ON THE COEFFICIENT RING OF A TORUS EXTENSION

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Introduction. S. Abhyankar, W. Heinzer and P. Eakin treated the following problem in [1]; if \( A[X] = B[Y] \), when is \( A \) isomorphic or identical to \( B \)? Replacing the polynomial ring by the torus extension we shall take up the following problem; if \( A[X, X^{-1}] = B[Y, Y^{-1}] \), when is \( A \) isomorphic or identical to \( B \)? We say that \( A \) is torus invariant (resp. strongly torus invariant) whenever \( A[X, X^{-1}] = B[Y, Y^{-1}] \) implies \( A = B \) (resp. \( A \approx B \)). The roles played by polynomial rings in [1] are played by the graded rings in our theory. A graded ring \( A = \sum A_i, i \in \mathbb{Z} \), with the property that \( A_i \neq 0 \) for each \( i \in \mathbb{Z} \), will be called a \( \mathbb{Z} \)-graded ring. Main results are the followings.

An affine domain \( A \) of dimension one over a field \( k \) is always torus invariant. Moreover \( A \) is not strongly torus invariant if and only if \( A \) has a graded ring structure. An affine domain of dimension two is not always torus invariant. We shall construct an affine domain of dimension two which is not torus invariant. Let \( A \) be an affine domain over \( k \) of dimension two. Assume that the field \( k \) contains all roots of “unity” and is of characteristic zero. If \( A \) is not torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring such that there exist invertible elements of non-zero degree.

In Section 1 we study elementary properties of graded rings. Especially we are interested in \( \mathbb{Z} \)-graded rings with invertible elements of non-zero degree. In Section 2 we discuss some conditions for \( A \) to be torus invariant. In Section 3 we give several sufficient conditions for an integral domain to be strongly torus invariant. Some relevant results will be found in S. Iitaka and T. Fujita [2]. Section 4 is devoted to the proof of the main results mentioned above. In Section 5 we fix an integral domain \( D \) and we treat only \( D \)-algebras and \( D \)-isomorphisms there. We shall prove the following two results. When \( A \) is a \( D \)-algebra of \( \text{tr. deg}_D A = 1 \) and \( A \) is not \( D \)-torus invariant, \( A \) is a \( \mathbb{Z} \)-graded ring such that \( D \) is contained in \( A \). If \( A \) is a \( \mathbb{Z} \)-graded ring such as \( D = A \), then the number of elements of the set of \( \{D \text{-isomorphic classes of } D\text{-algebras } B \text{ such that } A[X, X^{-1}] = B[Y, Y^{-1}] \} \) is \( \Phi(d) \), where \( d \) is the smallest positive integer among the degrees of units in \( A \) and \( \Phi \) is the Euler function.

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1. Some properties of graded rings

Let $R$ be commutative ring with identity. The ring $R$ is said to be a graded ring if $R$ is a graded module, $R = \sum R_i$, and $R_i R_j \subseteq R_{i+j}$.

**Lemma 1.1.** Let $R$ be a graded domain. Then we have the following.

1. The unity element of $R$ is homogeneous.
2. If $a$ is homogeneous and $a = bc$, then $b$ and $c$ are both homogeneous. In particular every invertible element is homogeneous.
3. If $R$ contains a field $k$, then $k$ is a subring of $R_0$.

**Proof.** (1) follows immediately from the relation $1^2 = 1$. The proof of (2) is easy and will be omitted. To prove (3) we can assume $k$ is different from $F_2$ by (1). Let $a$ be an element of $k$ different from 1. Then $1-a$ is homogeneous from (2). The unity 1 is homogeneous of degree 0 by (1). Hence $a$ should be homogeneous of degree 0.

We call a graded ring $R = \sum R_i$ to be a $\mathbb{Z}$-graded ring if $i \in \mathbb{Z}$, for some $i \in \mathbb{Z}^+$ and $\mathbb{Z}^-$.

**Proposition 1.2.** Let $R$ be a $\mathbb{Z}$-graded domain. Let $S = \{ i \in \mathbb{Z}; R_i \neq 0 \}$. Then $S = n\mathbb{Z}$ for a certain integer $n$.

**Proof.** Since $R$ is a domain, $S$ is a semi-group. Hence (1.2) is immediately seen by the following lemma.

**Lemma 1.3.** Let $S \subseteq \mathbb{Z}$ be a semi-group. If $S \cap \mathbb{Z}^+ \neq 0$ and $S \cap \mathbb{Z}^- \neq 0$, then $S$ is a subgroup of $\mathbb{Z}$.

If $R$ is a $\mathbb{Z}$-graded domain, then we may assume $R_i = 0$ for any $i \in \mathbb{Z}$.

**Proposition 1.4.** Let $R$ be a graded ring. If there is an invertible element $x$ in $R_i$, then $R = R_0[x, x^{-1}]$.

**Proof.** For any $r \in R_0$, $r = r(x^{-1}x^*) = rx^{-n}x^*$ and $rx^{-n}$ is in $R_0$, therefore $r \in R_0[x^*]$. Hence $R = R_0[x, x^{-1}]$.

**Corollary.** Let $R$ be a $\mathbb{Z}$-graded domain. If $R_0$ is a field, so $R = R_0[x, x^{-1}]$ for every $x \in R_i$, $x \neq 0$.

**Proof.** Choose non-zero elements $x \in R_i$, and $y \in R_i$. Since $R_0$ is a field, $0 \neq xy$ is invertible, therefore $x$ and $y$ are units in $R$, hence $R = R_0[x, x^{-1}]$.

2. Torus invariant rings

A ring $A$ is said to be torus invariant provided that $A$ has the following property:

If there exist a ring $B$, a variable $Y$ over $B$, and a variable $X$ over $A$ such that $A[X, X^{-1}]$ is isomorphic to $B[Y, Y^{-1}]$. 

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*Note:* The content provided is a partial transcription of the document, focusing on the key sections. For a complete and accurate representation, the entire document should be considered.
\( \Phi: A[X, X^{-1}] \to B[Y, Y^{-1}] \),
then \( A \) is always isomorphic to \( B \).

Especially if we have always \( \Phi(A) = B \) in such case, we say that the ring \( A \) is strongly torus invariant.

To show \( A \) is torus invariant (resp. strongly torus invariant) it suffices to prove that \( A \) is isomorphic to \( B \) (resp. \( A = B \)) under the assumption: \( A[X, X^{-1}] = B[Y, Y^{-1}] \).

