ON STABLE JAMES NUMBERS OF STUNTED COMPLEX OR QUATERNIONIC PROJECTIVE SPACES

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Following James [7] we denote the stunted complex \( (F=C) \) or quaternionic \( (F=H) \) projective spaces by \( FP_{n+k,k} \) (or \( P_{n+k,k} \)) for positive integers \( n \) and \( k \), that is

\[
FP_{n+k,k} = FP_{n+k}/FP_n = FP_{n+k-1}/FP_{n-1}.
\]

Let \( d \) be the dimension of \( F \) over the real number field. Let \( i: S^d = FP_{n+1,1} \to FP_{n+k,k} \) be the inclusion. By stable James number \( F\{n, k\} \) we mean the order of the cokernel of

\[
\text{deg} = i^*: \{FP_{n+k,k}, S^{sd}\} \to \{S^{sd}, S^{sd}\} = Z
\]

where \( \{X, Y\} \) denotes the group of stable maps from a pointed space \( X \) to an other pointed space \( Y \). In the previous papers [5, 8, 9, 10] we used the notations \( k_s(FP_{n+k-1}, S^{sd}) \) instead of \( F\{n, k\} \) and estimated \( F\{1, k\} \).

The first purpose of this note is to determine \( F\{n, k\} \) for small \( k \), that is, we shall determine \( H\{n, k\} \) for \( k \leq 4 \), estimate them for \( k=5 \), determine \( C\{n, k\} \) for \( k \leq 8 \) and estimate them for \( k=9 \) and 10. These shall be done in \( \S 2 \) and \( \S 3 \). The second purpose is to show that \( F\{n, k\} \) can be identified with the James numbers defined by James in [6]. This shall be done in \( \S 4 \).

An application of this note to \( F \)-projective stable stems shall be given in [11].

In this note we work in the stable category of pointed spaces and stable maps between them, and we use Toda's notations of stable stems and Toda brackets in [14] freely.

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1. Preliminaries

In what follows we shall be working with both real \( K \)-cohomology theory \( KO^* \) and complex \( K \)-cohomology theory \( K^* \). We use the following notations. \( KO^* \) and \( K^* \) denote both the \( K \)-functors and the coefficient rings. By the same letter \( \xi=\xi_n \) we denote the canonical \( F \)-line bundle over \( FP_n \),
the underlying complex or real vector bundle of it. Put $z = \xi - d/2 \in K(FP_n)$ and $t = (-1)^{d/2} c_d(z) \in H^d(FP_n; \mathbb{Z})$, where $c_d(z)$ denotes the $m$-th Chern class of $\xi$. Put also $\xi = \xi - 1 \in K^0(\mathbb{P}^n) \cong K_0^+(\mathbb{P}^n)$. The formal power series $\phi_F(x)$ are defined to be $\exp(x) - 1$ for $F = C$ or $\exp(\sqrt{x}) + \exp(-\sqrt{x}) - 2$ for $F = H$. The rational numbers $\alpha_F(n, j)$ are defined by $(\phi_F^{j}(x))^{a_{n, j}} = \sum_{n} \alpha_F(n, j)x^n$.

$ch: K(\quad \quad) \to H^*(\quad ; Q)$ denotes the Chern character. Then the followings are well known.

**Proposition 1.1.**

(i) $K(FP_n) = Z[\xi]/(\xi^n)$.

(ii) $KO^*(\mathbb{P}^n) = KO^*[\bar{\xi}]/(\bar{\xi}^n)$ and $\bar{\xi} \mid_{\mathbb{P}^n-1} = \bar{\xi} - 1$.

(iii) $H^*(FP_n; \mathbb{Z}) = Z[t]/(t^n)$.

(iv) $ch(z) = \phi_F(t)$.

Let $i = i_l: FP_{n+k+1+k+l} \subset FP_{n+k+1+k+l}$ be the inclusion for $l > 0$, $q = q_m: FP_{n+k+1+k+l} \to FP_{n+k+1+k+l}$ the canonical quotient map for $0 < m < k$, $p = p_n: S^{ad-1} \to FP_n$ the Hopf bundle projection, and $p = p_n: S^{(n+k)d-1} \to FP_{n+k}$ the composition of $p_n$ and $q_n - 1: FP_{n+k} \to FP_{n+k+1+k+l}$. Let $G_k$ denote the $k$-stem of the stable groups of spheres. Let $e_G: G \to \mathbb{Q}/\mathbb{Z}$ or $e'_R: G \to \mathbb{Q}/\mathbb{Z}$ be the Adams' complex or real $e$-invariant respectively [1]. Then we have

**Proposition 1.2 (Adams[1]).** $e_G: G_1 \to \mathbb{Z}, e_R: G_2 \to \mathbb{Z}, e_G: G_3 \to \mathbb{Z}, e_R: G_4 \to \mathbb{Z}, e_G: G_5 \to \mathbb{Z}, e_R: G_6 \to \mathbb{Z}, e_G: G_7 \to \mathbb{Z}$, and $e_R: G_8 \to \mathbb{Z}$ are isomorphisms, while there is a split exact sequence

$$0 \to \mathbb{Z}_2 \to G_15 \to \mathbb{Z}_59 \to 0.$$ 

In [10] we obtained the following.

**Proposition 1.3.** For $f \in \{FP_{n+k}, S^{ad}\}$ we have

$$e_G(f \circ C_{n+k}) = -\deg(f)\alpha_F(n, k).$$

Since $e_G = 2e'_R$ on $(8k+3)$-stems [1], $e'_R$ gives more precise informations about 2-primary components, so we compute $e'_R(f \circ C_{n+k})$ for the case of $F = H$ and $k \equiv 1 \text{ mod}(2)$ or $F = C$ and $k \equiv 2 \text{ mod}(4)$.

We use the following notations. Let $g_c \in K(S^2)$ and $g_c \in K(S^0)$ denote the Bott generators. $\psi^k$ denotes the Adams operation. Let $c: KO^* \to K^*$ be the complexification and $r: K^* \to KO^*$ the real restriction. Put $x_0 = r(x) \in KO(CP_n)$ and $x_j = r(g_c^j) \in KO^{2j}(CP_n)$. Put also $y_{2k} = g_c^{k+1} \in KO^{2k}$ and $y_{2k+1} \in KO^{2k+1}$ the generator satisfying $c(y_{2k+1}) = 2g_c^{k+2}$ for integer $k$. For $f \in \{X, Y\}, C(f)$ denotes the mapping cone of $f$.

We consider the case of $F = H$ and $k \equiv 1 \text{ mod}(2)$ or $F = C$ and $k \equiv 2 \text{ mod}(4)$.
Given \( f \in \{ FP_{n+k,k}, S^{nd} \} \), we have the commutative diagram

\[
\begin{array}{ccc}
S^{(n+k)d-1} & \xrightarrow{p_{n+k,k}} & FP_{n+k,k} \\
\downarrow & & \downarrow f' \\
S^{(n+k)d-1} & \xrightarrow{f \circ p_{n+k,k}} & S^{nd} \\
\end{array}
\]

Applying \( KO^{nd} \) and \( K^{nd} \) to this diagram; since \( KO^{nd}(S^{(n+k)d-1}) = K^{nd}(S^{(n+k)d-1}) \) \( = K^{nd-1}(S^{nd}) = 0 \) and \( KO^{nd-1}(FP_{n+k,k}), K^{nd-1}(FP_{n+k,k}) \) and \( KO^{nd-1}(S^{nd}) \) are finite groups, we have the following commutative diagram in which the horizontal sequences are exact.

\[
\begin{array}{ccc}
0 & \xleftarrow{c} & KO^{nd}(FP_{n+k,k}) \\
& \xleftarrow{f^*} & KO^{nd}(FP_{n+k+1,k+1}) \\
& \xleftarrow{c} & KO^{nd}(S^{(n+k)d}) \\
0 & \xleftarrow{c} & K^{nd}(FP_{n+k,k}) \\
& \xleftarrow{f^*} & K^{nd}(FP_{n+k+1,k+1}) \\
& \xleftarrow{c} & K^{nd}(S^{(n+k)d}) \\
0 & \xleftarrow{c} & KO^{nd}(S^{nd}) \\
& \xleftarrow{f^*} & KO^{nd}(C(f \circ p_{n+k,k})) \\
& \xleftarrow{c} & KO^{nd}(S^{(n+k)d}) \\
0 & \xleftarrow{c} & K^{nd}(C(f \circ p_{n+k,k})) \\
& \xleftarrow{f^*} & K^{nd}(S^{(n+k)d}) \\
& \xleftarrow{c} & K^{nd}(S^{(n+k)d}) \\
\end{array}
\]

We can choose generators \( a, b \in KO^{nd}(C(f \circ p_{n+k,k})) \) and \( a', b' \in K^{nd}(C(f \circ p_{n+k,k})) \) such that \( a' = c(a), 2b' = c(b), j^*(a') \) generates \( K^{nd}(S^{nd}) = Z \) and \( f'^*(b') = g_c^{-(nd/2)z^{n+k}} \).

