A CHARACTERIZATION OF THE SIMPLE GROUP $J_4$

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(Received June 12, 1979)
(Revised March 19, 1980)

In [4] Janko describes the properties of a simple group of order $2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ denoted by $J_4$. It has exactly one conjugacy class of elements of order 3 and if $\pi$ is one of them, then the centralizer of $\pi$ in $J_4$ is isomorphic to the 6-fold cover of the Mathieu group $M_{22}$. We show in this paper that these properties characterize the group $J_4$, we prove namely the

**Theorem A.** Let $G$ be a finite group containing an element $\pi$ of order 3 such that $C_G(\pi)$ is isomorphic to the 6-fold cover $M_{22}$. If $G$ is not 3-normal then $G$ is isomorphic to $J_4$.

In the first section we shall list some properties of the 6-fold cover of $M_{22}$, which will be needed in the proof. The second section is then devoted to the proof of Theorem A. In the last section we remark that the following holds:

**Theorem B** There exists no simple group $G$ which is not 3-normal and contains an element $\pi$ such that $C_G(\pi)$ is isomorphic to the triple cover of $M_{22}$.

The Frattini subgroup of a group $X$ is denoted by $D(X)$. The other notation is hopefully standard.

In the whole paper with the exception of the last section $G$ denotes a simple group satisfying the assumptions of Theorem A and $\pi$ is an element of $G$ of order 3 such that $C_G(\pi)$ is isomorphic to the 6-fold cover of $M_{22}$.

1. Some known results and structure of $N_G(\langle \pi \rangle)$

We first list some well known results which will be used in the proof of our theorems.

**Lemma 1.1** (Gaschütz). Let $A$ be an abelian normal subgroup of the group $X$ contained in the subgroup $B$ of $X$ with $(|X: B|, |A|) = 1$. Then if $A$ has a complement in $B$, $A$ has a complement in $X$.

Proof. See [1].
Lemma 1.2 (Thompson). If the group $X$ admits a fixed-point-free automorphism of prime order then $X$ is nilpotent.

Proof. See [2; 10.2.1].

Lemma 1.3 (Thompson). Let $T_0$ be a maximal subgroup of an $S_2$-subgroup of the group $X$. If $X$ does not have a subgroup with index two then all involutions of $X$ are conjugate to elements of $T_0$ in $X$.

Proof. See [10. Lemma 5.38].

Lemma 1.4 (Burnside). Let $P$ be an $S_p$-subgroup of the group $X$ and assume that $N_X(P)=C_X(P)$. Then $X$ has a normal $p$-complement.

Proof. See [2; 7.4.3].

Lemma 1.5. Let $P$ be a $p$-group and let $Q$ be a noncyclic abelian $q$-group of automorphisms of $P$, $q$ a prime distinct from $p$. Then $P=\langle C_p(x) \mid 1=x \in Q \rangle$.

Proof. See [2; 5.3.16].

Lemma 1.6. Any involution $t$ of the group $X$ which does not lie in the maximal normal 2-subgroup of $X$ inverts a nontrivial element of $X$ of odd order.

Proof. Let $t$ be an involution of $X$ with $t \notin 0_x(X)$. Then there exists a conjugate $t_1$ of $t$ in $X$ such that the dihedral group $\langle t, t_1 \rangle$ is not a 2-group by [2; 3.8.2]. Since the index of the cyclic subgroup $\langle t, t_1 \rangle$ has index two in $\langle t, t_1 \rangle$ we see that $0(\langle t, t_1 \rangle)$ is nontrivial and is inverted by $t$ since $t$ inverts $t,t_1$.

The following three lemmas are taken from [4; (2.1), (2.3), (2.4)].

Lemma 1.7. Let $X \cong M_{22}$ and let $T$ be an $S_2$-subgroup of $X$. Then $T$ possesses precisely two distinct elementary abelian subgroups $E_1$ and $E_2$ of order 16 and they are both normal in $T$. We have $N_X(E_1)$ is a splitting extension of $E_1$ by $A_6$, $N_X(E_2)$ is a splitting extension of $E_2$ by $S_5$ and $N_X(E_i)$ acts transitively on $E_i^*$. The group $X$ has the order $2^7 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ and exactly one conjugacy class of involutions in $X$ with the representative $e \in E_1$ and we have $C_X(e)=C(e) \cap N_X(E_1)$. An $S_2$-subgroup $P$ of $X$ is elementary abelian of order 9 and we have $C_X(P)=P$ and $N_X(P)=PQ$ where $Q$ is quaternion and acts regularly on $P$. The group $X$ has exactly one conjugacy class of elements of order 3 and if $\sigma$ is one of them, then $C_X(\sigma) \cong \langle \sigma \rangle \times A_4$ and $N_X(\langle \sigma \rangle)=\langle \sigma \rangle B$ where $B \approx S_4$.