(2.0) We begin with some elementary observations. Assume that

\[
(1) \quad R = A[X, X^{-1}] = B[Y, Y^{-1}].
\]

Then \( X \) and \( Y \) are units of \( R \). It follows from (1.1) that we have

\[
(2) \quad X = vY^{f} \text{ and } Y = uX^{f}, \quad v \in B \text{ and } u \in A,
\]
or equivalently

\[
(3) \quad v = u^{-f}X^{1-ff} \text{ and } u = v^{-f}Y^{1-ff}.
\]

In the rest of our paper we shall use the letters \( u \) and \( v \) to denote the elements of \( A \) and \( B \) respectively satisfying the relations (2) and (3) whenever we encounter the situation (1).

(2.1) The element \( u \) is in \( B \) if and only if \( ff' = 1 \). In this case we have \( A[X, X^{-1}] = B[X, X^{-1}] \), thus we have \( A \approx B \).

Proof is easy and is omitted.

**Proposition 2.2.** Let \( k \) be a field and \( A \) be a \( k \)-algebra. If \( A^{*} \) (the set of all invertible elements in \( A \)) = \( k^{*} \), then the ring \( A \) is torus invariant.

Proof. Let \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). By (1.1) the field \( k \) is contained in \( B \). Since \( A^{*} = k^{*} \), the unit element \( u \) of \( A \) is in \( k \), hence in \( B \). It follows from (2.1) that \( A \) is torus invariant.

**Proposition 2.3.** Let \( A = A_{0}[t_{1}, t_{2}, \ldots, t_{n}, (t_{1}t_{2}\cdots t_{n})^{-1}] \) where \( t_{i}'s \) are independent variables over \( k \)-algebra \( A_{0} \) and \( A^{*} = k^{*} \), then \( A \) is torus invariant.

Proof. Let \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). Then by the lemma (1.1) \( Y = uX^{f} \) and \( X = vY^{f} \). Since \( u \) is invertible in \( A = A_{0}[t_{1}, t_{2}, \ldots, t_{n}, (t_{1}t_{2}\cdots t_{n})^{-1}] \), \( Y = rt_{1}t_{2}\cdots t_{n}, \quad r \in A_{0}^{*} = k^{*} \). We may assume that \( r = 1 \), so \( Y = t_{1}t_{2}\cdots t_{n}X^{f} \).

On the other hand as \( t_{i} \) is invertible in \( R = B[Y, Y^{-1}] \), \( t_{i} = b_{i}Y^{f}, \quad b_{i} \in B^{*} \). Then we have that

\[
ff' + \sum e_{i}f_{i} = 1.
\]

Therefore the following natural homomorphism is surjective.
Since $Z$ is P.I.D., we can construct a basis of $Z^{(n+1)}$ containing this vector $(f', e_1, \ldots, e_n)$. Put this basis

$$e_0 = (f', e_1, \ldots, e_n)$$

$$e_i = (f_i, f_{i1}, \ldots, f_{in})$$

and put $u_i = t^{i_1} \cdots t^{i_n} X^{l_i}$.

$$R = A[u_1, \ldots, u_n, (u_1 \cdots u_n)^{-1}] [Y, Y^{-1}] = A[X, X^{-1}] = B[Y, Y^{-1}]$$.

Therefore $A$ is isomorphic to $B$. Hence $A$ is torus invariant.

(2.4) Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. An ideal $I$ of $R$ is said to be vertical relative to $A$ if there exists an ideal $J$ of $A$ such that $JR = I$. If $J$ is an ideal of $A$ such that $JR$ is vertical relative to $B$, then we will simply say that $J$ is vertical relative to $B$. If $A$ is a $k$-affine domain, the prime ideals defined by the singular locus of Spec $A$ are vertical relative to $B$.

**Proposition 2.5.** Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. If there exists a maximal ideal of $A$ which is vertical relative to $B$, then $A[X, X^{-1}] = B[X, X^{-1}]$. In particular $A$ and $B$ are isomorphic.

**Proof.** Let $m$ be a maximal ideal of $A$ which is vertical relative to $B$. Then there exists an ideal $n$ of $B$ such that $mR = nR$. Therefore $R/mR = A/m[X, X^{-1}] = B/n[Y, Y^{-1}]$, where $X = vY$ and $Y = uX$. Since $m$ is a maximal ideal, $A/m$ is a field. Hence $\overline{u}$ is in $B/n$ by (1.1). Therefore we obtain $f = \pm 1$ by (2.1). Thus $A$ is isomorphic to $B$.

**Corollary 2.6.** Let $A$ be a $k$-affine domain with isolated singular points, then $A$ is torus invariant.

### 3. Strongly torus invariant rings

In this section we investigate strongly torus invariant rings.

**Proposition 3.1.** Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If $Q(A) \subseteq Q(B)$, then $A = B$, where $Q(R)$ is the total quotient field of $R$.

**Proof.** Let $x$ be an element of $A$, then there exist two elements $b$ and $b'$ of $B$ such as $x = b/b'$. Hence $b = b'x$. In the graded ring $B[Y, Y^{-1}]$ the elements $b$ and $b'$ are homogeneous of degree zero, thus $x$ is also degree zero. Hence we have $A \subseteq B$. Let $b$ be an element of $B$. Then $b = \sum a_j X^j$, $a_j \in A$. By (2) and (2.0) we have that $b = \sum a_j v^j Y^{j_f}$. If $f = 0$, then $X \in B$. Thus $A[X, X^{-1}] \subseteq B$, and...
it's a contradiction, hence \( f \neq 0 \). Since \( a_jv^i \in B \) and \( Y \) is a variable over \( B \), \( b = a_0 \in A \). Thus \( A = B \).

**Corollary 3.2.** Let \( \bar{A} \) denote the integral closure of \( A \). If \( \bar{A} \) is strongly torus invariant, then \( A \) is also so.

Proof. It is easily seen that if \( A[X, X^{-1}] = B[Y, Y^{-1}] \) then \( \bar{A}[X, X^{-1}] = \bar{B}[Y, Y^{-1}] \). Since \( \bar{A} \) is strongly torus invariant, \( \bar{A} = \bar{B} \). Hence \( Q(A) = Q(B) \), and we have that \( A = B \).

**Proposition 3.3.** Let \( A \) be a domain with \( J(A) \neq 0 \), where \( J(D) \) is the Jacobson radical of a ring \( D \). Then \( A \) is strongly torus invariant.

Proof. Let \( a \) be a non-zero element of \( J(A) \). Then \( 1 + a \) is unit, so in the graded ring \( B[Y, Y^{-1}] \), \( 1 + a \) is homogeneous. Since the “unity 1” is a homogeneous element of degree 0, the element \( a \) is also so. Thus the element \( a \) is contained in \( B \).

Let \( x \) be any element of \( A \). Since \( xa \) is contained in \( J(A) \), \( xa \) is in \( B \). Hence \( A \) is contained in \( Q(B) \). By (3.1), we have that \( A = B \).