Here we identify \( K^{nd}(FP_{n+k+1,k+1}) \) with the free subgroup of \( K^{nd}(FP_{n+k+1}) \) generated by \( g_c^{-(nd/2)z^n}, g_c^{-(nd/2)z^{n+k}}, \ldots, g_c^{-(nd/2)z^{n+k}} \). Hence we can put

\[
f'^*(a') = g_c^{-(nd/2)\sum_{i=0}^k a_i z^{n+i}}
\]

for some integers \( a_i \). Then by the proof of (1.1) of [10] we have

\[
a_i = \deg(f)\alpha_F(n, i) \quad \text{for} \quad 0 \leq i \leq k-1,
\]

(1.4)

And we have

**Proposition 1.5.** In case of \( F=H \) and \( k \equiv 1 \mod(2) \) or \( F=C \) and \( k \equiv 2 \mod(4) \) we have

(i) \( e_k'(f \circ p_{n+k,k}) = \frac{1}{2} a_k - \frac{1}{2} \deg(f)\alpha_F(n, k) \),

(ii) if \( F=H \), \( a_k \equiv 0 \mod(2) \),

(iii) if \( F=C \), \( n \equiv 1 \mod(2) \) and \( \deg(f) \) is known, \( a_k \mod(2) \) is computable.
Proof. First consider the case of $F=H$ and $n \equiv 0 \mod(2)$. By Bott periodicity we can use $\widetilde{KO}$ and $\widetilde{K}$ instead of $KO^n$ and $K^n$. Then we have
\[ \psi^2(a) = 4^s a + \lambda b \]
for some integer $\lambda$, and
\[ e^l_k (f \circ p_{s+k,k}) = \lambda / (4^s (4^s - 1)) \].
We have
\[ \psi^2(a') = c(\psi^2(a)) = 4^s a' + 2\lambda b' \]
\[ \psi^2(f^*(a')) = \psi^2(\sum_{i=0}^{s} a_i z^{n+i}) = \sum_{i=0}^{s} a_i (z^2 + 4z)^{n+i} \]
\[ = \sum_{j=0}^{k} \sum_{i=0}^{s} a_i (z^{n+i}) 4^{s+2i} - j \lambda \chi_{n+j} \]
\[ \psi^2(f^*(a')) = f^*(\psi^2(a')) = f^*(4^s a' + 2\lambda b') \]
\[ = 4^s \sum_{i=0}^{s} a_i z^{n+i} + 2\lambda \chi_{n+k} \].
Comparing the coefficients of $z^{n+k}$, we have
\[ 2\lambda = 4^s (4^s - 1)a_k + \sum_{i=0}^{k-1} a_i (2z^{n+i}) 4^{s+2i} - k \].
Then by (1.4) we have
\[ e^l_k (f \circ p_{s+k,k}) = \frac{1}{2} a_k - \frac{1}{2} \deg(f) \alpha_B(n, k) \]
as desired. Next we show (ii). Put $f^*(a) = \sum_{i=0}^{k} d_i y_{n+i} \xi^{s+i}$. Then
\[ c(f^*(a)) = \sum_{i=0}^{k} d_i c(y_{n+i}) (c(\xi))^{s+i} = \sum_{i=0}^{k} d_i \xi_i \gamma_{2i} (-2^{n+i}) (g_{c}^{2i} \chi^{n+i} \]
where $\xi_i = 1$ (if $i$ is even) or $2$ (if $i$ is odd). We have also
\[ c(f^*(a)) = f^*(c(a)) = \sum_{i=0}^{k} a_i z^{n+i} \]
Therefore $a_k = d_k \xi_k = 2d_k$.

In case of $F=H$ and $n \equiv 1 \mod(2)$, (i) and (ii) can be proved by the quite parallel arguments to the above. We omit the details.

For $F=C$ (i) can be proved by the same methods as the above. We only prove (iii). First we consider the case of $n \equiv 3 \mod(4)$. Put $n = 4m + 3$ and $k = 4l + 2$. By Bott periodicity we can use $\widetilde{KO}^2$ and $\widetilde{K}^2$ instead of $\widetilde{KO}^n$ and $\widetilde{K}^n$. By Theorem 2 of Fujii [4], it is easily seen that $\widetilde{KO}^2(CP_{4m+4l+6,4l+1})$ can
be identified with the free subgroup of $\widehat{KO}^{-2}(CP_{4m+4l+6})$ generated by $z_1z_0^{2m+1}$, $z_1z_0^{2m+2}$, $\ldots$, $z_1z_0^{2m+2l+2}$. So we can put $f^*(a) = \sum_{i=0}^{2l+1} d_i z_0^{2m+1+i}$ for some integers $d_i$. Then

$$c(f^*(a)) = \sum_{i=0}^{2l+1} d_i (c(z_1)(c(z_0)))^{2m+1+i} = g_c \sum_{i=0}^{2l+1} d_i (z - \bar{z})(z + \bar{z})^{2m+1+i}$$

where $\bar{z} = -z^2 + z^3 + \cdots$. We have also

$$c(f^*(a)) = f^*(c(a)) = g_c \sum_{i=0}^{4l+2} a_i z^{4m+3+i}.$$

So we have

$$\sum_{i=0}^{4l+2} a_i z^{4m+3+i} = \sum_{i=0}^{2l+1} d_i (2z - z^2 + z^3 - \cdots)(z^2 - z^3 + \cdots)^{2m+1+i}.$$

Calculating this equation over the mod 2 integers, we have

$$\sum_{i=0}^{4l+2} a_i z^{4m+3+i} \equiv \sum_{i=0}^{2l+1} d_i (z^2 + z^3 + \cdots)^{2m+2+i} \mod(2, z^{4m+4l+6})$$

$$\equiv \sum_{j=0}^{2l+1} \sum_{i=0}^{2l+1} d_i (2m+1+j-i)z^{4m+4l+j} \mod(2),$$

since $(x^2 + x^3 + \cdots)^n = \sum_{j=2}^{\infty} (j-1)j^x$. Then

$$a_i \equiv \sum_{j=0}^{2l+1} d_j (\frac{2m+1+j}{2m+4l+2}) \mod(2) \quad \text{for } 1 \leq i \leq 4l+2.$$  

By (1.4) and (1.6) for $1 \leq i \leq 4l+1$, $d_j \mod(2)$ is determined for $0 \leq j \leq 2l$, so the equation

$$a_{4l+2} \equiv \sum_{j=0}^{2l+1} d_j (\frac{2m+4l+2-j}{2m+4l+j}) \mod(2)$$

$$\equiv \sum_{j=0}^{2l+1} d_j (\frac{2m+4l+1-j}{2m+4l+2j+2}) \mod(2)$$

determines $a_{4l+2} \mod(2)$, here we use the fact $(\frac{2t}{2t-1}) \equiv 0 \mod(2)$ for any $t$. Next we consider the case of $n \equiv 1 \mod(4)$. Put $n = 4m+1$. We use $\widehat{KO}^{-6}$ and $\widehat{K}^{-6}$ instead of $\widehat{KO}^{2n}$ and $\widehat{K}^{2n}$. Then we can put $f^*(a) = \sum_{i=0}^{2l+1} d_i z_0^{2m+1+i}$ for some integers $d_i$. By the same arguments as the above we have

$$a_i = \sum_{j} d_j (\frac{2m+i-j-1}{2m+j}) \mod(2) \quad \text{for } 1 \leq i \leq 4l+2$$

and in particular

$$a_{4l+2} = \sum_{j} d_j (\frac{2m+4l+2-i}{2m+4l+j}) \mod(2).$$

These and (1.4) determine $a_{4l+2} \mod(2)$. This completes the proof.
To compute \( F\{n, k\} \) by inductive step on \( k \) we prepare the followings.

**Proposition 1.8.** \( F\{n, k\} \) is a divisor of \( F\{n, k+1\} \).

Proof. It is trivial by definition.

**Proposition 1.9.** For \( f \in \{FP_{n+k, k}, S^{nd}\} \) with \( \deg(f)=F\{n, k\} \) we have
\[
F\{n, k\} \# e_c(f \circ p_{n+k, k}) \mid F\{n, k+1\} \mid F\{n, k\} \# (f \circ p_{n+k, k})
\]
where \( \# g \) denotes the order of \( g \) and \( a \mid b \) implies that \( a \) is a divisor of \( b \).

Proof. Choose \( f' \in \{FP_{n+k+1, k+1}, S^{nd}\} \) with \( \deg(f')=F\{n, k+1\} \). Since \( i_1 \circ p_{n+k, k}=0 \), we have
\[
0 = e_c(f' \circ i_1 \circ p_{n+k, k}) = -\deg(f' \circ i_1) \alpha_F(n, k)
= -F\{n, k+1\} \alpha_F(n, k) = -F\{n, k\} \alpha_F(n, k) F\{n, k+1\} / F\{n, k\}
= -e_c(f \circ p_{n+k, k}) F\{n, k+1\} / F\{n, k\}
\]
Hence the first part of the conclusion is obtained. Since \( (\#(f \circ p_{n+k, k})) f \circ p_{n+k, k}=0 \), there exists \( h \in \{FP_{n+k+1, k+1}, S^{nd}\} \) with \( h \circ i_1=(\#(f \circ p_{n+k, k})) f \). Then \( \deg(h)=\deg(f) \# (f \circ p_{n+k, k})=F\{n, k\} \# (f \circ p_{n+k, k}) \). Since \( \deg(h) \) is a multiple of \( F\{n, k+1\} \), the second part of the conclusion follows.

**Proposition 1.10.** For \( f \in \{FP_{n+k, k}, S^{nd}\} \) with \( \deg(f)=F\{n, k\} \) there exists \( h \in \{FP_{n+k, k-1}, S^{nd}\} \) with \((F\{n, k+1\} / F\{n, k\}) f \circ p_{n+k, k}=h \circ i_1 \circ p_{n+k, k} \).

Proof. Consider the exact sequence
\[
\cdots \to \{FP_{n+k, k-1}, S^{nd}\} \xrightarrow{q_1^*} \{FP_{n+k, k}, S^{nd}\} \xrightarrow{\deg} \{FP_{n+1, 1}, S^{nd}\} \to \cdots
\]
Take \( f' \in \{FP_{n+k+1, k+1}, S^{nd}\} \) with \( \deg(f')=F\{n, k+1\} \). Then \( \deg((F\{n, k+1\} / F\{n, k\}) f \circ i_1)=0 \). So there exists \( h \in \{FP_{n+k, k-1}, S^{nd}\} \) with \( q_1^*(h)=(F\{n, k+1\} / F\{n, k\}) f \circ i_1 \) by exactness. Then \( h \circ q_1 \circ p_{n+k, k}=(F\{n, k+1\} / F\{n, k\}) f \circ p_{n+k, k} \) as desired.

**Proposition 1.11.** \( C\{2n, 2k\} \) is a divisor of \( H\{n, k\} \).

Proof. Consider the commutative diagram
\[
\begin{array}{ccc}
CP_{2n+2k, 2k} & \supset & CP_{2n+1, 1} = S^{4n} \\
S^{4n+4k-1} \downarrow {\pi} & & \downarrow {\pi'} \\
HP_{n+k, k} & \supset & HP_{n+1, 1} = S^{4n}
\end{array}
\]
in which all maps are the canonical ones. For our purpose it suffices to show that \( \pi' \) is a homotopy equivalence. Indeed this holds because in the following
commutative diagram $\pi^*$ is an isomorphism.

\[ H^{\ast\ast}(CP_{2n+2k}; Z) \xleftarrow{q^*} H^{\ast\ast}(CP_{2n+2k}; Z) \xrightarrow{\pi^*} H^{\ast\ast}(S^u; Z) \]

\[ H^{\ast\ast}(HP_{n+k}; Z) \xrightarrow{q^*} H^{\ast\ast}(HP_{n+k}; Z) \xrightarrow{\pi^*} H^{\ast\ast}(S^u; Z). \]

Next we compute $e$-invariants of some elements.