Lemma 1.8. Let $X \cong \text{Aut}(M_{22})$ so that $X' \cong M_{22}$ and $|X: X'|=2$. The group $X$ possesses exactly two conjugacy classes of involutions which are contained in $X-X'$ with the representatives $t_1$ and $t_2$. If $E_1$ and $E_2$ are the only elementary abelian subgroups of rank 4 of an $S_2$-subgroup of $X'$ as described in (1.7) then $t_1$ and $t_2$ can be chosen to lie in $C_X(E_2)=A=\langle E_2, t_2 \rangle$ which is elementary abelian of
order 32. Then \( N_X(A) = AB \) where \( B \subseteq X' \), \( B \cong S_5 \) and \( B \) operates transitively on \( E_3 \) and operates on \( A - E_2 \) in two orbits of sizes 10 and 6 represented respectively by \( t_1 \) and \( t_2 \). We have \( N_X(E_1) \) is a splitting and faithful extension of \( E_1 \) by \( S_5 \).

**Lemma 1.9.** Every maximal subgroup of the simple group \( M_{24} \) is isomorphic to one of the following groups:

- \( PSL(2, 23), M_{24}, \text{Aut}(M_{24}), \text{Aut}(M_{12}), PSL(2, 7) \).
- The holomorph of an elementary abelian group of order 16,
- An extension of \( M_{24} \) by \( S_3 \).
- A splitting and faithful extension of an elementary abelian group of order 64 by a subgroup \( Y \) where \( |0_3(Y)| = 3 \), \( Y/0_3(Y)^S \cong S_6 \), \( Y/Y'| \cong 2 \), \( Y'' = Y' \) and \( C_Y(0_3(Y)) = Y' \).
- A splitting and faithful extension of an elementary abelian group of order 64 by \( S_3 \times PSL(2, 7) \).

In the next lemma we list some properties of \( N_G(\langle \pi \rangle) \) which can be easily deduced from (1.7) and (1.8) and are essentially proved in [4]. Throughout the paper we shall fix the notation which will be introduced in the following lemma.

**Lemma 1.10.** The following hold in \( G \):

(i) Let \( H = N_G(\langle \pi \rangle) \). Then \( |H: H'| = 2, H'' = H' = C_H(\pi), Z(H') \) is cyclic of order 6, \( H'/Z(H') \cong M_{22} \) and \( H/Z(H') \cong \text{Aut}(M_{22}) \). Let us denote the involution in \( Z(H') \) by \( z \).

(ii) Let \( T \) be an \( S_2 \)-subgroup of \( H \) and let \( T_0 = T \cap H' \). Then \( T_0 \) contains exactly two elementary abelian subgroups \( E_1 \) and \( E_2 \) of rank 5. These are normal in \( T \) and we have \( C_H(E_i) = E_i \langle \pi \rangle, i = 1, 2 \) and

\[
N_H(E_i) = E_i B_i \quad \text{where } B_i \text{ is isomorphic to the triple cover of } A_6,
C_H(E_1) = E_1 \langle \pi \rangle, N_H(E_1)/C_H(E_1) \cong S_6,
N_H(E_2) = E_2 \langle \pi \rangle \times B_2 \quad \text{where } B_2 \cong S_5 \text{ and acts transitively on } (E_2/\langle z \rangle)^*,
N_H(E_2) = \langle \pi \rangle E_2^* B_2 \quad \text{where } E_2^* \text{ is an abelian group of order } 64, B_2 \text{ normalizes } E_2^*,
\langle z \rangle \geq D(E_2^*), E_2^* \text{ is elementary abelian if and only if there exist involutions in } H - H' \text{ and if so then } E_2^* \text{ is the only elementary abelian subgroup of } T \text{ of order } 64.
\text{Furthermore } E_2^* \cap H' = E_2 \text{ and } E_2^* \leq T.
\]

(iii) Let \( P \) be an \( S_2 \)-subgroup of \( B_1 \). Then \( P \) is an \( S_2 \)-subgroup of \( G \). \( P = \langle \pi, \sigma, \tau \rangle \) is extraspecial of order 27 and exponent 3, where the generators of \( P \) are chosen in such a way that \( C_2(\tau) = \langle z \rangle \) and \( C_2(\sigma) = E_0 \) is of order 8. We have \( N_H(P) = \langle z \rangle \times P Q \) where \( Q \) is quaternion and acts regularly on \( P/\langle \pi \rangle \). There exists exactly one conjugacy class of elements of order 3 in \( H - \langle \pi \rangle \) and exactly one conjugacy class of subgroups of order 9 in \( H' \) represented by \( M = \langle \pi, \sigma \rangle \). We have \( C_6(M) = M \times E_0 \) and \( N_H(M)/C_H(M) \cong S_2 \).