**Corollary 3.4.** If \( A \) is a local domain, then \( A \) is strongly torus invariant.

**Proposition 3.5.** Let \( A \) be an affine ring over a field \( k \) and let \( A[X, X^{-1}] = B[Y, Y^{-1}] \). Then \( A = B \) if and only if every maximal ideals of \( A \) is vertical relative to \( B \).

Proof. It suffices to prove the “if” part of the (3.5). By (3.3) we may assume that \( J(A) = 0 \). Let \( x \) be an element of \( B \) and let \( x = \sum_{j=1}^{t} a_j X^j \), where \( s < t \), \( a_j \in A \) and \( a_s \neq 0 \) and \( a_t \neq 0 \). For any maximal ideal \( m \) of \( A \) there exists a maximal ideal \( n \) of \( B \) such as \( mR = nR \), where \( R = A[X, X^{-1}] \). Let \( \bar{x} \) denote the residue class of \( x \) in \( B/n \). Then \( \bar{x} \) is algebraic over the coefficient field \( k \), hence there exist elements \( \lambda_0, \lambda_1, \cdots \lambda_{s-1} \) in \( k \), such that \( f(x) = x^s + \lambda_1 x^{s-1} + \cdots + \lambda_{s-1} x + \lambda_0 \in nR = mR \). If \( t \neq 0 \), then the highest degree term of \( f(x) \) with respect to \( X \) is \( a_t x^t \in mR \), thus \( a_t \) is contained in \( m \) for every maximal ideal in \( A \). Since \( J(A) = 0 \), \( a_t = 0 \). It’s a contradiction. Therefore \( t = 0 \). By the same way, we have that \( s = 0 \), hence \( x \) is in \( A \). Thus \( A = B \).

We denote the subring generated by all the units of \( A \) by \( A_u \).

**Proposition 3.6.** Let \( A \) be a \( k \)-affine domain with an isolated singular point. If \( A \) is algebraic over \( A_u \) then \( A \) is strongly torus invariant.

Proof. Let \( A[X, X^{-1}] = B[Y, Y^{-1}] \) and let \( m \) be the maximal ideal defined by the isolated singular point. Then there exists a maximal ideal \( n \) of \( B \) such as \( mR = nR \). Let \( a \) be a unit element of \( A \). In the graded ring \( B[Y, Y^{-1}] \), the
element \( a \) is also invertible, so \( a = bY^j \) for some invertible element \( b \) in \( B \) and a certain integer \( j \). Since \( A/m \) is algebraic over \( k \), there exist elements \( \lambda_0, \lambda_1, \ldots, \lambda_n \in k \) such that \( \lambda_0 a^0 + \cdots + \lambda_n a^n = mR = nR \). If \( j \neq 0 \), \( \lambda_0 b^n \) is in \( n \), hence \( b \) is not invertible, it's a contradiction. Thus we have that \( A \subseteq B \). By the following lemma our proof is over.

**Lemma 3.7.** Let \( A[X, X^{-1}] = B[Y, Y^{-1}] \). If \( A \) is algebraic over \( A \cap B \), then \( A = B \).

Proof. Since \( A \) is algebraic over \( A \cap B \), \( A \) is also algebraic over \( B \), but \( B \) is algebraically closed in \( B[Y, Y^{-1}] \), therefore \( A \) is contained in \( B \). Thus we have that \( A = B \).

Let \( A \) be an integral domain containing a field \( k \). We denote the set of all automorphisms of \( A \) over \( k \) by \( \text{Aut}_k(A) \).

**Proposition 3.8.** Let \( A \) be an integral domain containing an infinite field \( k \). If \( \text{Aut}_k(A) \) is a finite set, then \( A \) is strongly torus invariant.

Proof. Let \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). Let \( \Phi_{\lambda}, \lambda \in k^* \), be an automorphism of \( R \) defined by \( \Phi_{\lambda}(Y) = \lambda Y \) and \( \Phi_{\lambda}(b) = b \) for \( b \in B \). Following the notation of (2.0) we have \( X = vY \), thus \( \Phi_{\lambda}(X) = \lambda'X \), therefore \( R = \Phi_{\lambda}(A) [X, X^{-1}] \). Let \( p \) be the projection \( A[X, X^{-1}] \to A \) defined by \( p(X) = 1 \) and \( i \) be the canonical injection \( A \to A[X, X^{-1}] \). Define \( \sigma_{\lambda} = q \circ \Phi_{\lambda} \circ i \). Then \( \sigma_{\lambda} \) is an endomorphism of \( A \).

We shall prove that \( \sigma_{\lambda} \) is surjective. Let \( x \) be an element of \( A \). Since \( R = \Phi_{\lambda}(A) [X, X^{-1}] \), there exist elements \( a_j \) of \( A \) such as \( x = \sum \Phi_{\lambda}(a_j)X^j \). Hence \( x = p(x) = \sum p\Phi_{\lambda}(a_j) \). Let \( x' = \sum a_j \in A \), then \( \sigma_{\lambda}(x') = \sum p\Phi_{\lambda}(a_j) = x \). Thus \( \sigma_{\lambda} \) is surjective. Next we shall show that \( \sigma_{\lambda} \) is injective. Since \( \Phi_{\lambda}^{-1}(X^{-1})R \cap \Phi_{\lambda}(A) = \Phi_{\lambda}^{-1}(X^{-1})R \cap A = 0 \), we have \( (X^{-1})R \cap \Phi_{\lambda}(A) = 0 \), therefore \( \sigma_{\lambda} \) is injective. Hence \( \sigma_{\lambda} \) is an automorphism of \( A \).

We shall prove that the set \( \{ \sigma_{\lambda} \mid \lambda \in k^* \} \) is infinite when \( A \neq B \). Since \( u = v^{-1}Y^{1-ff'} \), \( \sigma_{\lambda}(u) = \lambda^{1-ff'}u \). Therefore our assertion is proved when \( 1-ff' \neq 0 \). Suppose \( ff' = 1 \). Then we may assume that \( R = A[X, X^{-1}] = B[X, X^{-1}] \). If \( A \subseteq B \), then \( A = B \), so there exists an element \( x \) of \( A \) not contained in \( B \), say \( x = \sum_{i=1}^t b_jX^j \), \( t > s \). Since \( \ker p = (X-1)R \) and \( (X-1)R \cap B = 0 \), \( p(b_j) \neq 0 \) for \( b_j \neq 0 \). Since \( \sigma_{\lambda}(x) = \sum p(b_j)\lambda^j \) and \( p(b_j) \neq 0 \) for some \( j \neq 0 \), the set \( \{ \sigma_{\lambda} \mid \lambda \in k^* \} \) is infinite.

