 \times S^{n-1})) = (-1)^{p/2} n! \frac{n}{n-p} v_1 v_{n-1} [S^1 \times S^{n-1}] \times$$

$$\tau(P_i) \cdots \tau(P_{i_0}) \tau(\mathcal{X}),$$

where $v_1$ is the volume of the unit sphere $S^1$.

Next we consider the Riemannian manifold $(S^1 \times S^{n-1})_k$ which is obtained from $S^1 \times S^{n-1}$ by the scale change $g \rightarrow k^2 g$. Since the Chern-Simons' TP form is invariant under the scale change, from Proposition 2.2 and Proposition 4.3 we have

**Proposition 4.4.** The characteristic classes of $F((S^1 \times S^{n-1})_k)$ are given by

$$r_p h_i \cdots h_n(F((S^1 \times S^{n-1})_k)) = (-1)^{p/2} n! \frac{n}{n-p} k^{n-p} v_1 v_{n-1} [S^1 \times S^{n-1}] \times$$

$$\tau(P_i) \cdots \tau(P_{i_0}),$$

$$r_p h_i \cdots h_n(F((S^1 \times S^{n-1})_k)) = (-1)^{p/2} n! \frac{n}{n-p} k^{n-p} v_1 v_{n-1} [S^1 \times S^{n-1}] \times$$

$$\tau(P_i) \cdots \tau(P_{i_0}) \tau(\mathcal{X}).$$

Now we are in a position to prove Theorem 4.1. In view of Proposition 4.4, it is enough to prove that the homomorphism

$$\psi: H_n(BR\Gamma_n; \mathbb{Z}) \rightarrow \mathcal{R}^{(n+1)/2}$$

defined by the characteristic classes $\{r_p\}_{0 \leq p \leq n, \ p \ even}$ is a surjection. Now the foliation $F((S^1 \times S^{n-1})_k)$ on $(S^1 \times S^{n-1})_k$ defines a homology class $\alpha_k \in H_n(BR\Gamma_n; \mathbb{Z})$
and by Proposition 4.4 its characteristic numbers are given by

\[ r_p(\alpha) = c_p k^{n-p} \]

for a non-zero \( c_p \) (\( 0 \leq p < n \), \( p \) even). We consider the homology class \( \alpha(k) = \alpha_{k(1)} + \alpha_{k(2)} + \cdots + \alpha_{k(\lfloor \frac{n+1}{2} \rfloor)} \) where \( k = (k(1), \ldots, k\left( \left\lfloor \frac{n+1}{2} \right\rfloor \right)) \) is an \( R^{(n+1)/2} \)-valued variable. The characteristic numbers of \( \alpha(k) \) are given by

\[ r_p(\alpha(k)) = \sum_i c_i k_i^{1-p} \]

where the sum ranges over \( i = 1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor \). Now let \( f: R^{(n+1)/2} \rightarrow R^{(n+1)/2} \) be the map defined by

\[
f(k(1), \ldots, k\left( \left\lfloor \frac{n+1}{2} \right\rfloor \right)) = (r_0(\alpha(k)), \ldots, r_{\left\lfloor \frac{n-1}{2} \right\rfloor}(\alpha(k))) \\
= (\sum_i c_0 k_i^n, \ldots, \sum_i c_{\left\lfloor \frac{n-1}{2} \right\rfloor} k_i^{n-2(\left\lfloor \frac{n-1}{2} \right\rfloor)})
\]

Then \( f \) is smooth and it is easy to see that the determinant of the Jacobian matrix of \( f \) is not constantly zero. Therefore we conclude that \( \text{Im} f \) contains an inner point. Hence \( \text{Im} \psi \) contains also an inner point. Since \( \text{Im} \psi \) is a subgroup of \( R^{(n+1)/2} \), it follows that \( \psi \) is surjective. This completes the proof. \( \square \)

**Remark 4.5** Lazairov and Pasternack [7] proved that certain characteristic classes for Riemannian foliations defined by them vary continuously by using the residue formula for zero-points of a Killing vector field.

If we use the sphere \( S^n \) instead of \( S^1 \times S^{n-1} \), then we can prove the following Theorems, which are refinements of Theorem 4.1.

**Theorem 4.6.** The characteristic classes \( \{ r_p \}_{0 \leq p < n, p \text{ even}} \) define a surjective homomorphism

\[ \pi_s(BR\Gamma^*_s) \rightarrow R^{(n+1)/2} \rightarrow 0. \]

**Theorem 4.7.** If \( n \) is even, then the characteristic classes \( \{ r_p \}_{0 \leq p < n, p \text{ even}} \) define a surjective homomorphism

\[ \pi_{2n-1}(BR\Gamma^*_n) \rightarrow R^{(n+1)/2} \rightarrow 0. \]
References


