ON THE COMMUTATIVITY OF THE RADICAL OF
THE GROUP ALGEBRA OF A FINITE GROUP

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Let $K$ be an algebraically closed field of characteristic $p>0$, and $G$ a finite
group of order $p^a m$ where $(p, m)=1$ and $a>0$. We denote by $J(KG)$ the radical
of the group algebra $KG$. In case $p$ is odd, D.A.R. Wallace [6] proved that
$J(KG)$ is commutative if and only if $G$ is abelian or $G'P$ is a Frobenius group
with complement $P$ and kernel $G'$, where $P$ is a Sylow $p$-subgroup of $G$ and
$G'$ the commutator subgroup of $G$. On the other hand, in case $p=2$, S. Koshi-
tani [1] has recently given a necessary and sufficient condition for $J(KG)$ to be
commutative. In this paper, we shall give alternative conditions for $J(KG)$
to be commutative.

If $J(KG)$ is commutative, then $G$ is a $p$-nilpotent group and a Sylow $p$-
subgroup of $G$ is abelian ([6], Theorem 2). We may therefore restrict our
attention to a $p$-nilpotent group. Now, we put $N=O_p'(G)$. For a central
primitive idempotent $\varepsilon$ of $KN$, we put $G_\varepsilon = \{g \in G \mid g\varepsilon g^{-1}=\varepsilon\}$. Let $a_i$ ($i=1,
2, \cdots, s$) be a complete residue system of $G(mod G_\varepsilon)$

$G = G_\varepsilon a_1 \cup G_\varepsilon a_2 \cup \cdots \cup G_\varepsilon a_s$.

Then K. Morita [2] proved the following:

Theorem 1. If $G$ is a $p$-nilpotent group, then $e=\sum_{i=1}^s \varepsilon^i$ is a central
primitive idempotent of $KG$ and $KG_\varepsilon$ is isomorphic to the matrix ring $(KP_\varepsilon)^f$
of degree $f$ over $KP_\varepsilon$ for some $f$, where $P_\varepsilon$ is a Sylow $p$-subgroup of $G_\varepsilon$.

In what follows, for a subset $S$ of $G$, we denote by $\hat{S}$ the element $\sum_{x \in S} x$ of
$KG$. By [5], Theorem, it holds that $J(KG)^2=0$ if and only if $p^a=2$. When this
is the case, $J(KG)$ is trivially commutative. Therefore we may restrict our
attention to the case $p^a \geq 3$. The following proposition contains [1], Theorem 2.

Proposition. If $G$ is a non-abelian group and $p^a \geq 3$, then the following
conditions are equivalent:

(1) $J(KG)$ is commutative.

(2) $(G'P)'=G'$ and $J(KG'P)$ is commutative.

(3) (i) $G'$ is a $p'$-group, and
(ii) each block of KG'P, which is not the principal block, is of defect 0 if \( p \neq 2 \) and of defect 1 or 0 if \( p = 2 \).

(4) (i) \( G' \) is a \( p' \)-group, and

(ii) for each \( x \in G' \), \( C_{a'P}(x) \) is a \( p' \)-group if \( p \neq 2 \) and its order is not divisible by 4 if \( p = 2 \).

Proof. (1)\( \Rightarrow \) (2): We put \( H = G'P \). Since \( H \) is a normal subgroup of \( G \), we have \( J(KH) \subseteq J(KG) \). Hence \( J(KH) \) is commutative, and so, by [6], Theorem 2, \( |H'| \) is not divisible by \( p \). Since \( J(KG) \) is commutative and \( J(KG'P) \supseteq J(KH) \supseteq H'J(KP) \), by [6], Lemma 3, we have \( \hat{G}'KG'P \supseteq J(KG) \supseteq H'J(KP) = \hat{H}'J(KP) \). Thus, we have \( G' \subseteq H'P \). Since \( G' \) is a \( p' \)-group by [6], Theorem 2, we have \( G' = H' \).

(2)\( \Rightarrow \) (3): Since \( J(KG'P) \) is commutative and \( (G'P)' = G' \), \( G' \) is a \( p' \)-group by [6], Theorem 2. Now, we put \( e_1 = |G'|^{-1}G' \), and \( e_2 = 1 - e_1 \). Then \( e_1 \) and \( e_2 \) are central idempotents of \( KG'P \). Thus we have \( J(KG'P) = e_1J(KG'P) \oplus e_2J(KG'P) \). Since \( J(KG'P) \) is commutative, by [6], Lemma 3, we have \( J(KG'P)^2 = e_1J(KG'P)^2 \oplus e_2J(KG'P)^2 = \hat{G}'KG'P = e_1KG'P \). Therefore \( e_1J(KG'P)^2 = 0 \), and so by Theorem 1, every non-simple block of \( e_1KG'P \) is isomorphic to the matrix ring over \( KD \), where \( K \) is of characteristic 2 and \( D \) is a group of order 2. Hence \( e_1KG'P \) is a direct sum of blocks of defect 0 or of defect 1 or 0 according as \( p \) is odd or 2. Since \( e_1KG'P(=e_1KP) \) is the principal block, we obtain (3).

(3)\( \Rightarrow \) (4): This is easy by [3], Theorem 4.

(4)\( \Rightarrow \) (3) is trivial.