**Lemma 1.12.** Suppose that there is a commutative diagram

\[
\begin{array}{cccc}
S^{(n+k)d-1} & \rightarrow & FP_{n+k, k} & \subset FP_{n+k+1, k+1} \\
\downarrow & & \downarrow L & \downarrow L' \\
S^{(n+k)d-1} & \rightarrow & FP_{n+k, k} & \rightarrow C(\tilde{p}) \\
\uparrow & & \uparrow \cup i & \uparrow i' \\
S^{(n+k)d-1} & \rightarrow & FP_{n+1, 1} & \rightarrow C(s)
\end{array}
\]

in which $L$ denotes the multiplication by non-zero integer $L$. Then

\[ e_C(s) = L \lbrace \sum_{j=1}^{k-1} (\tilde{p}^j d^k) C_j \rbrace / d^k (d^k - 1) \]

where $C_j = C_j(n, k)$ is the coefficient of $x^{n+k}$ in $(\phi_F(x))^{n+j}$.

Proof. Applying $\tilde{K}$ to the above diagram we have the following commutative diagram in which the horizontal sequences are exact.

\[
\begin{array}{cccc}
0 \leftarrow & \tilde{K}(FP_{n+k, k}) & \leftarrow & \tilde{K}(FP_{n+k+1, k+1}) \leftarrow \tilde{K}(S^{(n+k)d}) \leftarrow 0 \\
\uparrow L^* & & \uparrow L'^* & \uparrow = \\
0 \leftarrow & \tilde{K}(FP_{n+k, k}) & \leftarrow & \tilde{K}(C(\tilde{p})) \leftarrow \tilde{K}(S^{(n+k)d}) \leftarrow 0 \\
\downarrow i^* & & \downarrow i'^* & \downarrow = \\
0 \leftarrow & \tilde{K}(S^{ad}) & \leftarrow & \tilde{K}(C(s)) \leftarrow \tilde{K}(S^{(n+k)d}) \leftarrow 0.
\end{array}
\]

Choose $a_j \in \tilde{K}(C(\tilde{p}))$ for $0 \leq j \leq k$ such that $L'^*(a_j) = Lx^{n+j}$ for $0 \leq j \leq k-1$ and $L'^*(a_k) = x^{n+k}$. Then $i'^*(a_0)$ and $i'^*(a_k)$ generate $\tilde{K}(C(s))$. We have

\[ \psi^2(i'^*(a_0)) = d^2i'^*(a_0) + \lambda i'^*(a_k) \]

for some $\lambda \in \mathbb{Z}$ and
We compute $\lambda$. We have

\[ L'^*(\psi^2(a_0)) = \psi^2(L'^*(a_0)) = \psi^2(Lz^n) = L(z^2 + dz)^n \]
\[ = L \sum_{j=0}^{k-1} (\zeta)d^{n-j}z^{n+j} \]
\[ = \sum_{j=0}^{k-1} (\zeta)d^{n-j}L_2z^{n+j} + L(\zeta)d^{n-k}z^{n+k} \]
\[ = L'^*(\sum_{j=0}^{k-1} (\zeta)d^{n-j}a_j + L(\zeta)d^{n-k}a_k) . \]

Since $L'^*$ is monomorphic, we have

\[ \psi^2(a_0) = \sum_{j=0}^{k-1} (\zeta)d^{n-j}a_j + L(\zeta)d^{n-k}a_k . \]

Next consider the following commutative diagram

\[ \xymatrix{ \tilde{K}(FP_{n+k+1}) \ar[r]^{ch} \ar[u]^{L'^*} & H^*(FP_{n+k+1}; \mathbb{Q}) \ar[u]^{L'^*} \\
\tilde{K}(C(p)) \ar[r]^{ch} \ar[u]^{i'^*} & H^*(C(p); \mathbb{Q}) \ar[u]^{i'^*} \\
\tilde{K}(C(s)) \ar[u]^{i'^*} \ar[r]^{ch} & H^*(C(s); \mathbb{Q}). } \]

Choose the generators $x_{n+j} \in H^{n+j}(C(p); \mathbb{Q})$ for $0 \leq j \leq k$ such that $L'^*(x_{n+j}) = L^{n+j}$ for $0 \leq j \leq k-1$ and $L'^*(x_{n+k}) = L^{n+k}$. Then for $1 \leq j \leq k-1$

\[ L'^*(ch(a_j)) = ch(L'^*(a_j)) = ch(Lz^{n+j}) = L(\phi_F(t))^{n+j} \]
\[ = L(t^{n+j} + \text{middle dim} + C, t^{n+k}) \]
\[ = L'^*(x_{n+j} + \text{middle dim} + LC_jx_{n+k}) \]

where the terms middle dim mean elements of middle dimensions. Since $L'^*$ is monomorphic, we have

\[ ch(a_j) = x_{n+j} + \text{middle dim} + LC_jx_{n+k} \text{ for } 1 \leq j \leq k-1, \]

and so

\[ ch(i'^*(a_j)) = i'^*(ch(a_j)) = LC_ji'^*(x_{n+k}) = ch(LC_ji'^*(a_k)) \]
\[ \text{ for } 1 \leq j \leq k-1 . \]

Since $ch$ is monomorphic now, we have

\[ i'^*(a_j) = LC_ji'^*(a_k) \text{ for } 1 \leq j \leq k-1 . \]
Then
\[ \psi^2(i^*(a_0)) = i^*(\psi^2(a_0)) = i^* \left\{ \sum_{j=0}^{k-1} (j) d^{n-j} a_j + L(\xi) d^{n-k} a_k \right\} \]
\[ = d^n i^*(a_0) + \left\{ \sum_{j=1}^{k-1} (j) d^{n-j} L C_j + L(\xi) d^{n-k} \right\} i^*(a_k) \]
\[ = d^n i^*(a_0) + L d^{n-k} \left\{ \sum_{j=1}^{k-1} (j) d^{n-j} C_j + (\xi) \right\} i^*(a_k). \]

Therefore we have
\[ \lambda = L d^{n-k} \left\{ \sum_{j=1}^{k-1} (j) d^{n-j} C_j + (\xi) \right\} \]
and
\[ e_c(s) = L \left\{ \sum_{j=1}^{k-1} (j) d^{n-j} C_j + (\xi) \right\} / d^n (d^k - 1). \]

This completes the proof.

As a corollary of the above lemma we have

**Proposition 1.13.** In the same situation as (1.12) we have

(i) if \((F, k) = (C, 1), s = L n \eta\) and in particular \(p_{n+1,1} = n \eta: S^{2n+1} \to CP_{n+1,1} = S^{2n}\),

(ii) if \((F, k) = (H, 2), e_c(s) = L n(5n-1)/2^2 \cdot 3^2 \cdot 5, \)

(iii) if \((F, k) = (C, 4), e_c(s) = L n(15n^2+30n^2+5n-2)/2^7 \cdot 3^2 \cdot 5, \)

(iv) if \((F, k) = (C, 5), e_c(s) = L n(3n^4+10n^2+5n^2-2n+216)/2^8 \cdot 3^2 \cdot 5. \)

**Proof.** Since
\[ \phi_F(x) = \begin{cases} x + x^2/2! + x^3/3! + \cdots & \text{for } F = C \\ 2x/2! + 2x^3/4! + 2x^5/6! + \cdots & \text{for } F = H, \end{cases} \]
we can easily compute \(e_c(s)\) for small \(k\) by elementary analysis, so we omit the details except (i). (i) follows from the fact that \(e_C: G_1 \to Z_2\) is an isomorphism and \(e_c(s) = \frac{1}{2} L n = e_c(L n \eta). \)

**Remark.** (i) is well known.

In case of \(F=H\) and \(k \equiv 1 \mod(2)\) or \(F=C\) and \(k \equiv 2 \mod(4)\) we have \(e_c(s) = 2 e_h(s)\) so the computation of \(e_h(s)\) may give more precise informations about the 2-primary components of the order of \(s\). We do not require the whole computations but we only compute \(e_h(s)\) for the case of \((F, k) = (H, 1)\) or \((C, 2)\). Let \(g_4 = p_2: S^4 \to S^4 = HP_2\) be the Hopf map. Put \(g_\infty = \{g_4\} \in G_3. \)
Then \(e_h(\infty) = 1/24\) and

**Proposition 1.14** (James [7]). \(p_{n+1,1} = n g_\infty: S^{4n+3} \to HP_{n+1,1} = S^{4n} \)
Proof. We have the short exact sequence
\[ 0 \to \tilde{K}_0(-8)(HP_{n+1,1}) \xrightarrow{i^*} \tilde{K}_0(-8)(HP_{n+2,2}) \xrightarrow{q^*} \tilde{K}_0(-8)(S^{2n+4}) \to 0. \]
It is easily seen by (1.1) that \( \tilde{K}_0(-8)(HP_{n+1,1}) = Z\{g \in \mathbb{F}_5, \nu \} \), \( \tilde{K}_0(-8)(HP_{n+2,2}) = Z\{g \in \mathbb{F}_5, \nu \} \), \( \tilde{K}_0(-8)(S^{2n+4}) = Z\{e\} \), \( i^*(g \in \mathbb{F}_5) = g \in \mathbb{F}_5 \) and \( q^*(e) = y_{-1}^{2n+1} \).
We have
\[ \psi^2(g \in \mathbb{F}_5) = \psi^2(g \in \mathbb{F}_5) = 2^4 g \{2^{4n} + n2^{4n-3} y_{-1}^{2n+1}\}. \]
Then
\[ e'_K(p_{n+1,1}) = 2^{n+1} n/(2^{n+6} - 2^{n+4}) = n/24 = e'_K(g_n). \]
This shows that \( p_{n+1,1} = ng_n \), since \( e'_K : G_3 \to Z_{24} \) is an isomorphism by (1.2).

