ON THE EQUIVARIANT HOMOTOPY OF STIEFEL MANIFOLDS

KATSUHIRO KOMIYA

(Received May 18, 1979)

1. Introduction and results

Throughout this paper $G$ denotes a compact Lie group, and $\Lambda$ denotes one of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ and the quaternions $\mathbb{Q}$. Let $E$ be a representation of $G$ over $\Lambda$. All representations considered in this paper are orthogonal if $\Lambda = \mathbb{R}$, unitary if $\Lambda = \mathbb{C}$, and symplectic if $\Lambda = \mathbb{Q}$. For a positive integer $m \leq \dim \Lambda E$, the Stiefel manifold $V^m(E)$ consists of all orthonormal $m$-frames in $E$, i.e.,

$$V^m(E) = \{(v_1, \ldots, v_m)| v_i \in E, \|v_i\| = 1 \text{ for } i = 1, \ldots, m, \text{ and } v_i \perp v_j \text{ if } i \neq j\}.$$

If $m=1$, then $V^1(E)$ is the unit sphere $S(E)$ in $E$. For any $g \in G$ and any orthonormal $m$-frame $(v_1, \ldots, v_m)$ in $E$, $(gv_1, \ldots, gv_m)$ is also an orthonormal $m$-frame in $E$. This induces a smooth $G$-action on $V^m(E)$.

Let $E'$ be another representation of $G$ over $\Lambda$. We are interested in the set of $G$-homotopy classes of $G$-maps from $S(E)$ to $V^m(E')$, $[S(E), V^m(E')]_G$. If $m=1$, this set is the set of $G$-homotopy classes of $G$-maps from sphere to sphere, $[S(E), S(E')]_G$, which was studied in Hauschild [1], Rubinsztein [3] and others. (I am grateful to the referee who informed me that there was a gap in the proof of Rubinsztein’s main theorem [3; Theorem 7.2]. This information leads to an improvement of the presentation of this paper.)

For any positive integer $n$, let

$$\Lambda^n = \Lambda \oplus \cdots \oplus \Lambda \quad (n \text{ summands})$$

be a representation with trivial $G$-action and with the standard inner product. We define a map

$$j: S(E') \to V^m(E' \oplus \Lambda^{m-1})$$

by $j(v) = (v, e_1, \ldots, e_{m-1})$ for $v \in S(E')$ and the canonical orthonormal $(m-1)$-frame $(e_1, \ldots, e_{m-1})$ in $\Lambda^{m-1}$. Then $j$ is a $G$-embedding, and induces a transformation
between $G$-homotopy sets. We are also interested in this transformation $j_\ast$.  

In the non-equivariant case we already know some facts about $j_\ast$. Clearly $S^{d_n-1} = S(\Lambda^*)$ where $d=1$ if $\Lambda = \mathbb{R}$, $d=2$ if $\Lambda = \mathbb{C}$, and $d=4$ if $\Lambda = \mathbb{Q}$. The map

$$j: S^{d_n-1} = S(\Lambda^*) \to V^\Lambda_n(\Lambda^* \oplus \Lambda^{n-1}) = V^\Lambda_n(\Lambda^{n+n-1})$$

defined above induces a group homomorphism

$$j_\ast: \pi_i(S^{d_n-1}) \to \pi_i(V^\Lambda_n(\Lambda^{n+n-1}))$$

between the $i$-th homotopy groups for an integer $i \geq 0$. We collect known results about the homomorphism $j_\ast$ in the following:

**Proposition 1** (See for example [2; Chapter 7]). (a) $j_\ast$ is an isomorphism in each case of the followings:

(i) $m=1$,
(ii) $0 \leq i \leq d_n-2$,
(iii) $\Lambda = \mathbb{R}$, and $i = n-1$ is even,
(iv) $\Lambda = \mathbb{C}$ or $\mathbb{Q}$, and $i = d_n-1$.