Next we shall give two cases of rings which are not strongly torus invariant. If \( A \) has a non-trivial locally finite iterative higher derivation \( \psi : A \to A[T] \), then \( A[T] = B[T] \), where \( B = \psi(A) \) and \( A \neq B \), as is proved in [4]. Hence we have that \( A[T, T^{-1}] = B[T, T^{-1}] \) and \( A \neq B \). If \( A \) is a graded ring, then \( A \) is not strongly torus invariant. Indeed, let \( X \) be a variable over \( A \) and let
$B_i = \{a_iX^i; a_i \in A_i\}$. Then $B_i$ is an $A_\sigma$-module contained in $A[X, X^{-1}]$. Let $B = \sum B_i$. Then $B$ is a graded ring and we easily see that $A[X, X^{-1}] = B[X, X^{-1}]$. We shall show that $X$ is a variable over $B$. Assume that there exist elements $b_0, b_1, \ldots, b_n$ in $B$ such that $b_n \neq 0$ and $b_nX^n + \cdots + b_1X + b_0 = 0$. By the definition of $B$ we denote $b_i = \sum a_{ij}X^i$, $a_{ij} \in A_i$. In the graded ring $A[X, X^{-1}]$ the homogeneous term of degree $t$ of this equation is that

$$(a_{n,t-n} + a_{n-1,t-n+1} + \cdots + a_{0,t})X^t = 0.$$ 

Since $A$ is a graded ring and $a_{ij}$ is a homogeneous element of degree $j$, we obtain $a_{ij} = 0$ for all index $i$ and $j$, hence $X$ is a variable over $B$.

By [4] we have that a $k$-algebra $A$ has a non-trivial locally finite iterative higher derivation if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_a = \text{Spec } k[T]$. We easily see that $A$ is a non-trivial graded ring if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_a = \text{Spec } k[T, T^{-1}]$.

**Proposition 3.9.** A $k$-algebra $A$ is not strongly torus invariant, if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_a$ or $G_m$.

Assume that $\text{Aut}_k(A)$ is an infinite group. If $\text{Aut}_k(A)$ has an algebraic group structure, then there exists the following exact sequence;

$$0 \rightarrow T \rightarrow \text{Aut}_k(A)_0 \rightarrow \theta \rightarrow 0$$

where $\text{Aut}_k(A)_0$ is the connected component containing the identity $I_A$, and $T$ is a maximal torus subgroup of $\text{Aut}_k(A)_0$ and $\theta$ is an abelian variety. Let $P$ be an arbitrary closed point of $\text{Spec}(A)$. If $T = 0$, then there exists a regular map

$\Phi: \text{Aut}_k(A)_0 \rightarrow \text{Spec}(A)$

$$\sigma \mapsto \sigma(P).$$

Since $\text{Im}(\Phi)$ is a projective variety contained in the affine variety $\text{Spec}(A)$, the set $\text{Im}(\Phi)$ consists of one point, it contradicts $\dim \text{Aut}_k(A)_0 > 0$. Hence we have that $T \neq 0$. Since $T \cong G_a$ or $G_m$, we have the following result:

**Proposition 3.10.** If $\text{Aut}_k(A)$ is not a finite set and has an algebraic group structure, then $A$ is not strongly torus invariant.

**4. Affine domains of dimension $\leq 2$**

Let $k$ be a field of characteristic zero which contains all roots of "unity". In this section let $A$ be an affine domain over $k$. We shall see that if $\dim A = 1$, then $A$ is always torus invariant. Moreover $A$ is not strongly torus invariant if and only if $\text{Aut}_k(A) \cong G_m$. Let $\dim A \geq 2$. Then $A$ is not always torus
invariant. But if an integrally closed domain $A$ is not a $\mathbb{Z}$-graded ring, then $A$ is torus invariant.

For the proof we need a lemma.

**Lemma 4.1.** Let $K$ be a finite separable algebraic field extension of a field $k$. If $A$ is a one-dimensional affine normal ring such that $k \subset A \subseteq K[X, X^{-1}]$, then $A$ is a polynomial ring or a torus ring over $k'$ where $k'$ is the algebraic closure of $k$ in $A$.

**Proof.** We may assume that $k = k'$. Following the similar device to the proof of (2.9) in [1, p 322], we have $Q(A) = k(\theta)$ for some element $\theta$ of $A$.

Since $k[\theta] \subseteq A \subset k(\theta)$, $A = k[\theta]$ or $A = k[\theta, \frac{1}{f(\theta)}]$ for some polynomial $f(\theta) \in k[\theta]$. Let $A = k[\theta, \frac{1}{f(\theta)}]$. Then we may assume that $f(\theta)$ has no multiple factors. The element $f(\theta)$ is invertible in $A$, so is also invertible in $K[X, X^{-1}]$. Thus we have $f(\theta) = \beta X^r$, $\beta \in K$, $\theta \in K[X, X^{-1}]$. We may assume that $r \geq 0$, if necessary, by replacing $X$ with $X^{-1}$. Then we easily see that $\theta \in K[X]$. The uniqueness of the irreducible decomposition in a polynomial ring implies that deg$_A f(\theta) = 1$, since the polynomial $f(\theta)$ has no multiple factors and $f(\theta) = \beta X^r$. Hence we may assume that $f(\theta) = \theta$ and we obtain $A = k[\theta, \frac{1}{\theta}]$.

Let $A$ be an integral domain. If $A$ is contained in $K[X, X^{-1}]$, then $A$ is a polynomial ring or a torus ring over $k'$.

**Proposition 4.2.** Let $A$ be a one-dimensional affine domain over a field $k$ of characteristics zero. Then we obtain that

1. $A$ is torus invariant,

2. $A$ is not strongly torus invariant if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_m$. If $A$ is not strongly torus invariant and $A$ is integrally closed, then $A$ is a polynomial ring or a torus ring over the algebraic closure of $k$ in $A$.

**Proof.** At first we shall prove (2). The sufficiency follows from (3.9). Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$ in which $A \cong B$. If $mR \cap A \neq 0$ for any maximal ideal $m$ of $B$, then $m$ is vertical relative to $A$, and we have $A = B$ by (3.5). Hence there exists a maximal ideal $m$ such as $mR \cap A = 0$. Since $ch k = 0$, $B/m = K$ is a finite separable algebraic field over $k$. The residue mapping of $R$ to $R/mR$ yields (up to isomorphism) $k \subset A \subseteq K[Y, Y^{-1}]$ where $Y$ is algebraically independent over $K$. Therefore $A$ is a polynomial ring or a torus ring by the lemma (4.1). Thus the automorphism group $\text{Aut}_k A$ contains a subgroup isomorphic to $G_m$.