Now consider the following commutative diagram in which the horizontal sequences are exact.

\[ \cdots \xrightarrow{p_{n+1,1}} \{S^{2n+1}, S^{2n-1}\} \xrightarrow{p_{n+1}} \{S^{2n+1}, CP_{n+1}\} \xrightarrow{i^*} \{S^{2n+1}, CP_{n+1}\} \xrightarrow{q^*} \{S^{2n+1}, S^{2n}\} \xrightarrow{q^*} \{S^{2n+1}, S^{2n}\} \xrightarrow{q^*} \{S^{2n+1}, S^{2n}\} \xrightarrow{q^*} \cdots \]

By (1.13) \( q_*(p_{n+1}) = m7 \). Then we have

**Proposition 1.15.** If \( L\equiv 0 \mod(2) \)

\[ q_*(i_*)^{-1}(Lp_{n+1}) = \begin{cases} \frac{1}{2} L(n-1)g_m & \text{for } n \text{ odd} \\ \{\frac{1}{2} L(n+2)g_m, \left(\frac{1}{2} L(n+2)+12\right)g_m\} & \text{for } n \text{ even.} \end{cases} \]

Proof. The above diagram shows that \( q_*(i_*)^{-1}(Lp_{n+1}) = (j_*)^{-1}(Lp_{n+1,2}) \). Since \( \{S^{2n+1}, S^{2n-1}\} = Z_2 \{\nu^3\} \) and \( p_{n+1}^{-1}(\nu^2) = (n-1)\nu^3 = 12(n-1)g_m \), \( (j_*)^{-1}(Lp_{n+1,2}) \) is a coset of the subgroup of \( \{S^{2n+1}, CP_{n+1}\} = G_3 \) generated by \( 12(n-1)g_m \). This coset consists of a single element if \( n \) is odd or two elements if \( n \) is even. In case of \( n \) being odd we have the following commutative diagram by the proof of
(1.11), (i) of (1.13) and (1.14).

\[
\begin{array}{c}
p_{n+1,2} \quad CP_{n+1,2} = S^{2n+2} \vee S^{2n} \supset CP_{n+1} = S^{2n-2} \\
S^{2n+1} \quad \text{=}
\end{array}
\]

(1/2)(n-1)g_{\infty} \rightarrow HP_{(n+1)/2,1} = S^{2n-2}

This diagram proves Proposition if \( n \) is odd. If \( n \) is even, we have the short exact sequence

\[0 \rightarrow \{S^{2n+1}, S^{2n-1}\} \rightarrow \{S^{2n+1}, S^{2n-2}\} \xrightarrow{j_*} \{S^{2n+1}, CP_{n+1,2}\} \rightarrow 0\]

since \( p_{n,1} = (n-1)\eta \) by (i) of (1.13). For our purpose it suffices to show that

\[(j_*)^{-1}(p_{n+1,2}) = \{(n/2+1)g_{\infty}, (n/2+13)g_{\infty}\} .\]

For any \( f \in (j_*)^{-1}(p_{n+1,2}) \) the equation

\[(*) \quad e'_k(f) = (n/2+1+12e)/24\]

implies this, because \( e'_k((n/2+1)g_{\infty}) = (n/2+1)/24 \). We prove (*). We use \( \tilde{KO}^{-2} \) if \( n \equiv 0 \mod(4) \) or \( \tilde{KO}^{-6} \) if \( n \equiv 2 \mod(4) \). The methods are quite parallel, so we only prove (*) for the case of \( n \equiv 0 \mod(4) \). Put \( n = 4m \). There is the following commutative diagram in which the horizontal sequences are exact.

\[
\begin{array}{c}
0 \leftarrow \tilde{KO}^{-2}(CP_{4m+1,2}) \leftarrow \tilde{KO}^{-2}(CP_{4m+2,2}) \leftarrow \tilde{KO}^{-2}(S^{6m+2}) \leftarrow 0 \\
\downarrow i^* \quad \downarrow i^* \quad \downarrow i^* \\
0 \leftarrow \tilde{KO}^{-2}(S^{6m+2}) \leftarrow \tilde{KO}^{-2}(C(f)) \leftarrow \tilde{KO}^{-2}(S^{6m+2}) \leftarrow 0
\end{array}
\]

By Theorem 2 of Fujii [4] it is easy to see that \( \tilde{KO}^{-2}(CP_{4m+1,2}) = Z \{z_1z_2^{2m-1}\} \), \( \tilde{KO}^{-2}(CP_{4m+2,2}) = Z \{z_1z_2^{2m-1}, z_1z_2^{2m}\} \), \( \tilde{KO}^{-2}(CP_{4m+1,1}) = Z \{w\} \) with \( 2w = z_1z_2^{2m-1} \) and \( \tilde{KO}^{-2}(CP_{4m+2,1}) = Z \{z_1z_2^{2m}\} \). Take \( a \in \tilde{KO}^{-2}((C(f)) \) with \( u^*(a) = w \). Then \( a \) and \( v^*(z_1z_2^{2m}) = i^*(z_1z_2^{2m}) \) generate \( \tilde{KO}^{-2}(C(f)) \). By definition \( 2a = i^*(z_1z_2^{2m-1}) + ev^*(z_1z_2^{2m}) \) for some integer \( e \). We have \( \psi^2(a) = 2^{4m}a + \lambda i^*(z_1z_2^{2m}) \) for some integer \( \lambda \), and \( e'_k(f) = \lambda/2^{4m+3} \). We have also

\[c(2a) = c(i^*(z_1z_2^{2m-1}) + ev^*(z_1z_2^{2m})) = g_{c}i^*\{2z^{4m-1} - (4m-1)z^{4m} + (4m^2+2e)z^{4m+1}\}\]

and

\[c(i^*(z_1z_2^{2m})) = 2g_{c}i^*(z^{4m+1})\]

and then
\[ c(\psi^2(2a)) = c(2^{4m+1}a + 2\lambda z^{2m}) = g c^i \{ 2^{4m+1}z^{4m-1} - 2^{4m}(4m - 1)z^{4m} + (2^{4m+2}m^2 + 2^{4m+1}e + 4\lambda)z^{4m+1} \} . \]

On the other hand
\[ c(\psi^2(2a)) = \psi^2(c(2a)) = \psi^2[g c^i \{ 2z^{4m-1} - (4m - 1)z^{4m} + (4m^2 + 2e)z^{4m+2} \}] = 2g c^i \{ 2z^{4m-1} - (4m - 1)z^{4m} + (4m^2 + 2e)z^{4m+1} \} = g c^i \{ 2^{4m+1}z^{4m-1} - 2^{4m}(4m - 1)z^{4m} + 2^{4m-1}(2^m + 2m + 1 + 16e)z^{4m+1} \} . \]

Comparing the coefficients of \( z^{4m+1} \), we have
\[ \lambda = 2^{4m-2}(2m + 1 + 12e) \]
and so
\[ e_k(f) = (2m + 1 + 12e)/24 . \]

This completes the proof.

In the sequel we shall need the explicit form of \( \alpha_F(n, k) \) for small \( k \). Since the expansion of \( \phi^{r'}(x) \) is known (see e.g. [10]), we can obtain the following by elementary calculations.

**Lemma 1.16.**

\[
\begin{align*}
\alpha_F(n, 0) & = 1, \\
\alpha_F(n, 1) & = -n/2^3, \\
\alpha_F(n, 2) & = n(5n+1)/2^3 \cdot 5, \\
\alpha_F(n, 3) & = -n(35n^2 + 231n + 382)/2^7 \cdot 3 \cdot 5 \cdot 7, \\
\alpha_F(n, 4) & = n(175n^3 + 2310n^2 + 10181n + 14982)/2^{11} \cdot 3^5 \cdot 5^2 \cdot 7, \\
\alpha_F(n, 5) & = -n(385n^4 + 8470n^3 + 69971n^2 + 257246n + 355128)/2^{13} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11, \\
\alpha_F(n, 1) & = -n/2, \\
\alpha_F(n, 2) & = n(3n + 5)/2^5 \cdot 3, \\
\alpha_F(n, 3) & = -n(n+2)(n+3)/2^4 \cdot 3, \\
\alpha_F(n, 4) & = n(15n^3 + 150n^2 + 485n + 502)/2^7 \cdot 3^5, \\
\alpha_F(n, 5) & = -n(3n^4 - 30n^3 + 785n^2 - 78n + 1240)/2^8 \cdot 3^2 \cdot 5, \\
\alpha_F(n, 6) & = n(63n^5 + 1575n^4 + 15435n^3 + 73801n^2 + 171150n + 152696) \\
& /2^{10} \cdot 3^4 \cdot 5 \cdot 7, \\
\alpha_F(n, 7) & = -n(9n^6 + 315n^5 + 4515n^4 + 33817n^3 + 139020n^2 + 295748n \\
& + 252336)/2^{11} \cdot 3^4 \cdot 5 \cdot 7, \\
\alpha_F(n, 8) & = n(135n^7 + 6300n^6 + 124110n^5 + 1334760n^4 + 8437975n^3 \\
& + 74777100n^2 + 68303596n + 138452016)/2^{15} \cdot 3^5 \cdot 5^2 \cdot 7, \\
\end{align*}
\]
\[ \alpha_c(n, 9) = -n(15n^8 + 900n^7 + 23310n^6 + 339752n^5 - 829745n^4 + 38354500n^3 \\
+ 27449684n^2 + 112877136n + 100476288)/2^{185} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 , \]

\[ \alpha_c(n, 10) = n(99n^9 + 7425n^8 + 244530n^7 + 4634322n^6 + 55598235n^5 \\
+ 436886945n^4 + 2242194592n^3 + 7220722828n^2 \\
+ 38722058672 - 15239326848)/2^{185} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 . \]

2. \( H\{n, k\} \) for \( k \leq 5 \)

The results of this section are summarized as follows.

**Theorem 2.1.**
(i) \( H\{n, 1\} = 1 \),
(ii) \( H\{n, 2\} = 24/(n, 24) \),
(iii) \( H\{n, 3\} = H\{n, 2\} \text{den}[H\{n, 2\} \alpha_H(n, 2)] \),
(iv) \( H\{n, 4\} = H\{n, 3\} \text{den}\left[ \frac{1}{2} H\{n, 3\} \alpha_H(n, 3) \right] \),
(v) \( H\{n, 5\}/(H\{n, 4\} \text{den}[H\{n, 4\} \alpha_H(n, 4)]) \\
= \begin{cases} 1 & \text{if } n \equiv 1 \text{ mod}(2^5) \text{ or } 34 \text{ mod}(2^6) \\
1 & \text{otherwise}, \end{cases} \)

where \( \text{den}(a) \) denotes the denominator of a rational number \( a \) when the fraction \( a \) is expressed in its lowest terms.

**Proof.** (i) is trivial.

By (1.14), \( \#p_{n+1, 1} = 24/(n, 24) \), since \( g_{\infty} = 24 \). Then \( H\{n, 2\} | 24/(n, 24) \) by (1.9). Choose \( f \in \{H_P\}_{n+2, 2}, S^{\infty}\) with \( \text{deg}(f) = H\{n, 2\} \). Then

\[ 0 = f \circ i_1 \circ p_{n+1, 1} = \text{deg}(f)p_{n+1, 1} = H\{n, 2\} p_{n+1, 1} . \]

Therefore \( 24/(n, 24) | H\{n, 2\} \). Hence (ii) follows.