Therefore

$$\pi_i(V^\Lambda_n(\Lambda^{n+n-1})) = \begin{cases} 0 & \text{csae (ii)} \\ \mathbb{Z} & \text{case (iii) or (iv)}. \end{cases}$$

(b) If $\Lambda = \mathbb{R}$, $m \geq 2$, and $i = n-1$ is odd, then $j_\ast$ is an epimorphism and

$$\pi_i(V^\Lambda_n(\Lambda^{n+n-1})) = \mathbb{Z}/2\mathbb{Z}.$$ 

To state our result in the equivariant case, let us define some notations. For any closed subgroup $H$ of $G$, $N(H)$ denotes the normalizer of $H$ in $G$, and $(H)$ denotes the conjugacy class of $H$ in $G$. Let $X$ be a $G$-space. For any $x \in X$, $G_x$ denotes the isotropy subgroup at $x$. The conjugacy class of an isotropy subgroup is called an orbit type. We put

$$X^H = \{ x \in X \mid H \subset G_x \},$$

$$X_H = \{ x \in X \mid H = G_x \},$$

and

$$X_{(H)} = \{ x \in X \mid (H) = (G_x) \}.$$ 

For a representation $E$ of $G$, $\mathcal{M}(E)$ denotes the set of orbit types appearing on $S(E)$. Choose a representative of each element of $\mathcal{M}(E)$, and denote by $\mathcal{M}_r(E)$ the set of those representatives. For any $H \in \mathcal{M}_r(E)$ there is a transformation

$$r_H: [S(E), S(E')]_G \to [S(E''), S(E''')]$$

restricting to the fixed point set by $H$, where $[ , ]$ denotes the non-equivariant
Our result is

**Theorem 2.** Let $E$, $F$ be representations of a compact Lie group $G$ over $\Lambda$. Let

$$j_* : [S(E), S(E \oplus F)]_G \rightarrow [S(E), V^*_\Lambda(E \oplus F \oplus \Lambda^{n-1})]_G$$

be the transformation induced from the $G$-map

$$j : S(E \oplus F) \rightarrow V^*_\Lambda(E \oplus F \oplus \Lambda^{n-1}) \cdot$$

Then

(a) $j_*$ is surjective,
(b) $j_*$ is bijective in particular in each case of the followings (i), (ii):
(i) $\Lambda = \mathbb{R}$, $\dim \mathbb{R} E^\mu$ is odd for any $H \in \mathbb{R}(E, F) = \{H \in \mathbb{R}_*(E) | \dim \mathbb{R} F^H = 0 \}$,
and
\[ r = \prod_{H \in \mathbb{R}(E, F)} r^H : [S(E), S(E \oplus F)]_G \rightarrow \prod_{H \in \mathbb{R}(E, F)} [S(E^H), S(E^H \oplus F^H)] \]

is injective,
(ii) $\Lambda = \mathbb{C}$ or $\mathbb{Q}$, and $r$ is injective,
(c) if $\dim \mathbb{R} E^\circ \geq 2$ then $[S(E), V^*_\Lambda(E \oplus F \oplus \Lambda^{n-1})]_G$ has a group structure and $j_*$ is a group homomorphism.

**Note.** The injectivity of $r$ is studied by several authors, e.g., Hauschild [1; Satz 4.5].

In the subsequent sections we prove Theorem 2. Section 2 is devoted to preliminary lemmas. Section 3 is devoted to proving the surjectivity of $j_*$, and section 4 is devoted to proving the injectivity of $j_*$. In section 5 we give a group structure to $[S(E), V^*_\Lambda(E \oplus F \oplus \Lambda^{n-1})]_G$ so that $j_*$ is a group homomorphism.

**2. Preliminary lemmas**

Let $E$, $F$ be representations of a compact Lie group $G$ over $\Lambda$, and let $M$ be a compact, smooth, free $G$-manifold with $\dim M \leq \dim S(E \oplus F)$. Consider the fibre bundles

$$B = M \times G V^*_\Lambda(E \oplus F \oplus \Lambda^{n-1}) \rightarrow M/G$$

with fibre $V^*_\Lambda(E \oplus F \oplus \Lambda^{n-1})$, and

$$B' = M \times G S(E \oplus F) \rightarrow M/G$$

with fibre $S(E \oplus F)$. The $G$-map
induces a bundle embedding \( j: B' \to B \). There is a one-to-one correspondence between \( G \)-maps from \( M \) to \( V^\omega(E \oplus F \oplus \wedge^{n-1}) \) and cross sections of \( B \), and there is also a one-to-one correspondence between their homotopies. This shows that the following two lemmas are equivalent:

**Lemma 3.** Let

\[ f: M \to V^\omega(E \oplus F \oplus \wedge^{n-1}) \]

be a \( G \)-map, and let

\[ P: \partial M \times [0, 1] \to V^\omega(E \oplus F \oplus \wedge^{n-1}) \]

be a \( G \)-homotopy with \( P_0 = f|\partial M \) and \( P_1(\partial M) \subset i(S(E \oplus F)) \). Then \( P \) extends to a \( G \)-homotopy

\[ Q: M \times [0, 1] \to V^\omega(E \oplus F \oplus \wedge^{n-1}) \]

with \( Q_0 = f \) and \( Q_1(M) \subset j(S(E \oplus F)) \).