Assume that $A$ is not integrally closed. Then prime divisors in $A$ of the conductor $t(A/A)$ are vertical relative to $B$. Hence we may assume $X = Y$ by
The above lemma (4.1) implies that \( \overline{A} = k'[t, t^{-1}] \) or \( \overline{A} = k'[t] \) where \( k' \) is the algebraic closure of \( k \) in \( \overline{A} \).

Firstly let \( \overline{A} = k'[t] \). Since \( \overline{A} = \overline{B} \), there exists an element \( s \) in \( \overline{B} \) such that \( t = f(s) + g(X) \) and \( s = g(t) + g(X) \) where \( f(s) = 1 \) and \( f(X), g(X) \in k'[X, X^{-1}] \). We may assume that \( t = X^s s + f(X) \) and \( s = X^{-s} t + g(X) \). Let \( \bar{n} \) be a prime divisor in \( \overline{A} \) of the conductor \( t(A) \).

Then there exists a maximal ideal \( \bar{m} \) of \( \overline{B} \) such that \( \bar{n} \bar{R} = \bar{m} \bar{R} \). Since \( \overline{A} / \bar{n} \) is algebraic over \( k' \), there exist elements \( \lambda_0, \lambda_1, \ldots, \lambda_{d-1} \in k' \) such that \( t^d + \lambda_{d-1} t^{d-1} + \cdots + \lambda_0 \in \overline{m} \). The constant term of this polynomial with respect to \( s \) is the following:

\[
f(X)^d + \lambda_{d-1} f(X)^{d-1} + \cdots + \lambda_0 \in \overline{n} k'[s][X, X^{-1}] .
\]

Therefore \( f(X) = f \in k' \). Hence we may assume that \( t = X^s s \). We shall show that \( A \) is a graded ring. Let \( a \) be an element of \( A \). Since a is contained in \( A = k'[t] \) and \( t = X^s s \), we have that \( a = \sum \lambda_i t^i = \sum \lambda_i s^i X^{jn} \), \( \lambda_i, s^i \in \overline{B} \). On the other hand, as the element \( a \) is contained in \( B[X, X^{-1}] \), \( a = \sum b_i X^i \), \( b_i \in B \). Comparing the coefficient of the each term in the following; \( \sum \lambda_i s^i X^{jn} = \sum b_i X^i \), we have \( b_i = \lambda_i s^i (i = jn) \) and \( b_i = 0 \) (\( i \notin jn \)). If \( b_i \neq 0 \), then \( b_i X^i = \lambda_i s^i X^{jn} = \lambda_i t^i \beta B[X, X^{-1}] \cap \overline{A} = A[X, X^{-1}] \cap \overline{A} = A \). Therefore \( A \) has a graded ring structure.

Secondary let \( \overline{A} = k'[t, t^{-1}] \). Then \( \overline{B} = k'[s, s^{-1}] \). Since \( t \) and \( s \) are invertible in \( \overline{R} \), we may assume that \( t = s X^p \) and \( s = t X^q \), then \( t = (s X^p) X^q - s t X^{p+q} \), therefore \( ij = 1 \). Hence we may assume \( t = s X^p \). By the same method as in the case \( \overline{A} = k'[t] \) we have that \( A \) is a graded ring.

Proof of (1). If \( A \) is not integrally closed, then the prime divisors of the conductor \( t(\overline{A}/A) \) are vertical relative to \( B \). Since non-zero prime ideals of \( A \) are maximal, the ring \( A \) is isomorphic to \( B \) by (2.5). If \( A \) is integrally closed and \( A \) is neither a polynomial ring nor a torus ring, then \( A \) is strongly torus invariant, hence \( A \) is torus invariant. If \( A \) is either a polynomial ring or a torus ring, \( A \) is torus invariant by (2.2) and (2.3).

Next we shall consider the case; the coefficient field \( k \) has all roots of “unity” and its characteristic is zero. Then we prove the following:

**Theorem 4.3.** Let \( A \) be an integrally closed \( k \)-affine domain of dimension two, where the field \( k \) has all roots of “unity” and \( ch k = 0 \). If \( A \) is not torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring which contains units of non-zero degree.

Proof. Assume that \( A \) is not torus invariant. Then there exist a \( k \)-algebra \( B \) and independent variables \( X, Y \) such that \( A \) is not isomorphic to \( B \) and \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). By (2.0) and (2.1) we obtain \( ff' \neq 1 \). We shall show
that it follows from $ff'\equiv 1$ that $A$ is a $Z$-graded ring. We may only consider the case $1-ff'>0$. Let $x$ be a $(1-ff')-th$ root of $u$ and let $y=x^{-f}/X$. Then $y_{-ff'}=v$. Since $(y_{-ff'})^{-ff'}=u$, $x=\lambda y_{-ff'}Y$ for some $(1-ff')-th$ root $\lambda$ of “unity”. From the relations; $y=x^{-f}/X$ and $Y=ux^f$, we have $\lambda=1$.

Since $y=x^{-f}/X$ and $x=y_{-ff'}Y$ are invertible, we have $A[x][X, X^{-1}]=B[y][Y, Y^{-1}]=A[x][y, y^{-1}]=B[y][x, x^{-1}]$. Define a surjective homomorphism $j: A[x][y, y^{-1}]ightarrow A[x]$ by $j(y)=1$. Let $A_0=j(B[y])\subseteq A[x]$. We shall show that $A[x]/A_0[x, x^{-1}]$ is a $\Gamma$-graded ring. We may only consider $\sigma$ such that it follows from $j\phi_1$ that $M_1\equiv 1$.

Let $a$ be an element of $A_0$. Then $a=\sum b_jx^j$, $b_j\in B$. Since $j(a)=a$ and $j(x)=x$, we have that $a=\sum j(b_j)x^j$, $j(b_j)\in A_0$. Thus $A[x]=A_0[x, x^{-1}]$ and $x$ is algebraically independent over $A_0$. By the same way $B[y]=B_0[y, y^{-1}]$.

Since the every $(1-ff')-th$ roots of “unity” is contained in $k$ and $ch k=0$ and $A$ is normal, the extension $A[x]/A$ is a Galois extension with a cyclic group $G=\langle \sigma \rangle$ (cf. [3] p 214). Indeed when $|G|=n$, $n(1-ff')$ and there exists a primitive $n-th$ root $\lambda$ of “unity” such that $\sigma(x)=\lambda x$ and the invariant subring $A[x]^\sigma=A$ and $A[x]\sigma=A+x+\cdots+Ax^{n-1}$ is a free $A$-module.

Since the element $u$ is a unit of $A$ and $ch(k)=0$, the extension $A[x]/A$ is étale. Since $A$ is a normal domain, $A[x]$, hence $A_0[x, x^{-1}]$, is also a normal domain. From this we see that $A_0$ is always normal.