Take \( f \in \{H_P\}_{n+2, 2}, S^{\infty}\) with \( \text{deg}(f) = H\{n, 2\} \). We have \( \#e_c(f \circ p_{n+2, 2}) = \#(f \circ p_{n+2, 2}) \), since \( e_c: G_7 \to Z_{360} \) is an isomorphism by (1.2). They by (1.9) \( H\{n, 3\} = H\{n, 2\} \cdot \#c(f \circ p_{n+2, 2}) \). By (1.3) \( e_c(f \circ p_{n+2, 2}) = -H\{n, 2\} \alpha_H(n, 2) \). Hence (iii) is obtained.

For any \( h \in \{H_P\}_{n+3, 2}, S^{\infty}\) we have

\[ e'_{k}(h \circ q_1 \circ p_{n+3, 3}) = -\frac{1}{2}\text{deg}(h \circ q_1)\alpha_H(n, 3) = 0 \]

by (1.5). Since \( e'_k: G_1 \to Z_{360} \) is an isomorphism by (1.2), \( h \circ q_1 \circ p_{n+3, 3} = 0 \).

Then by (1.10), for \( f \in \{H_P\}_{n+3, 3}, S^{\infty}\) with \( \text{deg}(f) = H\{n, 3\} \), \( \#(f \circ p_{n+3, 3}) \) is a divisor of \( H\{n, 4\}/H\{n, 3\} \). Conversely (1.9) implies that \( \#(f \circ p_{n+3, 3}) \) is a multiple of \( H\{n, 4\}/H\{n, 3\} \). Hence \( \#(f \circ p_{n+3, 3}) = H\{n, 4\}/H\{n, 3\} \). On the other hand \( e'_{k}(f \circ p_{n+3, 3}) = -\frac{1}{2}H\{n, 3\} \alpha_H(n, 3) \) by (1.5). Hence \( \#(f \circ p_{n+3, 3}) = \text{den}\left[ \frac{1}{2} H\{n, 3\} \alpha_H(n, 3) \right] \). Therefore
\[ H\{n, 4\}/H\{n, 3\} = \text{den}\left[ \frac{1}{2} H\{n, 3\} \alpha_H(n, 3) \right] \]

and this implies (iv).

For the proof of (v) we prepare a lemma.

**Lemma 2.2.** If \( n \equiv 0 \) or \( 3 \mod(4) \), the image of \( p_{n+4,2}^* : \{HP_{n+4,2}, S^{4n}\} \rightarrow \{S^{4n+15}, S^{4n}\} \) contains the element \( \eta \kappa \in G_{15} \).

The proof of (2.2): Since all Toda brackets which appear in the proof have zero indeterminacies, we have

\[ \eta \kappa = \langle \epsilon, 2t, \nu^t \rangle = \langle \epsilon, 2\nu, \nu \rangle = \langle \epsilon, 2g_{\omega}, g_{\omega} \rangle . \]

Consider the diagram

\[ \begin{array}{ccc}
S^{4n+14} & \xrightarrow{(n+3)g_{\omega}} & S^{4n+11} \\
\downarrow & & \downarrow \\
P_{n+3,1} & \xrightarrow{HP_{n+3,1}=S^{4n+8}} & P_{n+4,1} \\
\downarrow & & \downarrow \\
\epsilon & \xrightarrow{i} & S^{4n+12} \\
\end{array} \]

By (1.14) \( p_{n+3,1}=(n+2)g_{\omega} \) and \( p_{n+4,1}=(n+3)g_{\omega} \). So \( p_{n+3,1} \circ (n+3)g_{\omega} = \epsilon \circ p_{n+3,1} = 0 \), since \( 2g_{\omega}^2 = \epsilon g_{\omega} = 0 \). Then there exists \( f \in \{HP_{n+4,2}, S^{4n}\} \) with \( f \circ i = \epsilon \), and by definition of Toda bracket

\[ f \circ p_{n+4,2} \subseteq \langle \epsilon, (n+2)g_{\omega}, (n+3)g_{\omega} \rangle \]

and

\[ \langle \epsilon, (n+2)g_{\omega}, (n+3)g_{\omega} \rangle = \frac{1}{2} (n+2) (n+3) \langle \epsilon, 2g_{\omega}, g_{\omega} \rangle \]

\[ = \frac{1}{2} (n+2) (n+3) \eta \kappa . \]

Thus \( f \circ p_{n+4,2} = \frac{1}{2} (n+2) (n+3) \eta \kappa \). Since the order of \( \eta \kappa \) is 2, the conclusion follows.

Now we prove (v). Take \( f \in \{HP_{n+4,4}, S^{4n}\} \) with \( \text{deg}(f) = H\{n, 4\} \). Then \( e_c(f \circ p_{n+4,4}) = -H\{n, 4\} \alpha_H(n, 4) \) by (1.3), and \( \#(f \circ p_{n+4,4}) = \#e_c(f \circ p_{n+4,4}) = 1 \) or 2 by (1.2). From (1.9) \( H\{n, 5\}/(H\{n, 4\} \text{den}[H\{n, 4\} \alpha_H(n, 4)]) = 1 \) or 2. And by (1.2), if \( \nu_3(H\{n, 4\} \alpha_H(n, 4)) \equiv -1 \), we have \( \#(f \circ p_{n+4,4}) = \#e_c(f \circ p_{n+4,4}) = \text{den}[H\{n, 4\} \alpha_H(n, 4)] \) and

\[ H\{n, 5\} = H\{n, 4\} \text{den}[H\{n, 4\} \alpha_H(n, 4)] , \]
where \( \nu_p(n/m) = \nu_p(n) - \nu_p(m) \) for a prime number \( p \) and integers \( m \) and \( n \). (1.16), (ii), (iii), (iv) and elementary analysis show that \( \nu_p(H\{n,4\}\alpha_H(n,4)) \geq 0 \) if and only if \( n \equiv 3 \mod(2^3), 1 \mod(2^5), 34 \mod(2^6) \) or \( 0 \) and \( (2^9) \). Consider the case of \( n \equiv 3 \mod(2^9) \) or \( 0 \mod(2^{10}) \). By (2.2) there exists \( h \in \{HP_{n+4,2}, S^{\omega}\} \) with \( h \circ p_{n+4,2} = \eta \kappa \). Then if \( f + h \circ q_2 \), say \( f' \), satisfies the conditions \( \#e(f \circ p_{n+4,2}) = \#(f' \circ p_{n+4,2}) \) and \( \deg(f') = H\{n,4\} \). Then by (1.3) \( \#e(f' \circ p_{n+4,2}) = \text{den}[H\{n,4\}\alpha_H(n,4)] \) and the conclusion (v) follows from (1.9).

3. \( C\{n,k\} \) for \( k \leq 10 \)

In this section we determine inductively \( C\{n,k\} \) for \( k \leq 8 \) and estimate them for \( k = 9 \) and \( 10 \). The results are as follows.

**Theorem 3.1.**

(i) \( C\{n,1\} = 1 \),

(ii) \( C\{n,2\} = 2(n/2) \),

\( \frac{24}{n, 24} \) if \( n \equiv 1 \mod(4) \)

(iii) \( C\{n,4\} = C\{n,3\} = \begin{cases} 12/(n, 3) & \text{if } n \equiv 1 \mod(8) \\ 6/(n, 3) & \text{if } n \equiv 5 \mod(8) \end{cases} \)

(iv) \( C\{n,5\} = C\{n,4\}\text{den}[C\{n,4\}\alpha_c(n,4)] \),

(v) \( C\{n,6\} = C\{n,5\}\text{den}[C\{n,5\}\alpha_c(n,5)] \)

\( \begin{cases} C\{n,5\} & \text{if } n \equiv 0 \mod(2), 1, 11 \text{ or } 27 \mod(32) \\ 2C\{n,5\} & \text{otherwise} \end{cases} \)

(vi) \( C\{n,7\} = \begin{cases} C\{n,6\}\text{den}[C\{n,6\}\alpha_c(n,6)] & \text{if } n \equiv 0 \mod(2) \text{ or } 19 \mod(32) \\ 2C\{n,6\}\text{den}[C\{n,6\}\alpha_c(n,6)] & \text{otherwise} \end{cases} \)

(vii) \( C\{n,8\} = C\{n,7\} \),

(viii) \( C\{n,9\}/(C\{n,8\}\text{den}[C\{n,8\}\alpha_c(n,8)]) \)

\( \begin{cases} 1 \text{ or } 2 & \text{if } n \equiv 3 \mod(2^7) \text{ or } 1 \mod(2^9) \\ 1 & \text{otherwise} \end{cases} \)

\( \begin{cases} 1 & \text{if } n \equiv 0, 6 \mod(2^3), 10, 12 \mod(2^4) \\ 18, 20 \mod(2^5), 34, 36 \mod(2^6) \text{ or } 4 \mod(2^7) \\ 1 \text{ or } 2 & \text{otherwise} \end{cases} \)

Proof. (i) is trivial. (ii) is proved by the same methods as the proof of (ii) of (2.1).

The proof of (iii): The first equality is a consequence of (1.9) and the fact \( G_3 = 0 \). We prove the second equality. Choose \( f \in \{CP_{n+2,2}, S^{2\omega}\} \) with \( \deg(f) = C\{n,2\} \). Then \( C\{n,3\}/C\{n,2\} \) is a divisor of \( \#(f \circ p_{n+2,2}) \) from (1.9), there exists \( h \in \{CP_{n+2,1}, S^{2\omega}\} \) with \( (C\{n,3\}/C\{n,2\})f \circ p_{n+2,2} = h \circ q_1 \circ p_{n+2,2} \) from (1.10), while \( q_1 \circ p_{n+2,2} = (n+1)\eta \) from (i) of (1.13), so \( C\{n,3\}/C\{n,2\} \) is a multiple
of \( \#(f \circ p_{n+2,2}) \) if \( n \) is odd, and therefore \( C\{n,3\}/C\{n,2\} = \#(f \circ p_{n+2,2}) \) if \( n \) is odd. From (1.5), \( e'_k(f \circ p_{n+2,2}) = \frac{1}{2} a_2 - \frac{1}{2} C\{n,2\} \alpha_c(n,2) \) for some integer \( a_2 \). If \( n \equiv 3 \) mod(4), say \( n=4m+3 \), \( a_2 \equiv 0 \) mod(2) by (1.6)', then \( e'_k(f \circ p_{n+2,2}) = -(4m+3)/(6m+7)/12 \) by (1.16) and (ii), hence \( \#(f \circ p_{n+2,2}) = \text{den}[(4m+3)/12] = 12/(n,24) \) by (1.2), and therefore the conclusion follows in this case since \( C\{n,2\} = 2 \). If \( n \equiv 1 \) mod(4), say \( n=4m+1 \), \( a_2 \equiv 1 \) mod(2) by (1.4), (1.7), (1.7)' and (ii), then \( e'_k(f \circ p_{n+2,2}) = -(12m-1)(m+1)/6 \) by (1.16) and (ii), hence \( \#(f \circ p_{n+2,2}) = \text{den}[(m+1)/6] \) and the conclusion follows easily in this case also.