**Lemma 4.** Let \( N = M/G \). Let \( s: N \to B \) be a cross section of \( B \), and let \( P: \partial N \times [0, 1] \to B \) be a homotopy of cross section of \( B \) with \( P_0 = s|\partial N \) and \( P_1(\partial N) \subset j(B') \). Then \( P \) extends to a homotopy of cross section of \( B \), \( Q: N \times [0, 1] \to B \), with \( Q_0 = s \) and \( Q_1(N) \subset j(B') \).

We give a proof of Lemma 4 making use of the obstruction theory. Refer to [4; Part III] for the obstruction theory.

**Proof of Lemma 4.** Since \( N \) is a smooth manifold, we obtain a triangulation of \( N \). Let \( n = \dim S(E \oplus F) \). Then \( \dim N \leq n \), and \( S(E \oplus F) \), which is the fibre of \( B' \), is \((n-1)\)-connected. So the cross section \( j^{-1}P_1 \) of \( B' \) extends to a cross section \( s_1: N \to B' \) of \( B' \). We see from Proposition 1 that \( V^\omega(E \oplus F \oplus \wedge^{n-1}) \) is also \((n-1)\)-connected. Let \( N^{n-1} \) denote the \((n-1)\)-skeleton of \( N \), which contains \( \partial N \). Then \( P \) extends to a homotopy of cross section,

\[ R: N^{n-1} \times [0, 1] \to B |N^{n-1} \]

with \( R_0 = s|N^{n-1} \) and \( R_1 = j s_1|N^{n-1} \). So, if \( \dim N < n \), the lemma is proved.

Now let \( \dim N = n \). Let \( B(\pi_n) \) and \( B'(\pi_n) \) be the bundles of coefficients associated with the bundles \( B \) and \( B' \) by the \( n \)-th homotopy group, respectively. Also let \( C^n(N; B(\pi_n)) \) and \( C^n(N; B'(\pi_n)) \) be the groups of \( n \)-cochains of \( N \) with coefficients in \( B(\pi_n) \) and \( B'(\pi_n) \), respectively. The bundle embedding \( j: B' \to B \) induces a group homomorphism

\[ j_*: C^n(N; B'(\pi_n)) \to C^n(N; B(\pi_n)) \]

We see from Proposition 1 that \( j_* \) is an epimorphism. Let
be the deformation $n$-cochain. (See [4; p 172].) There is $d' \in C^*(N; B'(\pi_n))$ with $\tilde{j}_*(d') = -d$. By [4; 33.9] there is a cross section $s_2$ of $B'$ such that $s_2$ agrees with $s_1$ on $N^{s-1}$ and $d(s_1, s_2) = -d'$, where $d(s_1, s_2)$ is the difference $n$-cochain. (Also see [4; p 172].) We see

$$d(\tilde{j}s_1, \tilde{j}s_2) = \tilde{j}_*(d(s_1, s_2)) = -d.$$

We define a homotopy of cross section of $B|N^{s-1}$,

$$S: N^{s-1} \times [0, 1] \to B|N^{s-1},$$

by

$$S(x, t) = \begin{cases} R(x, 2t) & \text{if } 0 \leq t \leq 1/2 \\ \tilde{j}s_2(x) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

By [4; 33.7],

$$d(s, s, \tilde{j}s_2) = d(s, R, \tilde{j}s_1) + d(\tilde{j}s_1, \tilde{j}s_2) = d - d = 0.$$

By this and [4; 33.8], $S$ extends to a homotopy of cross section of $B, T: N \times [0, 1] \to B$, with $T_0 = s$ and $T_1 = \tilde{j}s_2$. Let $\partial N \times [0, 1] \subset N$ be a collar of $\partial N$ in $N$. Then we define a homotopy $Q: N \times [0, 1] \to B$ as, for $x \in N$ and $t \in [0, 1]$,

$$Q(x, t) = T(x, t) \quad \text{if } x \in N - \partial N \times [0, 1],$$

$$Q(x, t) = T(x, t/(2-r)) \quad \text{if } x = (y, r) \in \partial N \times [0, 1] \text{ and } 2t + r \leq 2,$$

$$Q(x, t) = T(x, (t+r-1)/r) \quad \text{if } x = (y, r) \in \partial N \times [0, 1], \quad 2t + r \geq 2$$

and $r \neq 0$.