We shall show that there exists a subring $A_0'$ in $A[x]$ such that $A[x]=A_0'[x, x^{-1}]$ and $\sigma(A_0')=A_0'$. If $A_0$ is strongly torus invariant, then $\sigma(A_0)=A_0';$ for $\sigma(A_0)[x, x^{-1}]=A_0[x, x^{-1}]$, therefore $A_0'$ satisfies the conditions. If $A_0$ is not strongly torus invariant, then $A_0'=k'[t]$ or $k'[t, t^{-1}]$ by (4.2). Firstly let $A_0=k'[t]$. Since $k'[x, x^{-1}][t]=k'[x, x^{-1}][\sigma(t)]$, we easily see that $\sigma(t)=\mu xt+f(x)$, $\mu \in k^*$. The order of $\sigma$ is $n$, i.e. $\sigma^n=I$ and $\sigma^s(t)=t$, $t=\mu xt+f(x)$, $\mu \in k^*$. Since $\lambda$ is a primitive $n-th$ root of “unity”, the integer $n=2m$. If $w$ is odd, say $n=2m+1$, then $\sigma^s(t)=\mu xt+f(x)$ and $\mu^{s-1}=1$. Let $f(x)=\sum_{i=0}^{s-1}f_ix^i$ and define the set $\Delta=\{j\in Z; \lambda^j \neq \mu \}$. Let $h(x)=\sum_{i=0}^{s-1}h_ix^i$, where $h_i=\mu f_i(\mu-x_i)$, and put $s=t+h(x)$. Then $\sigma(s)=\mu s+\sum_{i=0}^{s}f_ix^i$, hence $\sigma^s(s)=\mu s+n\mu^{s-1}(\sum_{i=0}^{s}f_ix^i)=s+n\mu^{s-1}(\sum_{i=0}^{s}f_ix^i)$. Since $\sigma^s(s)=s$, we have $\sigma(s)=\mu s$. We set $A_0'=k'[s, s^{-1}]$, then $A_0'$ satisfies the conditions.

Secondary let $A_0=k'[t, t^{-1}]$. Since $k'[x, x^{-1}][t, t^{-1}]=k'[x, x^{-1}][\sigma(t), \sigma(t)^{-1}]$, we easily see that $\sigma(t)=\mu xt$ or $\sigma(t)=\mu xt^{-1}$, $\mu \in k^*$. Case (i); $\sigma(t)=\mu xt$. Since $\sigma^s(t)=\mu^s(1+(-n+1)\mu x^t)$ and $\sigma^s(t)=t$, we have that $\sigma(t)=\mu t$, so $\sigma(A_0)=A_0$.

Case (ii); $\sigma(t)=\mu xt^{-1}$. If $n$ is odd, say $n=2m+1$, then $\sigma^s(t)=\mu^s\lambda^{im}s^{-1}$, but this is impossible for $\sigma^s(t)=t$. Therefore $n$ is even, say $n=2m$. Then $\sigma^s(t)=\lambda^s t$. Since $\lambda$ is a primitive $n-th$ root of “unity”, the integer $i$ is even, say $i=2j$. Let $s=x^{-t}$ and $A_0'=k'[s, s^{-1}]$. Then $A_0'$ satisfies the conditions.
Next we shall show that $A$ has a $\mathbb{Z}$-graded ring structure. Let $a$ be an element of $A$. Since $a \in A_i[x, x^{-1}]$, $a = \sum a_i x^i$. Then $a = \sigma(a) = \sum \sigma(a_j) x^j$ and $\sigma(a_i) \in A_i$. Comparing the coefficient of each term in the equality; $\sum a_i x^i = \sum \sigma(a_j) x^j$, we have that $a_i = \sigma(a_j) x^j$. Then $\sigma(a) = \sum \sigma(a_j) x^j$, and $\sigma(a) x^j \in A_j$. Comparing the coefficient of each term in the equality; $2^i = \sum \sigma^i$, we have that $a_i = \sigma(a_j) x^j$ then $\sigma(a) x^j \in A_j$. Thus $a_i x^j$ is an element of $A$. Therefore $A$ is a graded ring. Since there exists units of non-zero degree, $A$ has a $\mathbb{Z}$-graded ring structure.

**Remark.** The converse of this theorem is false. Indeed we find by (2.3) that the ring $k[T][X, X^{-1}]$ is a $\mathbb{Z}$-graded ring with respect to $X$ which is torus invariant.

**Example.** We shall construct an example of an affine dimension $A$ of dimension two which is not torus invariant.

Let $D$ be an integrally closed domain of dimension one over an algebraically closed field $k$. Let $a$ be a non-unit of $D$ and $\alpha^\mu = a$, $\alpha \in D$. Assume that $D$ is noetherian and $D[\alpha]$ is strongly torus invariant. Since an affine domain of dimension one whose totally quotient field has a positive genus is strongly torus invariant by (4.2), this assumption can be satisfied for a suitable choice of $D$. Let $T$ be a variable over $D$ and $A = D[\alpha T, T]$. Let $X$ be a variable over $A$ and $S = T^2 X$ and $Y = T^5 X^2$. Let $B = D[\alpha S, S^5, S^{-5}]$. Since $T = S^{-2} Y$ and $X = S^5 Y^{-2}$, we have that $A[X, X^{-1}] = B[Y, Y^{-1}]$. By (1.1) invertible elements in the graded ring $A$ are homogeneous. Since $D^* = k^*$, we obtain $A^* = \{\eta T^{\eta i}; \eta \in k^* \text{ and } i \in \mathbb{Z}\}$. Hence the quotient $A^*/k^*$ is generated by $T^\eta$. Similarly $B^*/k^*$ is generated by $S^\eta$. We shall show that $A$ is not isomorphic to $B$. We assume that there exists an isomorphism $\sigma$ of $A$ to $B$. Since $\sigma$ is a group-isomorphism of $A^*$ to $B^*$, we have $\sigma(T^\eta) = \mu S^\eta$ or $\sigma(T^\eta) = \mu S^{-\eta}$, $\mu \in k^*$. We shall only consider the case: $\sigma(T^\eta) = \mu S^\eta$, since the proof of the other case is the similar. Let $\sigma$ be an isomorphism of $A[T]$ to $B[S]$ defined by $\sigma = \sigma$ on $A$ and $\sigma(T) = \xi S$, $\xi^\eta = \mu$. Then we have that $D[\alpha][S, S^{-1}] = \sigma(D[\alpha]) [S, S^{-1}]$, therefore $\sigma(D[\alpha]) = D[\alpha]$; for $D[\alpha]$ is strongly torus invariant. Since $\sigma(D) \subseteq D[\alpha] \cap B = D$, we have $\sigma(D) = D$, therefore we easily see that $\sigma$ is an isomorphism as graded rings. Thus we have $\sigma((\alpha T) D) = (\alpha^\mu S) D$, hence $\sigma(a) \in a^\mu D$. Since the element $a$ is not a unit, $a^\mu D \subseteq a D$, thus $\sigma(a) D \subseteq a D \subseteq a D$, so $a D \subseteq \sigma^{-1}(a) D$, hence we have a proper ascending chain $\{\sigma^{-n}(a) D\}$, but it contradicts the netherian assumption of $D$. Hence $A$ is not torus invariant.