Next we consider the case of \( n \) being even. Take \( f \in \{CP_{n+3,3}, S^2n\} \) with \( \text{deg}(f) = C\{n,3\} \). First we show that \( C\{n,3\} \) is a multiple of \( 24/(n,24) \). Since arguments are quite parallel we only consider the case of \( n \equiv 0 \) mod(4). Put \( n=4m \) and consider the commutative diagram

\[
\begin{array}{ccc}
\widetilde{K}_0(CP_{4m+3,3}) & \xrightarrow{c} & \widetilde{K}(CP_{4m+3,3}) \\
\uparrow f^* & & \uparrow f^* \\
\widetilde{K}_0(S^{2m}) & \xrightarrow{c} & \widetilde{K}(S^{2m}) \\
\end{array}
\]

We can put \( f^*(g^n_k) = d_0\bar{z}^m + d_1\bar{z}^{m+1} \) for some integers \( d_0 \) and \( d_1 \). We have

\[
c(f^*(g^n_k)) = d_0(\bar{z}^m + \bar{z}) + d_1(\bar{z}^{m+1}) = d_0\bar{z}^m - 2d_1m\bar{z}^{m+1} + ((2m^2+m)d_0 + d_1)\bar{z}^{m+2},
\]

\[
c(f^*(g^n_k)) = d_0\bar{z}^m + a_1\bar{z}^{m+1} + a_2\bar{z}^{m+2}
\]

for some integers \( a_0, a_1 \) and \( a_2 \). Comparing the coefficients of the powers of \( z \), by (1.4) we have

\[
d_0 = a_0 = C\{n,3\},
\]

\[
(2m^2+m)d_0 + d_1 = a_2 = C\{4m,3\} \alpha_c(4m,2) = C\{4m,3\}m(12m+5)/6
\]

and so \( d_1 = -C\{4m,3\}m/6 \). Thus \( C\{4m,3\} \) is a multiple of \( \text{den}(m/6) = 24/(4m,24) \) as desired. Second we show that \( C\{n,3\} \) is a divisor of \( 24/(n,24) \). We define \( h: \text{CP}_{n+2,2} = S^{2n} \vee S^{2n+2} \to S^{2n} \) by \( h|_{S^{2n}} = 24/(n,24) \) and

\[
h|_{S^{2n+2}} = \begin{cases} 0 & \text{if } n \equiv 0 \text{ mod}(16) \\ \eta^2 & \text{for other even } n \end{cases}
\]

Since \( p_{n+2,2} = 2 \text{ng} \vee \eta, h\circ p_{n+2,2} = (12n/(n,24))\eta^m + \eta|_{S^{2n+2}} = 0 \). Hence there exists \( f' \in \{CP_{n+3,3}, S^{2n}\} \) with \( f'|_{CP_{n+2,2}} = h \). Clearly \( \text{deg}(f') = 24/(n,24) \), so \( C\{n,3\} \) is a divisor of \( 24/(n,24) \). Thus \( C\{n,3\} = 24/(n,24) \) if \( n \) is even. This completes the proof of (iii).

**The proof of (iv):** By (1.3), \( e_c(h\circ q_{1}\circ p_{n+4,4}) = 0 \) for any \( h \in \{CP_{n+4,3}, S^{2n}\} \) and then \( h\circ q_{1}\circ p_{n+4,4} = 0 \) by (1.2). So by (1.3), (1.9) and (1.10)
$$C\{n,5\}/C\{n,4\} = \#(f \circ p_{n+5,5}) = \text{den}[C\{n,4\} \alpha_c(n,4)].$$

The proof of (v): First consider the case of $n \equiv 1 \mod(2)$. Choose $f \in \{CP_{n+5,5}, S^{2n}\}$ with $\deg(f) = C\{n,5\}$. Recall that $G_9 = Z_2\{7\} \oplus Z_2\{7\} \oplus Z_3\{\mu\}$ and the kernel of $e_c: G_9 \to Q/Z$ is $Z_2\{7\} \oplus Z_2\{7\}$. Hence, if $e_c(f \circ p_{n+5,5}) = 0$, we can choose $h \in \{CP_{n+5,1}, S^{2n}\} = G_8$ with $(f+h \circ q) p_{n+5,5} = 0$, because $g_c p_{n+5,5} = p_{n+5,4} = \eta$ by (i) of (1.13). Since $\deg(f+h \circ q) = \deg(f) = C\{n,5\}$, by (1.9) we have

$$C\{n,6\} = C\{n,5\} = C\{n,5\} \# e_c(f \circ p_{n+5,5}).$$

If $e_c(f \circ p_{n+5,5}) \neq 0$, (1.9) implies

$$C\{n,6\} = 2C\{n,5\} = C\{n,5\} \# e_c(f \circ p_{n+5,5}).$$

Since $C\{n,5\}$ and $\alpha_c(n,5)$ are known, we can easily compute $\text{den}[C\{n,5\} \alpha_c(n,5)]$ by elementary analysis. Indeed

$$\# e_c(f \circ p_{n+5,5}) = \begin{cases} \text{den}[C\{n,5\} \alpha_c(n,5)] & \\
1 \text{ if } n \equiv 1, 11 \text{ or } 27 \mod(32) & \\
2 \text{ for other odd } n. & 
\end{cases}$$

This completes the proof of (v) if $n$ is odd.

Suppose that $n$ is even. It is easy to see that $\text{den}[C\{n,5\} \alpha_c(n,5)] = 1$. From (1.8) and (1.11)

$$C\{n,5\} \mid C\{n,6\} \mid H\{n/2,3\}.$$ 

By the previous calculations $C\{n,5\}$ and $H\{n/2,3\}$ are coincide if $n \equiv 0 \mod(4), 6, 10$ or $14 \mod(16)$, so $C\{n,5\} = C\{n,6\}$ in this case, while if $n \equiv 2 \mod(16)$ the odd components are coincide but

$$2 = \nu_2(C\{n,5\}) \leq \nu_2(C\{n,6\}) \leq \nu_2(H\{n/2,3\}) = 3.$$ 

Put $n = 16m + 2$. We construct a commutative diagram in which $\deg(f) = C\{16m+2,5\}$.
By (i) of (1.13), $q_{16m+5}p_{16m+7}=p_{16m+7,1}=0$ and so by (1.15) we have

$$q_{16m+7}(i_1^*i_1^{-1}(p_{16m+7}) = ((8m+4)g_\omega, (8m+16)g_\omega) .$$

Take $s_1 \in (i_1^*i_1^{-1}(p_{16m+7}) \subseteq \{S_{32m+13}, CP_{16m+6}\}$ with $q_{16m+4}s_1=(8m+16)g_\omega$. Put $s_1=q_{16m+1}^*s_1$. Then

$$q_3^3s_1 = q_{16m+4}^3s_1 = (8m+16)g_\omega = 0 .$$

Hence there exists $s_2 \in \{S_{32m+13}, CP_{16m+5,3}\}$ with $i_1^*s_2 = s_1$. Since $q_2^*s_2 \in G_3=0$, there exists $s_3 \in \{S_{32m+13}, CP_{16m+4,2}\}$ with $i_1^*s_3 = s_2$. Next we define $h$ by $h|_{S_{32m+4}} = C\{16m-2,4\}$ and $h|_{S_{32m+6}} = h^2$. Since $p_{16m+4,3}=(8m+1)g_\omega \vee \eta$ by the proof of (1.11), (1.14) and (i) of (1.13), we have

$$h \circ p_{16m+4,2} = C\{16m+2,4\} \cdot (8m+1)g_\omega + \eta^3$$

$$= \frac{24(8m+1)g_\omega + 12g_\omega}{16m+2,24}$$

$$= 0 .$$

So there exists $h' \in \{CP_{16m+5,3}, S_{32m+4}\}$ with $h' \circ i = h$. Since $h' \circ p_{16m+5,3} \in G_3=0$, there exists $h'' \in \{CP_{16m+6,4}, S_{32m+4}\}$ with $h'' \circ i = h'$. By (1.2), (1.3) and (iv) we have

$$\#(h'' \circ p_{16m+6,4}) = \#C(h'' \circ p_{16m+6,4})$$

$$= \text{den} \{\deg(h'') \alpha_c(16m+2,4)\}$$

$$= C\{16m+2,5\}/C\{16m+2,4\} .$$

Hence there exists $f \in \{CP_{16m+7,5}, S_{32m+4}\}$ with $(C\{16m+2,5\}/C\{16m+2,4\})h'' = f \circ i$ and $\deg(f) = \deg(h'')C\{16m+2,5\}/C\{16m+2,4\} = C\{16m+2,5\}$. This completes the construction of the above diagram.