(3)\( \Rightarrow \) (1): Since \( G'P \) is a normal subgroup of \( G \) and \( [G: G'P] \) is not divisible by \( p \), we have \( J(KG) = J(KG'P)KG \). We put \( e_1 = |G'|^{-1}G' \), and \( e_2 = 1 - e_1 \). Then \( e_1 \) and \( e_2 \) are central idempotents of \( KG \) and \( J(KG) = e_1J(KG'P) \cdot KG \oplus e_2J(KG'P)KG \). Since \( e_1J(KG'P)KG \subseteq \hat{G}'KG \), \( e_2J(KG'P)KG \) is a central ideal of \( KG \) by [4], Lemma 5. By Theorem 1, every block of \( e_2KG'P \) is isomorphic to the matrix ring over \( KD \), where \( D \) is a \( p' \)-group. From our assumption, every non-simple block of \( e_2KG'P \) has the radical of square zero. Thus, \( J(KG) \) is commutative.

Remark. The condition (4) of Proposition for \( p \) odd is equivalent to the condition of Wallace’s result ([6]) that \( G'P \) is a Frobenius group with complement \( P \) and kernel \( G' \).

Now, in case \( p = 2 \), we shall give the conditions for \( J(KG) \) to be commutative.

Theorem 2. Assume that \( p = 2 \), \( 2^s \geq 4 \) and \( G' \neq 1 \). Then the following conditions are equivalent:
(1) $J(KG)$ is commutative.

(2) $G'$ is of odd order and $|P \cap P^h| \leq 2$ for every $h \in G'P - P$.

(3) $G'$ is of odd order and $C_{G'}(s)/\langle s \rangle$ is either a 2-group or a Frobenius group with complement $P/\langle s \rangle$ for every involution $s$ of $P$.

Proof. (1)$\Rightarrow$(2): Suppose that $J(KG)$ is commutative. Then, by Proposition, $G'$ is of odd order. Let $h$ be an arbitrary element of $G'P - P$, and $x$ an arbitrary element of $P \cap P^h$. Then $hxh^{-1}x^{-1} \in P \cap G' = 1$, and so $x \in C_{G'}(h)$. Thus, $P \cap P^h \subseteq C_{G'}(h)$. Since we may assume that $h \in G' - 1$, we obtain $|P \cap P^h| \leq 2$ by Proposition.

(2)$\Rightarrow$(3): Let $s$ be an arbitrary involution of $P$ such that $C_{G'}(s) \neq P$. Then $P \cap P^s = \langle s \rangle$ for $x \in C_{G'}(s) - P$, and so $C_{G'}(s)/\langle s \rangle$ is a Frobenius group with complement $P/\langle s \rangle$.

(3)$\Rightarrow$(1): Let $x$ be an element of $G' - 1$, and $S$ a Sylow 2-subgroup of $C_{G'}(x)$. Suppose that $S \neq 1$. Then $S \subseteq P^u$ for some $u \in G'P$, and $x \in C_{G'}(S) \subseteq C_{G'}(s)$ for every involution $s$ of $S$. Hence, $C_{G'}(s)$ is not a 2-group, and so $C_{G'}(s)/\langle s \rangle$ is a Frobenius group with complement $P/\langle s \rangle$. Thus, we have $S \subseteq P^u \cap P^s = \langle s \rangle$, and hence $|C_{G'}(x)|$ is not divisible by 4, which implies (1) by Proposition.

Corollary. Assume that $p=2, 2^s \geq 4$ and $G' \neq 1$. If $J(KG)$ is commutative, then a Sylow 2-subgroup of $G$ is a cyclic group or an abelian group of type $(2, 2^s-1)$.

Proof. Suppose that $J(KG)$ is commutative. Then, by Theorem 2, $|P \cap P^h| \leq 2$ for every $h \in G'P - P$. If $P \cap P^h = 1$ for all $h \in G'P - P$, then $G'P$ is a Frobenius group with complement $P$ and kernel $G'$. Hence $P$ is cyclic. On the other hand, if $P \cap P^h = \langle s \rangle$ for some $h \in G'P - P$ and some involution $s$ of $P$, then $hsh^{-1}s^{-1} \in P \cap G' = 1$, and so $h \in C_{G'}(s)$ and $h \in P$. Therefore $C_{G'}(s)/\langle s \rangle$ properly contains $P$. Hence, $C_{G'}(s)/\langle s \rangle$ is a Frobenius group with complement $P/\langle s \rangle$ by the condition (3) of Theorem 2. Hence $P/\langle s \rangle$ is cyclic, and so $P$ is a cyclic group or an abelian group of type $(2, 2^s-1)$.

Remark. In case $G$ is a non-abelian group and $p^s \geq 3$, S. Koshitani [1] proved that if $J(KG)$ is commutative, then $N_G(P)$ is abelian. This is included in the following proposition: Let $G$ be a non-abelian group, and $p^s \geq 3$. If $J(KG)$ is commutative then $G$ is a semi-direct product of $G'$ by (abelian) $N_G(P)$.

Proof. It is easy to see $G = G'N_G(P)$. Suppose that $J(KG)$ is commutative. Let $x$ be a $p'$-element of $N_{G'}(P)$. Since $G'P$ is a $p$-nilpotent group, $N_{G'}(P)$ is the direct product of $P$ and a normal $p'$-subgroup, and so $C_{G'}(x)$ contains $P$. Hence, by Proposition (4), we have $x=1$, which implies that $G' \cap N_G(P) = 1$. 
References


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