(4.4) Now let $A = \sum A_i$ be an integrally closed $\mathbb{Z}$-graded domain which contains invertible elements of non-zero degree. Let $e$ be an invertible element of $A$ with the smallest positive degree $d$. Let $a$ be a unit of $A$, then $a$ is an homogeneous elements with $\deg a = jd$ for some integer $j$, and there exists an element $\xi$ of $A^*_d$ such as $a = \xi \eta$. Let $i$ be any positive integer and $x$ be one of the $ijd$ roots of $a$, say $x^{ijd} = a$. Since $A[x]$ is a $\mathbb{Z}$-graded ring with the
invertible elements \( x \) of degree one, \( A[x] = A_0[x, x^{-1}] \) by (1.4) where \( A_0 \) contains \( A \). Let \( f \) and \( f' \) be integers such as \( ff' + ij d = 1 \) and let \( X \) be a variable over \( A \). Put \( y = x^{-f'} \) and \( X = aX' \). Then \( x = y^{-f'}Y \) and \( X = y^{ijd}Y' \). Therefore \( A_0[x, x^{-1}] [X, X^{-1}] = A_0[y, y^{-1}] [Y, Y^{-1}] \). Since the every \( n \)-th roots of "unity" is contained in \( k \) and \( A \) is integral closed, the extension \( A[x]/A \) is a Galois extension with a cyclic group \( G = \langle \sigma \rangle \). Indeed \( |G| = df \) and there exists a primitive \( di-th \) root \( z \) of "unity" such as \( \sigma(z) = \lambda x \), and \( (A[z])^n = A \). Since \( A_0 \) is algebraic over \( A_0 \), \( \sigma(A_0) \) is also so, hence \( \sigma(A_0) \) is algebraic over \( A_0 \), but \( A_0 \) is algebraically closed in \( A_0[x, x^{-1}] \), therefore \( \sigma(A_0) = A_0 \). Since \( \sigma(y) = \lambda^{-f'y} \), \( \sigma \) is an automorphism of \( A_0[y, y^{-1}] \). Let \( B = A_0[y, y^{-1}]^\sigma \) and \( \sigma \) be an automorphism of \( A_0[x, x^{-1}] [X, X^{-1}] \) defined by \( \sigma(X) = X \) and \( \sigma = \sigma \) over \( A_0[x, x^{-1}] \). Since \( \sigma(Y) = Y \) and \( \sigma(X) = X \), we obtain \( B[Y, Y^{-1}] = A_0(y, y^{-1}) [Y, Y^{-1}]^\sigma = A_0[x, x^{-1}] [X, X^{-1}]^\sigma = A[X, X^{-1}] \).

**Proposition 4.5.** Let \( A \) be an integrally closed \( k \)-affine domain of dimension 2. If \( A[X, X^{-1}] = B[Y, Y^{-1}] \) and \( ff' \neq 1 \), then \( A \) has a \( \mathbb{Z} \)-graded ring structure and \( B \) is isomorphic to one of algebras constructed in (4.4).

Proof. The first statement is already mentioned in the proof of (4.3) and we obtained \( A_0[x, x^{-1}] [X, X^{-1}] = A_0[y, y^{-1}] [Y, Y^{-1}] \) and \( \sigma(A_0) = A_0 \). Let \( B' = A_0[y, y^{-1}]^\sigma \). Then \( B' \) is one of algebras in (4.4). Since \( B'[Y, Y^{-1}] = B[Y, Y^{-1}] \), \( B \) is isomorphic to \( B' \).

5. D-torus invariant

Let \( D \) be an integral domain containing a field \( k \) of characteristic zero and \( A \) be a \( D \)-algebra. The ring \( A \) is called \( D \)-torus invariant if \( A[X, X^{-1}] = B[Y, Y^{-1}] \) for a certain \( D \)-algebra \( B \) and independent variables \( X \) and \( Y \), then we have always \( A \approx_{D} B \). Then we have the following result:

**Proposition 5.1.** Let \( A \) be an integrally closed domain over \( D \) and \( \text{tr. deg}_D A = 1 \). If \( A \) is not \( D \)-torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring containing units of non-zero degree.

Proof. Let \( A[X, X^{-1}] = B[Y, Y^{-1}] \), where \( B \) is a \( D \)-algebra and not \( D \)-isomorphic to \( A \). By (2.0) and (2.1) we easily see that \( ff' = 1 \). Then we may assume \( 1 - ff' > 0 \). Let \( x \) be a \((1 - ff') - \text{th} \) root of \( u \) and \( y = x^{-f'}X \). Then we have that \( A[x] = A_0[x, x^{-1}] \) and \( B[y] = B_0[y, y^{-1}] \) as the proof of (4.3), where \( A_0 \) and \( B_0 \) are respectively subalgebras of \( A[x] \) and \( B[y] \) containing \( D \). Let \( \sigma \) be a generator of the cyclic Galois group of the extension \( A[x]/A \). We shall show that \( \sigma(A_0) = A_0 \). Since \( \text{tr. deg}_D A_0[x, X^{-1}] = 1 \), \( A_0 \) is algebraic over \( D \), thus \( \sigma(A_0) \) is also so. Since \( A_0 \) is algebraically closed in \( A_0[x, x^{-1}] \), we have that \( \sigma(A_0) = A_0 \). Following the similar devise to the proof of (4.3) we obtain
that $A$ is a $\mathbb{Z}$-graded ring, and $D$ is contained in $A$.

In the following we shall consider the case where $A$ is a $\mathbb{Z}$-graded ring and $A_0=D$. We consider only $D$-isomorphisms of $D$-algebras.

**Theorem 5.2.** Let $A$ be an integrally closed $\mathbb{Z}$-graded ring. Assume that the subring $A_0$ contains an algebraically closed field $k$ and that $A_0^n=k^n$. Let $d$ be the smallest positive integer among the set of degrees of units in $A$. Then the number of the isomorphic classes of $A_i$-algebra as $B$ such that $A[X, X^{-1}]=B[Y, Y^{-1}]$ equals to $\Phi(d)$, where $\Phi$ is the Euler function.