$Q$ is well-defined, and is an extension of $P$ with the desired property.

Q.E.D.

3. The surjectivity of $j_*$

Let $E, F$ be representations of a compact Lie group $G$ over $\Lambda$, and let

$$j_*: [S(E), S(E \oplus F)]_G \to [S(E), V^\Lambda(E \oplus F \oplus \Lambda^{s-1})]_G$$

be the transformation induced from the $G$-map

$$j: S(E \oplus F) \to V^\Lambda(E \oplus F \oplus \Lambda^{s-1}).$$

The purpose of this section is to prove the surjectivity of $j_*$. Since $j$ is an embedding, it suffices to prove the following fact:
Lemma 5. Let

$$f: S(E) \to V_m(E \oplus F \oplus \Lambda^{n-1})$$

be a G-map, and let N be a compact smooth G-submanifold of S(E) with \(\dim N = \dim S(E)\). Let

$$T: N \times [0, 1] \to V_m(E \oplus F \oplus \Lambda^{n-1})$$

be a G-homotopy with \(T_0 = f|N\) and \(T_1(N) \subset j(S(E \oplus F))\). Then T extends to a G-homotopy

$$R: S(E) \times [0, 1] \to V_m(E \oplus F \oplus \Lambda^{n-1})$$

with \(R_0 = f\) and \(R_1(S(E)) \subset i(S(E \oplus F))\).

Proof. \(\mathcal{M}(E)\) is a finite set. Let us number its elements

$$\mathcal{M}(E) = \{(H_1), (H_2), \ldots, (H_a)\}$$

in such a way that if \(i < k\) then \((H_i) \subseteq (H_k)\). Consider the following Assertion:

**Assertion.** There are compact smooth G-submanifolds \(M_0, M_1, \ldots, M_a\) of S(E) such that

\[
\dim M_i = \dim S(E) \quad \text{for} \quad i = 0, 1, \ldots, a, \quad M_0 \supset N, \quad \text{and} \quad \text{Int} M_i \supset M_{i-1} \cup S(E|_{Hi}) \quad \text{for} \quad i = 1, 2, \ldots, a.
\]

Furthermore there are G-homotopies \(R^{(0)}, R^{(1)}, \ldots, R^{(a)}\) such that

- \(R^{(i)}, M_i \times [0, 1] \to V_m(E \oplus F \oplus \Lambda^{n-1})\) for \(i = 0, 1, \ldots, a\),
- \(R^{(i)}|M_i = f|M_i\) for \(i = 0, 1, \ldots, a\),
- \(R^{(i)}(M_i) \subset j(S(E \oplus F))\) for \(i = 0, 1, \ldots, a\),
- \(R^{(0)}|N \times [0, 1] = T\), and
- \(R^{(i)}|M_{i-1} \times [0, 1] = R^{(i-1)}\) for \(i = 1, 2, \ldots, a\).

Lemma 5 follows from the Assertion since \(M_a = S(E)\). In the following we prove the Assertion.

\(N\) and \(T\) satisfy the conditions for \(M_0\) and \(R^{(0)}\), respectively. Suppose that \(M_0, \ldots, M_{i-1}, R^{(0)}, \ldots, R^{(i-1)}\) are constructed. Put

$$M = (S(E) - \text{Int} M_{i-1})^{H_i} \subset S(E^{H_i}) - \text{Int} M_{i-1}^{H_i}.$$  

Then \(M\) is a compact smooth manifold with boundary \(\partial M = M \cap \partial M_{i-1}\). Moreover \(M\) is \(N(H_i)\)-invariant, and all isotropy subgroups on \(M\) are \(H_i\). So \(M\) becomes a free \(N(H_i)/H_i\)-manifold. Regard \(E^{H_i}\) and \(F^{H_i}\) as representations of \(N(H_i)/H_i\). By Lemma 3 there is an \(N(H_i)/H_i\)-homotopy
such that
\[ Q_0 = f|\partial M, \]
\[ Q_1(M) \supset i((S(E_{H_i} \oplus F_{H_i}))_{H_i}), \quad \text{and} \]
\[ Q|\partial M \times [0, 1] = R^{d-1}|\partial M \times [0, 1]. \]