Now we proceed to the proof of (v). We may write $s_3 = s_3 \setminus q_1^*s_3$ for some $s_3 \in \{S_{32m+13}, S_{32m+4}\}$. By (iii) of (1.13)

$$e(c(q_3^*s_3) = (16m+3)(3840m^3+2640m^2+590m+43)/2! \cdot 3 \cdot 5$$

so by (1.2) $q_1^*s_3$ is divisible by 2. Then

$$f \circ p_{16m+7,5} = f \circ 3p_{16m+7,5}, \text{ since } 2G_3 = 0 ,$$

$$= (C\{16m+2,5\}/C\{16m+2,4\})h \circ s_3$$

$$= (C\{16m+2,5\}/C\{16m+2,4\})(C\{16m+2,4\})s_3^3 + \eta^2 \circ q_1^*s_3$$

$$= (C\{16m+2,5\}/C\{16m+2,4\})(0+0), \text{ since } C\{16m+2,4\} \equiv 0 \mod(2)$$

$$\quad \text{ and } 2\eta = 0$$

$$= 0 .$$

Thus by (1.9), $C\{16m+2,6\} = C\{16m+2,5\}$. This completes the proof of (v).
The proof of (vi): First consider the case of \( n \) being odd. For any \( h \in \{CP_{n+6,6}, S^{2n}\} \), by (i) of (1.5) we have

\[ e'_h(h \circ q \circ p_{n+6,6}) = \frac{1}{2} a \]

for some integer \( a \). By (1.6) and (1.7) \( a \) is even. Then \( h \circ q \circ p_{n+6,6} = 0 \) by (1.2). Thus (1.9) and (1.10) imply

\[ C \{n, 7\} = C \{n, 6\} \#(f \circ p_{n+6,6}) \]

for \( f \in \{CP_{n+6,6}, S^{2n}\} \) with \( \deg(f) = C \{n, 6\} \). Again by (i) of (1.5)

\[ e'_h(f \circ p_{n+6,6}) = \frac{1}{2} a + \frac{1}{2} C \{n, 6\} a \]

for some integer \( a \), and by the proof of (iii) of (1.5) we have

\[ a \equiv 0 \mod(2) \quad \text{if} \quad n \equiv 3 \mod(4) \quad \text{or} \quad 33 \mod(64) \]

for other odd \( n \).

Then since \( \#(f \circ p_{n+6,6}) \) is equal to \( e'_h(f \circ p_{n+6,6}) = \text{den} \left[ \frac{1}{2} a + \frac{1}{2} C \{n, 6\} a \right] \)

by (1.2), elementary analysis draws the conclusion for odd \( n \) by (iii), (iv), (v) and (1.16).

Next suppose that \( n \) is even. Choose \( f \in \{CP_{n+6,6}, S^{2n}\} \) with \( \deg(f) = C \{n, 6\} \).

(1.2) says that \( e_c = 2e'_h : G_1 \to Q/Z \) is monomorphic on the odd component, so (vi) is true about the odd components by (1.3) and (1.9). So we only see the 2-primary part. Recall that \( G_1 = Z \{5\} \oplus Z_{63} \). By (1.3), (1.16) and elementary analysis show that

\[ \nu_2(\# \circ (f \circ p_{n+6,6})) \leq 2 \]

If \( \nu_2(\# \circ (f \circ p_{n+6,6})) = 0 \), \( \nu_2(\#(f \circ p_{n+6,6})) \leq 1 \) by (1.2) and (1.5). If \( \nu_2(\#(f \circ p_{n+6,6})) = 0 \), the result follows by (1.9). If \( \nu_2(\#(f \circ p_{n+6,6})) = 1 \), we have

\[ f \circ p_{n+6,6} = 4 \xi \mod(\text{odd components}) \]

Since \( 4 \xi = \mu \gamma^2 \) and \( p_{n+6,1} = q_5 \circ p_{n+6,6} = n \),

\[ (f + \mu \gamma q_5) p_{n+6,6} = 0 \mod(\text{odd components}) \]

Clearly \( \deg(f + \mu \gamma q_5) = \deg(f) = C \{n, 6\} \), so the result follows again by (1.9). If \( \nu_2(\# \circ (f \circ p_{n+6,6})) = u = 1 \) or 2,

\[ \nu_2(C \{n, 6\}) + u \leq \nu_2(C \{n, 7\}) \]

by (1.9), and

\[ \nu_2(\#(f \circ p_{n+6,6})) = u + 1 \]
by (1.2) and (1.5), so
\[ f \circ p_{n+6,6} \equiv 2^{n+6,6} \mod (2^{n+6,6}, \text{odd components}) \]
and then
\[ (2^n f + \mu \eta q_5) \circ p_{n+6,6} \equiv 0 \mod (\text{odd components}). \]

Put \[ \#((2^n f + \mu \eta q_5) \circ p_{n+6,6}) = 2m + 1. \] Then there exists \( h \in \{ CP_{n+7,7}, S^{2n+10} \} \) with \( h \mid CP_{n+6,6} = (2m + 1) (2^n f + \mu \eta q_5) \). Clearly \( \deg(h) = 2^n (2m + 1) \deg(f) = 2^n (2m + 1) \cdot C \{ n, 6 \} \). Since \( \deg(h) \) is a multiple of \( C \{ n, 7 \} \), we have
\[ \nu_2(C \{ n, 7 \}) \leq \nu_2(C \{ n, 6 \}) + u \]
and hence
\[ \nu_2(C \{ n, 7 \}) = \nu_2(C \{ n, 6 \}) + u \]
\[ = \nu_2(C \{ n, 6 \}) + \nu_2(\#c(f \circ p_{n+6,6})) \]
\[ = \nu_2(C \{ n, 6 \}) \text{den}[C \{ n, 6 \} \alpha_c(n, 6))] \]
as desired. This completes the proof of (vi).

**The proof of (vii):** Since \( G_{13} = Z_3 \{ \alpha_1, \beta_1 \} \), \( C \{ n, 8 \}/C \{ n, 7 \} = 1 \) or 3 by (1.9). In case of \( n \) being even, the relations
\[ C \{ n, 7 \} \mid C \{ n, 8 \} \mid H \{ n/2, 4 \} \]
and the previous calculations show that the 3-components of the first and the third are equal so that the 3-components of these three are equal. Thus \( C \{ n, 8 \} = C \{ n, 7 \} \) if \( n \) is even.

Choose \( h \in \{ CP_{n+7,7}, S^{2n+10} \} \) with \( \deg(h) = C \{ n+5, 2 \}. \) Then
\[ e_c(h \circ q_5 \circ p_{n+7,7}) = -C \{ n+5, 2 \} \alpha_c(n+5, 2) \]
\[ = -(n+5)(3n+20)/(12(n+5,2)) \]
so by (1.2)
\[ \#(h \circ q_5 \circ p_{n+7,7}) \equiv 0 \mod(3) \text{ if and only if } n \equiv 1 \mod(3). \]
Therefore if \( n \equiv 1 \mod(3) \), the image of
\[ p_{n+7,7}^* = (q_5 \circ p_{n+7,7})^*: \{ CP_{n+7,2}, S^{2n+10} \} \rightarrow \{ S^{2n+13}, S^{2n+10} \} = G_3 \]
contains \( Z_3 \{ \alpha_1 \}. \)

Take \( f \in \{ CP_{n+7,7}, S^{2n} \} \) with \( \deg(f) = C \{ n, 7 \}. \) Suppose that \( n \equiv 1 \mod(3). \)
If \( f \circ p_{n+7,7} = 0, C \{ n, 8 \} = C \{ n, 7 \} \) by (1.9). If \( f \circ p_{n+7,7} \neq 0, \) that is \( f \circ p_{n+7,7} = \pm \beta \alpha_1, \)
the above implies that there exists \( h' \in \{ CP_{n+7,2}, S^{2n+10} \} \) with \( h' \circ q_5 \circ p_{n+7,7} = \mp \alpha_1, \)
and we have
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\[(f + \beta_1 \circ h' \circ q_3) \circ p_{n+7,7} = 0,\]
\[\deg(f + \beta_1 \circ h' \circ q_3) = \deg(f) = C\{n, 7\}\]

and so by (1.9)
\[C\{n, 8\} = C\{n, 7\}.\]

Therefore \(C\{n, 8\} = C\{n, 7\}\) if \(n \equiv 1 \mod(3)\).

We must prove (vii) for the case of \(n \equiv 1 \mod(6)\). Put \(n = 6m + 1\). Take \(f \in \{CP_{6m+8,7}, S^{12m+2}\}\) with \(\deg(f) = C\{6m+1,7\}\). By the same methods as the proof of (v) we can construct a commutative diagram

\[
\begin{array}{ccc}
S^{12m+15} & \xrightarrow{s_1} & CP_{6m+8,7} \\
\downarrow & & \downarrow 2 \\
CP_{6m+7,6} & \xleftarrow{s_2} & CP_{6m+8,7} \\
\downarrow 4 & & \downarrow 4 \\
CP_{6m+5,4} & \xleftarrow{i} & CP_{6m+6,5} \\
\end{array}
\]

Take \(a \in \{CP_{6m+5,4}, S^{12m+2}\}\) with \(\deg(a) = C\{6m+1,4\}\) and \(b \in \{CP_{6m+3,2}, S^{12m+2}\}\) with \(\deg(b) = C\{6m+1,2\} = 2\). Consider the diagram

\[
\begin{array}{ccc}
\{S^{12m+6}, S^{12m+2}\} & \xrightarrow{\gamma} & \{CP_{6m+5,4}, S^{12m+2}\} \\
\downarrow & & \downarrow \approx \\
\{S^{12m+8}, S^{12m+2}\} & \xrightarrow{\gamma} & \{S^{12m+4}, S^{12m+2}\} \\
\end{array}
\]

in which the horizontals and the vertical are the parts of suitable Puppe exact sequences. Then \(a\) generates a free part of \(\{CP_{6m+5,4}, S^{12m+2}\}\) which is of rank 1, and so
\[f \circ i \circ i = (\deg(f)/\deg(a))a + q^*(e)\]
\[= (C\{6m+1,7\}/C\{6m+1,4\})a + q^*(e)\]

for some \(e \in \{S^{12m+8}, S^{12m+2}\} = G_6\). Then
\[ 2f \circ p_{6m+8,7} = 8f \circ p_{6m+8,7}, \text{ since } G_{12} = Z_3 \]
\[ = f \circ \text{i.o.i.o.s}_3 \]
\[ = (C \{6m+1,7\} / C \{6m+1,4\}) a \circ s_3 + e \circ q \circ s_3 \]
\[ = (C \{6m+1,7\} / C \{6m+1,4\}) a \circ s_3, \text{ since } G_{6} \circ G_7 = 0. \]

By the previous calculations and elementary analysis it follows that

\[ \nu_3(C \{6m+1,7\}) = \begin{cases} 3 & \text{if } m \equiv 1 \text{ or } 2 \text{ mod}(3) \\ 2 & \text{if } m \equiv 3 \text{ or } 6 \text{ mod}(9) \\ 1 & \text{if } m \equiv 0 \text{ mod}(9) \end{cases} \]

so if \( m \equiv 0 \text{ mod}(9) \) we have

\[ C \{6m+1,7\} / C \{6m+1,4\} \equiv 0 \text{ mod(3)} \]

and so

\[ f \circ p_{6m+8,7} = 0 \]

and then by (1.9)

\[ C \{6m+1,8\} = C \{6m+1,7\} \text{ if } m \equiv 0 \text{ mod}(9). \]

Next suppose that \( m \equiv 0 \text{ mod}(9) \). By (iii) of (1.13) we can easily see that

\[ \nu_3(\#_{eC}(g_2 \circ s_3)) = 0. \]

So by (1.13) and the same methods as the proof of (v), we can construct a commutative diagram

Then

\[ f \circ p_{6m+8,7} = 640f \circ p_{6m+8,7} \]
\[ = f \mid C_{6m+3,2} \circ s_5 \]
\[ = (\deg(f)/\deg(b))b \circ s_5 \]
\[ = (C\{6m+1,7\}/2)b \circ s_5 \]
\[ = 0, \text{ since } C\{6m+1,7\} \equiv 0 \mod(6) \]

so by (1.9)
\[ C\{6m+1,8\} = C\{6m+1,7\} \text{ if } m \equiv 0 \mod(9). \]

This completes the proof of (vii).