Proof. Let $i$ be an integer such as $1 \leq i < d$ and $(i, d)=1$. Since $(i, d)=1$, $ij+dh=1$ for some integers $j$ and $h$. Moreover we may assume $h \geq 0$. Fix a unit $\epsilon$ of degree $d$. Let $x$ be one of the $d$-th roots of $\epsilon$. Then we have that $A[x]=A_0[x, x^{-1}]$ for a subring $A_0$ by (1.4). Let $\sigma$ be a generator of the cyclic Galois group of the extension $A[x]/A$. Then $\sigma(x)=\lambda x$, where $\lambda$ is a primitive $d$-th root of “unity”. Since $A_0$ is algebraic over $A_0$ and algebraically closed in $A_0[x, x^{-1}]$, we obtain $\sigma(A_0)=A_0$. Let $X$ be a variable over $A$ and let $y=x^{-1}X$ and $Y=\epsilon^{d}x^{i}$. Then we have that $A_0[y, y^{-1}] [X, X^{-1}]=A_0[y, y^{-1}] [Y, Y^{-1}]$. Define $B_i=A_0[y, y^{-1}]^\sigma$ and let $\sigma$ be an isomorphism of $A_0[x, x^{-1}][X, X^{-1}]$ defined by $\sigma(X)=X$ and $\sigma=\sigma$ on $A_0[x, x^{-1}]$. Since $Y=\epsilon^{d}X$, $\sigma(Y)=Y$, therefore we obtain that $A[X, X^{-1}]=B[Y, Y^{-1}]$. We can easily see that $B_i$ is a $X$-graded ring and $(B_i)_0=A_0$. Especially we have $B_1\cong A$.

Let $i_1$ and $i_2$ be integers such as $1 \leq i_1 < i_2 < d$ and $(i_1, d)=(i_2, d)=1$. Let $B'=A_0[y, y^{-1}]^\sigma$ and $B''=A_0[z, z^{-1}]^\sigma$ where $\sigma(y)=\lambda^{-i_1}y$ and $\sigma(z)=\lambda^{-i_2} z$, i.e., $B'=B_i$ and $B''=B_{i_2}$. We shall show that $B'$ and $B''$ are not isomorphic. Assume that there exists an $A_0$-isomorphism $\psi$ of $B'$ to $B''$. Let $a$ be a unit in $B'$ of non-zero degree, say degree $a=n$, $n \neq 0$. Let $b$ be a homogeneous element of $B'$ and degree $b=t$. Then we have $b^n=ra'$ for an element $r$ in the coefficient ring $A_0$, hence $\psi(b^n)=\psi(b)^n=r\psi(a')$. Since $r$ and $\psi(a')$ are homogeneous, $\psi(b)$ is also homogeneous by (1.1), therefore $\psi$ is an isomorphism as graded rings.

Let $c$ be a homogeneous element in $B'$ of degree one. Then $c=s_1y$ for an element $s_1$ in $A_0$. Since $\sigma(c)=c$ and $\sigma(y)=\lambda^{-i_1}y$, we have $\sigma(s_1)=\lambda^{i_1}s_1$ hence $s_1'$ is in $B'$. Since $\psi(s_1y)=s_1z$ for an element $s_2$ in $A_0$. Since $\sigma(s_2z)=s_2z$ and $\sigma(z)=\lambda^{-i_2}z$, we have $\sigma(s_2)=\lambda^{i_2}s_2$, hence $s_2'$ is in $B''$. By the relations, $s_1'\psi(y^d)=\psi((s_1y)^d)=\psi(s_1y)^d=s_1'z^d$, we obtain $s_2'=\psi(y^d)\zeta^{-d}x^d$. Since $\psi(y^d)\zeta^{-d}$ is an invertible element in $B''$ and degree zero, we have $\zeta=\psi(y^d)\zeta^{-d} \in A_0^\times=k^\times$, therefore we have $s_2'=\eta s_1$, for some $\eta \in k$, $\eta^d=\zeta$. Hence $\sigma(s_2)=\lambda^{i_2}s_2$, but it contradicts the fact that $\sigma(s_2)=\lambda^{i_2}s_2$ and $\lambda$ is a primitive $d$-th root of “unity”. Therefore $B' \cong B''$.

Finally we shall show that if $A[X, X^{-1}]=B[Y, Y^{-1}]$ then $B$ is isomorphic to $B_i$ for some $i$ satisfying $0 < i < d$ and $(i, d)=1$. The invertible element $u$
in (2.0) is homogeneous. Let \( n \) be the degree of \( u \). If \( n=0 \), then \( A \) is isomorphic to \( B \) by (2.1), hence \( B \cong B_1 \). Assume \( n \neq 0 \). Let \( c \) be a non-zero homogeneous element of degree 1 and put \( \eta = c^* u^{-1} \). Then \( \eta \) is an element of \( A_0 \).

In the graded ring \( B[Y, Y^{-1}] \) the elements \( u \) and \( \eta \) are homogeneous, hence \( c \) is also homogeneous, thus we denote \( c=bY^j \) for some element \( b \) in \( B \) and some integer \( j \). Then we obtain that \( c^* = \eta u^{-1} \). On the other hand we have \( c^* = \eta u^{-1} Y^{1-ff'} \) by (2.0). Therefore we have \( 1-ff' = nj \).

By the minimality of \( d \) we obtain \( n=ld \) for some integer \( l \) and \( u=\xi e \), \( \xi \in A^*_g = k^* \). Since the field \( k \) is algebraically closed, we may assume \( \xi = 1 \), then the \( d \)-th root \( x \) of \( e \) is an \( n \)-th root of \( u \). Since the element \( \lambda \) is a primitive \( d \)-th root of "unity", there exists the unique integer \( i \) such that \( \lambda^{-i} = \lambda^{-i} \), \( 0 < i < d \), then \( (i, d) = 1 \) since \( (f, d) = 1 \). Let \( y' = x^{-f} Y^j \) and \( B' = (A_0][y', y'^{-1}])^g \). Then \( \sigma(y') = \lambda^{-i} y' = \lambda^{-i} y' \), hence \( B' = B_i \). We can easily show that \( x = y^{-f'} Y^j \), therefore we obtain \( A_0[x, x^{-i}] [X, X^{-i}] = A_0[y', y'^{-1}] [Y, Y^{-1}] \). Since \( \sigma(X) = X \) and \( \sigma(Y) = Y \), we have \( A[X, X^{-i}] = B_i[Y, Y^{-1}] \), hence \( B[Y, Y^{-1}] = B_i[Y, Y^{-1}] \). Thus we have \( B \cong B_i \).

References