Since \( G(M) = G \times_{N(E)} M \), we may extend \( Q \) to a \( G \)-homotopy
\[ Q': G(M) \times [0, 1] \to G(V^d_m(E_{H_i} \oplus F_{H_i} \oplus \Lambda^{n-1})) \subset V^d_m(E \oplus F \oplus \Lambda^{n-1}) \]
such that
\[ Q'_0 = f|G(M), \]
\[ Q'_1(G(M)) \supset i(S(E \oplus F)), \quad \text{and} \]
\[ Q'|\partial G(M) \times [0, 1] = R^{d-1}|\partial G(M) \times [0, 1]. \]

Applying [3; Lemma 1.1] to the \( G \)-manifold \( A = S(E) - \text{Int} \, M_{i-1} \) and the submanifold \( G(M) \) of \( A \), we obtain compact \( G \)-submanifolds \( K, L \) of \( A \) such that
(i) \( K \cup L = A \),
(ii) \( \partial L = L \cap K \),
\( \partial K = \partial L \cup \partial A = \partial L \cup \partial M_{i-1}, \)
\( \partial M_{i-1} \cap \partial L = \phi, \)
(iii) \( \partial M_{i-1} \cup G(M) \subset K \), and
(iv) \( K \) is a mapping cylinder of some \( G \)-map
\[ \psi: \partial L \to \partial M_{i-1} \cup G(M). \]

Put \( M_i = M_{i-1} \cup K \) in \( S(E) \). Then \( M_i \) is a compact smooth \( G \)-submanifold of \( S(E) \) with \( \dim M_i = \dim S(E) \), and with \( \text{Int} \, M_i \supset M_{i-1} \cup S(E)_{(H_i)} \). According to (iv), let us denote a point of \( K \) by the form \([y, s], \) where \( y \in \partial L \) and \( s \in [0, 1] \). Under this form \([y, 1] = y \) and \([y, 0] = \psi(y). \) We define a \( G \)-homotopy
\[ R^{d}(i): M_i \times [0, 1] \to V^d_m(E \oplus F \oplus \Lambda^{n-1}) \]
as the following: For \((x, t) \in M_i \times [0, 1],\)

\[
R^{(i)}(x, t) = R^{(i-1)}(x, t) \quad \text{if} \quad x \in M_{i-1},
\]

\[
R^{(i)}(x, t) = f([y, s-2t]) \quad \text{if} \quad x = [y, s] \in K \quad \text{and} \quad 2t \leq s,
\]

\[
R^{(i)}(x, t) = R^{(i-1)}(\psi(y), (2t-s)/(2-s)) \quad \text{if} \quad x = [y, s] \in K,
\]

\[
\psi(y) \in \partial M_{i-1} \quad \text{and} \quad s \leq 2t,
\]

\[
R^{(i)}(x, t) = Q'(\psi(y), (2t-s)/(2-s)) \quad \text{if} \quad x = [y, s] \in K,
\]

\[
\psi(y) \in G(M) \quad \text{and} \quad s \leq 2t.
\]

\(M_i\) and \(R^{(i)}\) constructed above satisfy the conditions in the Assertion. Thus this completes the proof. Q.E.D.

Now let \(X, Y\) be \(G\)-spaces, and let \(x_0 \in X^G, y_0 \in Y^G\). Denote by

\[
[(X, x_0), (Y, y_0)]_G
\]

the set of \(G\)-homotopy classes rel. \(x_0\) of \(G\)-maps \(f: X \to Y\) with \(f(x_0) = y_0\).

The following Proposition is required in section 5.

**Proposition 6.** Let \(E, F\) be representations of \(G\), and let \(x_0 \in S(E^G), y_0 \in S(E^G \oplus F^G)\). Then

\[
j_*: [(S(E), x_0), (S(E \oplus F), y_0)]_G \to [(S(E), x_0), (V^\Lambda_m(E \oplus F \oplus \Lambda^{m-1}), j(y_0))]_G
\]

is surjective.