The proof of (viii): Take \( f \in \{C_{n+8,8}, S^{2n}\} \) with \( \deg(f) = C\{n,8\} \). First consider the case of \( n \) being even. By (i) of (1.13) \( p_{n+8,i} = q_7 \circ p_{n+8,8} = \eta \). Then \( f \) or \( f + \kappa q_n \), say \( f' \), satisfies
\[ \#(f' \circ p_{n+8,8}) = \#_c(f' \circ p_{n+8,8}) = \text{den}[C\{n,8\} \alpha_c(n,8)], \]
\[ \deg(f') = \deg(f) = C\{n,8\} \]
by (1.2), and so the conclusion follows from (1.9). Next suppose that \( n \) is odd. By (1.2)
\[ \#(f \circ p_{n+8,8})/\#_c(f \circ p_{n+8,8}) = 1 \text{ or } 2. \]

By the previous calculations and elementary analysis we have
\[ \nu_2(\text{den}[C\{n,8\} \alpha_c(n,8)]) = 0 \text{ if and only if } n \equiv 3 \mod(2^7) \text{ or } 1 \mod(2^9). \]
Therefore if \( n \equiv 3 \mod(2^7) \) and \( 1 \mod(2^9) \), by (1.2) we have
\[ \#(f \circ p_{n+8,8}) = \#_c(f \circ p_{n+8,8}) = \text{den}[C\{n,8\} \alpha_c(n,8)] \]
and so the conclusion follows.

The proof of (ix): Since \( 2G_{17} = 0 \), by (1.9) we have
\[ C\{n,10\}/C\{n,9\} = 1 \text{ or } 2. \]

In case of \( n \) being even, by the following relations and an elementary analysis conclusion follows if \( n \equiv 0 \mod(2^3), 10, 12, 14 \mod(2^4), 18, 20, 22 \mod(2^5), 34, 36 \mod(2^6) \) or \( 4 \mod(2^7) \)
\[ C\{n,9\}/C\{n,10\}/H\{n/2,5\}. \]

If \( n \equiv 6 \mod(2^9) \), the conclusion follows from the same methods as the proof of (vii).

4. Relations with other James numbers

In this section we use the notations and terminologies of James [6,7] freely.
Consider the fibration of Stiefel manifolds
\[ O_{n-1,k-1} \rightarrow O_{n,k} \overset{p}{\rightarrow} O_{n,1} = S^{nd-1} \]
and the cofibration of quasi-projective spaces
\[ Q_{n-1,k-1} \rightarrow Q_{n,k} \overset{q}{\rightarrow} Q_{n,1} = S^{nd-1} \]
where \( n > k > 0 \). Following James [6] we define non-negative integers \( O\{n,k\} \), \( O'\{n,k\} \), \( Q\{n,k\} \) and \( Q'\{n,k\} \) by the equations
\[
\begin{align*}
\pi_{nd-1}(O_{n,k}) &= O\{n,k\} \pi_{nd-1}(S^{nd-1}), \\
\pi_{nd-1}(O'_{n,k}) &= O'\{n,k\} \pi_{nd-1}(S^{nd-1}), \\
\pi_{nd-1}(Q_{n,k}) &= Q\{n,k\} \pi_{nd-1}(S^{nd-1}), \\
\pi_{nd-1}(Q'_{n,k}) &= Q'\{n,k\} \pi_{nd-1}(S^{nd-1})
\end{align*}
\]
here \( \pi_m^*(X) = \{S^m, X\} \) for a pointed space \( X \). We have

**Lemma 4.1.** \( O\{n,k\} | Q\{n,k\}, O'\{n,k\} | O\{n,k\} \) and \( Q'\{n,k\} | Q\{n,k\} \).

**Proof.** The first conclusion follows from the commutative diagram
\[
\begin{array}{ccc}
Q_{n,k} & \xrightarrow{q} & Q_{n,1} \\
\cap & & \cap \\
O_{n,k} & \xrightarrow{p} & O_{n,1}
\end{array}
\]
and the others follow immediately by definition.

Let \( M_k(F) \) be the order of the canonical \( F \)-line bundle over \( FP_k \) in the \( J \)-group \( J(FP_k) \) [3] which was determined by Adams-Walker [2] and Sigrist-Suter [13]. We have

**Lemma 4.2.** \( Q'\{n,k\} = O'\{n,k\} \).

**Proof.** For any \( m \) with \( m \equiv 0 \mod(M_k(F)) \) there exists \( S^0 \)-section \( w: Q_{m,1} \rightarrow Q_{m,k} \), that is, \( q \circ w = 1 \). By James [7] we have the diagram
\[
\begin{array}{cccccccc}
Q_{m,1} & \xrightarrow{1*} & Q_{m,k} & \xrightarrow{w*1} & Q_{m,k} & \xrightarrow{g'} & Q_{m+1,k} \\
\downarrow & & \downarrow & & \downarrow & & \\
1*q & & 1*p & & q*p & & q \\
\end{array}
\]
\[
\begin{array}{cccccccc}
Q_{m,1} & \xrightarrow{1*} & Q_{m,1} & \xrightarrow{w*1} & Q_{m,k} & \xrightarrow{g'} & Q_{m+1,k} \\
\end{array}
\]

in which \( g' \circ (w*1) \circ (1*i) \) is a homotopy equivalence by (7.3) of [7], the first
square is commutative, the second is homotopy commutative and the third is homotopy commutative up to sign from quasi-projective case of (5.2) of [7]. Applying \( \pi_{(m+n)d-1} \) to this diagram we have the following diagram

\[
\begin{array}{c}
\pi_{(m+n)d-1}(Q_{n,k}) \xrightarrow{i_*} \pi_{(m+n)d-1}(O_{n,k}) \xrightarrow{g_*} \pi_{(m+n)d-1}(Q_{m,k} \ast O_{n,k}) \xrightarrow{q_*} \\
\downarrow q_* \downarrow p_* \downarrow (q \ast p)_* \downarrow q_* \\
\pi_{(m+n)d-1}(Q_{n,k}) = \pi_{(m+n)d-1}(O_{n,k}) \xrightarrow{(q \ast p)_*} \pi_{(m+n)d-1}(Q_{m,k} \ast O_{n,k}) \xrightarrow{q_*} \pi_{(m+n)d-1}(S^{m+n})
\end{array}
\]

in which the first and second squares are commutative and the third is commutative up to sign. Hence \( Q^j \{m-n,k \} \mid O^j \{n,k \} \mid Q^j \{n,k \} \). Since \( Q^j \{m+n,k \} = Q^j \{n,k \} \), the conclusion follows.

We have also

**Lemma 4.3.** If \( n \geq 2(k-1)+2/d \), then

\[ Q^j \{n,k \} = O^j \{n,k \} = O \{n,k \} = Q \{n,k \}. \]

Proof. Since \( Q_{n,k} \) and \( O_{n,k} \) are \((n-k+1)d-2\) connected, the canonical homomorphisms \( \pi_{(m+n)d-1}(Q_{n,k}) \xrightarrow{i_*} \pi_{(m+n)d-1}(O_{n,k}) \) and \( \pi_{(m+n)d-1}(O_{n,k}) \xrightarrow{q_*} \pi_{(m+n)d-1}(O_{n,k}) \) are epimorphisms if \( n \geq 2(k-1)+2/d \). Then \( Q^j \{n,k \} = Q \{n,k \} \) and \( O^j \{n,k \} = O \{n,k \} \) in this case, and the conclusion follows from (4.2).

Atiyah [3] proved that \( Q_{n,k} \) and \( P_{k-n,k} \) are \( S \)-duals. His proof gives the following precise theorem.

**Theorem 4.4.** For any \( j \) with \( jM \{F \} \geq n \), there exists a \((djM \{F \} - 1)\)-duality \( u \in \{Q_{jM \{F \} - n+k,k} \mid P_{n,k} \}, S^{d(jM \{F \} - 1)} \} \).

Consider the cofibrations

\[
S^{(n-k)d} \xrightarrow{i_*} P_{n,k} \rightarrow P_{n,k-1} \rightarrow S^{(n-k)d+1} \\
S^{md-2} \xrightarrow{q_*} Q_{m-1,l-1} \subset Q_{m,l} \rightarrow S^{md-1}
\]

We have

**Proposition 4.5.** If \( jM \{F \} \geq n \), \((djM \{F \} - 1)\)-dual of \( i : S^{(n-k)d} \rightarrow P_{n,k} \) is \( q: Q_{jM \{F \} - n+k,k} \rightarrow S^{d(jM \{F \} - n+k)d-1} \), and hence \( F \{n-k,k \} = Q^j \{jM \{F \} - n+k,k \} \).

Proof. By Puppe exact sequences associated with the above cofibrations it is easily seen that \( \{S^{(n-k)d}, P_{n,k} \} \) and \( \{Q_{jM \{F \} - n+k,k}, S^{d(jM \{F \} - n+k)d-1} \} \) are infinite cyclic groups with generators \( i \) and \( q \) respectively. Then the conclusion follows from (4.4).
As a corollary of (4.3) and (4.5) we have

**Theorem 4.6.** $F\{n,k\}$ is equal to $O\{jM_k(F) - n, k\}$ if $jM_k(F) \geq n + 2k - 2 + 2/d$.

In case of $F = C$, Sigrist [12, Théorème I] proved that a prime number $p$ is a factor of $O\{m, l\}$ if and only if $p$ is a factor of $M_k(C)/(m, M_k(C))$. His proof is valid for the case of $F = H$, since $M_k(H)$ is known [13]. Then by (4.6) we have

**Proposition 4.7.** A prime number $p$ is a factor of $F\{n, k\}$ if and only if $p$ is a factor of $M_k(F)/(n, M_k(F))$.

**References**