**Proof.** Let

\[
f: S(E) \to V^\Lambda_m(E \oplus F \oplus \Lambda^{m-1})
\]

be a \(G\)-map with \(f(x_0) = j(y_0)\). Let \(D\) be a \(G\)-invariant, top-dimensional, small disc in \(S(E)\) with \(x_0\) as its center. We may deform \(f\) to a \(G\)-map \(f'\) such that \(f'(D) = j(y_0)\) and \(f' = f\) rel. \(x_0\). By Lemma 5 there is a \(G\)-homotopy

\[
R: S(E) \times [0, 1] \to V^\Lambda_m(E \oplus F \oplus \Lambda^{m-1})
\]

such that

\[
R_0 = f',
\]

\[
R_1(S(E)) \subset j(S(E \oplus F)), \quad \text{and}
\]

\[
R(D \times [0, 1]) = j(y_0).
\]

Then \(f'\) is \(G\)-homotopic to \(R_1\) rel. \(x_0\). This proves the Proposition. Q.E.D.

4. The injectivity of \(j_*\)

Let \(E, F\) be representations of a compact Lie group \(G\) over \(\Lambda\), and let
be the transformation induced from the $G$-map
\[ j: S(E \oplus F) \to V^\Lambda_\infty(E \oplus F \oplus \Lambda^{n-1}). \]

The purpose of this section is to prove the injectivity of $j_*$ under the assumption (i) or (ii) in Theorem 2.

For any closed subgroup $H$ of $G$, let
\[ j^H = j|S(E^H \oplus F^H): S(E^H \oplus F^H) \to V^\Lambda_\infty(E^H \oplus F^H \oplus \Lambda^{n-1}). \]

The following diagram is commutative:

\[
\begin{array}{ccc}
[S(E), S(E \oplus F)]_G & \xrightarrow{j_*} & [S(E), V^\Lambda_\infty(E \oplus F \oplus \Lambda^{n-1})]_G \\
r^H_H & & \downarrow r^H_H \\
[S(E^H), S(E^H \oplus F^H)] & \xrightarrow{j_*^H} & [S(E^H), V^\Lambda_\infty(E^H \oplus F^H \oplus \Lambda^{n-1})]
\end{array}
\]

where $r^H_H$ and $r^H_H'$ are the transformations restricting to the fixed point set by $H$. Now suppose
\[ j_*(\alpha) = j_*(\beta) \]
for $\alpha, \beta \in [S(E), S(E \oplus F)]_G$. Then, by the commutativity of the above diagram,
\[ j_*^H r^H_H(\alpha) = j_*^H r^H_H(\beta) \]
for any closed subgroup $H$ of $G$. Proposition 1 implies that $j_*^H$ is an isomorphism under the assumption (i) or (ii) in Theorem 2. Thus
\[ r^H_H(\alpha) = r^H_H(\beta) \]
for any $H$. Hence $r(\alpha) = r(\beta)$. By the assumption $r$ is injective, hence $\alpha = \beta$. Thus $j_*$ is injective.

5. The group structure

Let $E, F$ be representations of a compact Lie group $G$ over $\Lambda$. Suppose $\dim_R E^G \geq 2$. Then, according to [3, Section 6], $[S(E), S(E \oplus F)]_G$ has a group structure. In the similar way we may give a group structure to
\[ [S(E), V^\Lambda_\infty(E \oplus F \oplus \Lambda^{n-1})]_G \]
so that
\[ j_*: [S(E), S(E \oplus F)]_G \to [S(E), V^\Lambda_\infty(E \oplus F \oplus \Lambda^{n-1})]_G \]
is a group homomorphism. To show this is the purpose of this section.
Lemma 7. Suppose \( \dim_R E^G = 2 \) and \( x_0 \in S(E^G) \). Let
\[
\omega: [0,1] \to S(E^G) \subset S(E)
\]
be a path with \( \omega(0) = \omega(1) = x_0 \). Then there is a \( G \)-homotopy
\[
H: S(E) \times [0,1] \to S(E)
\]
such that
\[
H_0 = H_1 = \text{Id}, \quad \text{and} \quad H(x_0,t) = \omega(t) \quad \text{for any} \quad t \in [0,1].
\]

Proof. Choose a homotopy
\[
J: (E^G) \times [0,1] \to (E^G) \subset S(E)
\]
such that
\[
J_0 = J_1 = \text{the inclusion}, \quad \text{and} \quad J(x_0,t) = \omega(t) \quad \text{for any} \quad t \in [0,1].
\]

Denote by \((E^G)^\perp\) the orthogonal complement of \( E^G \) in \( E \), and denote a point of \( E \) by the form \( x + y \) where \( x \in E^G \) and \( y \in (E^G)^\perp \). Define
\[
H: S(E) \times [0,1] \to S(E)
\]
as
\[
H(x+y,t) = \|x\| J(x/\|x\|,t) + y \quad \text{if} \quad x \neq 0, \quad \text{and} \quad H(x+y,t) = y \quad \text{if} \quad x = 0.
\]

Then \( H \) is a \( G \)-homotopy with the desired property. Q.E.D.

Lemma 8. Suppose \( \dim_R E^G \geq 2 \), \( x_0 \in S(E^G) \) and \( y_0 \in S(E^G \oplus F^G) \). Then the natural transformations
\[
\psi_1: [(S(E), x_0), (S(E \oplus F), y_0)]_G \to [S(E), S(E \oplus F)]_G
\]
and
\[
\psi_2: [(S(E), x_0), (V^*_n(E \oplus F \oplus \Lambda^{n-1}), j(y_0))]_G \to [S(E), V^*_n(E \oplus F \oplus \Lambda^{n-1})]_G
\]
are bijective.

Proof. Consider the commutative diagram:
\[
\begin{array}{ccc}
[(S(E), x_0), (S(E \oplus F), y_0)]_G & \xrightarrow{\psi_1} & [S(E), S(E \oplus F)]_G \\
 j_* & & j_* \\
[(S(E), x_0), (V^*_n(E \oplus F \oplus \Lambda^{n-1}), j(y_0))]_G & \xrightarrow{\psi_2} & [S(E), V^*_n(E \oplus F \oplus \Lambda^{n-1})]_G
\end{array}
\]
In [3; Section 6], \( \psi_1 \) is already seen to be bijective. The two \( j_* \) are surjective by the arguments in section 3. So it follows that \( \psi_2 \) is surjective.

It only remains to show that \( \psi_2 \) is injective. Suppose

\[
\psi_2(\alpha) = \psi_2(\beta)
\]

for \( \alpha, \beta \in [(S(E), x_0), (V_n(E \oplus F \oplus \Lambda^{n-1}), j(y_0))]_G \). Since \( j_* \) is surjective, there are \( G \)-maps

\[
f, g: S(E) \to S(E \oplus F)
\]

such that \( f(x_0) = y_0, g(x_0) = y_0 \), and \( jf, jg \) are representatives of \( \alpha, \beta \), respectively. There also is a \( G \)-homotopy

\[
K: S(E) \times [0, 1] \to V_n(E \oplus F \oplus \Lambda^{n-1})
\]

with \( K_0 = ff \) and \( K_1 = jg \). Define a path

\[
\omega: [0, 1] \to V_n(E^G \oplus F^G \oplus \Lambda^{n-1})
\]

by \( \omega(t) = K(x_0, t) \) for \( t \in [0, 1] \). Then

\[
\omega(0) = \omega(1) = j(y_0).
\]

By Proposition 1 there is a path

\[
\omega': [0, 1] \to S(E^G \oplus F^G)
\]

such that

\[
\omega'(0) = \omega'(1) = y_0, \quad \text{and} \quad \omega = j\omega' \text{ rel. } \{0, 1\}.
\]

Let \( D \) be a \( G \)-invariant, top-dimensional, small disc in \( S(E) \) with \( x_0 \) as its center, and let \( D' = \frac{1}{2}D \). By radius contraction we may deform \( K \) to a \( G \)-homotopy

\[
K': S(E) \times [0, 1] \to V_n(E \oplus F \oplus \Lambda^{n-1})
\]

such that \( K'(x, t) = j\omega'(t) \) for \( x \in D' \) and \( t \in [0, 1] \). Moreover, if we put \( f' = K_0' \) and \( g' = K_1' \), then

\[
f'(S(E)) \subseteq i(S(E \oplus F)) ,
\]

\[
g'(S(E)) \subseteq i(S(E \oplus F)) ,
\]

and \( f', g' \) are \( G \)-homotopic to \( ff, jg \) rel. \( x_0 \), respectively.

So, to show \( \alpha = \beta \) we must show that \( f' \) is \( G \)-homotopic to \( g' \) rel. \( x_0 \).

(i) Suppose \( \dim_R E^G \oplus F^G > 2 \). By Proposition 1, \( j\omega' \) is homotopic to the constant path at \( j(y_0) \) rel. \( \{0, 1\} \). From this we may deform \( K' \) to a \( G \)-homotopy

\[
K'': S(E) \times [0, 1] \to V_n(E \oplus F \oplus \Lambda^{n-1})
\]
such that
\[ K_0' = f', \quad K_1' = g', \quad \text{and} \]
\[ K''(x_0, t) = j(y_0) \quad \text{for any} \quad t \in [0, 1]. \]

Therefore \( f' \) is \( G \)-homotopic to \( g' \) rel. \( x_0 \).

(ii) Suppose \( \dim_R E^G \oplus F^G = 2 \). Define a path
\[ \omega'': [0, 1] \to S(E^G \oplus F^G) \]
by \( \omega'' = (\omega')^{-1} \), i.e., \( \omega''(t) = \omega'(1 - t) \). Applying Lemma 7 to the path \( \omega'' \), there is a \( G \)-homotopy
\[ H: S(E \oplus F) \times [0, 1] \to S(E \oplus F) \]
such that
\[ H_0 = H_1 = \text{Id}, \quad \text{and} \]
\[ H(y_0, t) = \omega''(t) \quad \text{for any} \quad t \in [0, 1]. \]

Define a \( G \)-homotopy
\[ L: S(E) \times [0, 1] \to V^\Lambda(E \oplus F \oplus \Lambda^{-1}) \]
as, for \( x \in S(E) \) and \( t \in [0, 1] \),
\[ L(x, t) = K'(x, 2t) \quad \text{if} \quad 0 \leq t \leq 1/2, \quad \text{and} \]
\[ L(x, t) = jH(j^{-1}g'(x), 2t - 1) \quad \text{if} \quad 1/2 \leq t \leq 1. \]

Then
\[ L_0 = f', \quad L_1 = g', \quad \text{and} \]
\[ L(x, t) = j\omega' \cdot j\omega''(t) \quad \text{for} \quad x \in D' \quad \text{and} \quad t \in [0, 1]. \]

\( j\omega' \cdot j\omega'' \) is homotopic to the constant path at \( j(y_0) \) rel. \( \{0, 1\} \). So we may deform \( L \) to a \( G \)-homotopy
\[ L': S(E) \times [0, 1] \to V^\Lambda(E \oplus F \oplus \Lambda^{-1}) \]
such that
\[ L'_0 = f', \quad L'_1 = g', \quad \text{and} \]
\[ L'(x_0, t) = j(y_0) \quad \text{for any} \quad t \in [0, 1]. \]

Therefore \( f' \) is \( G \)-homotopic to \( g' \) rel. \( x_0 \).

Q.E.D.

Now suppose \( \dim_R E^G \geq 2, \) \( x_0 \in S(E^G) \) and \( y_0 \in S(E^G \oplus F^G) \). Let \( \lambda \) be the real one-dimensional subspace of \( E \) spanned by \( x_0 \), and let \( \lambda^\perp \) be the orthogonal complement of \( \lambda \) in \( E \). We may identify \( S(E) \) with a nonreduced suspension
\[ \Sigma S(\lambda^\perp) = [0, 1] \times S(\lambda^\perp)/\sim. \]
Under this identification \( x_0 = [0, x] \) and \( -x_0 = [1, x] \) for \( x \in S(\lambda^1) \). Let \( Y \) be one of \( S(E \oplus F) \) and \( V_\Delta^a(E \oplus F \oplus \Lambda^{a-1}) \). Put \( z_0 = y_0 \) if \( Y \) is the former, and \( z_0 = j(y_0) \) if \( Y \) is the latter. Then we may give a group structure to \([S(E), Y]_G\) as follows. Let \([f], [g] \in [S(E), Y]_G\). By Lemma 8 we may choose \( f \) and \( g \) in such a way that \( f(-x_0) = z_0 \) and \( g(x_0) = z_0 \). Define \( h: S(E) \to Y \) as, for \([t, x] \in \Sigma S(\lambda^1) = S(E)\),

\[
\begin{align*}
    h([t, x]) &= f([2t, x]) & \text{if} & \quad 0 \leq t \leq 1/2, \\
    h([t, x]) &= g([2t-1, x]) & \text{if} & \quad 1/2 \leq t \leq 1.
\end{align*}
\]

Define \([f] + [g] = [h]\). This gives a group structure to \([S(E), Y]_G\), and the transformation

\[
    j_*: [S(E), S(E \oplus F)]_G \to [S(E), V_\Delta^a(E \oplus F \oplus \Lambda^{a-1})]_G
\]

becomes a group homomorphism. We note that this group structure does not depend on the choice of \( x_0 \in S(E^c) \) and \( y_0 \in S(E^c \oplus F^c) \).

References


Department of Mathematics
Yamaguchi University
Yamaguchi 753, Japan