(iv) \( \text{We have } C_H(E_0) = E_0 M. \)

(v) \( E_1 \cap E_2 \) is of order 8 and we have \( C_H(E_1 \cap E_2) = \langle \pi \rangle \times E_1 E_2 \) and
$N_{H'}(E_1 \cap E_2) = C_{H'}(E_1 \cap E_2) U$ where $U \cong S_3$ and $O_3(U)$ acts regularly on $E_1 E_2/\langle z \rangle$. Furthermore $E_1 \cap E_2 \leq T' = \langle T' \cap E_1, T' \cap E_2 \rangle$ and $T' \cap E_i, i=1, 2$, are the only elementary abelian subgroups of $T'$ of rank four. So $E_1 \cap E_2$ is normal in $N_\circ(T)$. We have $C(E_i) \cap E_2 = E_1 \cap E_2$.

Proof. $C_\circ(\pi)$ is isomorphic to the 6-fold cover $M_{22}$, i.e. $C_\circ(\pi)' = C_\circ(\pi)$, $Z(C_\circ(\pi))$ is cyclic of order 6 and $C_\circ(\pi)/Z(C_\circ(\pi)) \cong M_{22}$.

Let $P$ be an $S_3$-subgroup of $C_\circ(\pi)$. Then $\langle \pi \rangle \subseteq Z(P)$ and $P$ does not split over $\langle \pi \rangle$ (1.1). So $D(P) = \langle \pi \rangle$ by (1.7) and hence $P$ is an $S_3$-subgroup of $G$. Let $R$ be an $S_3$-subgroup of $N(P) \cap C_\circ(\pi)$. Since $R$ operates transitively on $\{P\}_{H \subseteq C_\circ(\pi)}$ by (1.7) we see that $P$ is extraspecial of order 27 and exponent 3. Furthermore we have $R/\langle z \rangle \cong Q_8$ where $z$ is the involution in $Z(C_\circ(\pi))$. So $R$ must split over $\langle z \rangle$ and we get $N(P) \cap C_\circ(\pi) = \langle z \rangle \times P_Q$ where $Q \cong Q_8$ and acts regularly on $P/D(P)$. In particular there exists exactly one conjugacy class of elements of order 3 in $C_\circ(\pi) - \langle \pi \rangle$ and hence exactly one conjugacy class of subgroups of order 9.

Since $G$ is not 3-normal, $\pi$ must be conjugate to an element in $P - \langle \pi \rangle$ and hence to $\pi^{-1}$. So $|N_\circ(\langle \pi \rangle) : C_\circ(\pi)| = 2$. Let $H = N_\circ(\langle \pi \rangle)$. Then $H' = C_\circ(\pi)$ and $H/Z(H')$ is isomorphic to $\text{Aut}(M_{22})$ or $Z_3 \times M_{22}$ since $|\text{Aut}(M_{22}) : M_{22}| = 2$ by (1.8). But the second case is not possible since otherwise there would exist a 2-element in $H - H'$ which operates trivially on $P/D(P)$ and inverts $P/D(P)$ and this is absurd. So $H/Z(H') \cong \text{Aut}(M_{22})$.

Let $T$ be an $S_3$-subgroup of $H$. Then all assertions of (ii) are proved in [4; Proposition 1 and 3] and we shall use them in the following.

Since an $S_3$-subgroup of $B_1 \subseteq H$ is also an $S_3$-subgroup of $H$ we can assume that $P \subseteq B_1$. By the action of the non-cyclic abelian 3-group $P/D(P)$ on the 2-group $E_1$ we see that there is an element $\sigma$ in $P$ with $E_0 = E_1 \cap C(\sigma)$ is elementary abelian of order 8 and an element $\tau$ in $P$ with $\langle z \rangle = E_1 \cap C(\tau)$. Let $M = \langle \pi, \sigma, \tau \rangle$. Then $C_\circ(M) = E_0 \times M$ and $N_{H'}(M)/C_{H'}(M) \cong S_3$ by (1.7). This completes the proof of the first three assertions of the lemma.

For the proof of (iv) let $\bar{P} = H'/Z(H')$, which is isomorphic to $M_{22}$. By (1.7) we have $C_{\bar{P}}(\sigma) = \langle \sigma \rangle \times E_0 \langle \sigma \rangle$ where $E_0 \langle \sigma \rangle$ is isomorphic to $A_4$ and $N_{\bar{P}}(\langle \sigma \rangle)/\langle \sigma \rangle$ is isomorphic to $S_4$. This gives that $C_{\bar{P}}(E_0) \cap N_{\bar{P}}(\langle \sigma \rangle) = E_0 \langle \sigma \rangle$. By Burnside's transfer theorem we get $C_{\bar{P}}(E_0) = \sigma'(C_{\bar{P}}(E_0)) \langle \sigma \rangle$. By the structure of $M_{22}$, $\bar{K} = 0_3'(C_{\bar{P}}(E_0))$ is a 2-group containing $E_1$.

Suppose that $K \neq E$. Then the non-trivial group $K/E_1$ is normalized by $\bar{P} = \langle \sigma, \pi \rangle$. Since $P$ is not cyclic there is by (1.5) a non-trivial element $x$ in $\bar{P}$ such that $C_{K/E_1}(x) = C_{\bar{K}}(E_1)/E_1 \neq 1$. As $\sigma$ operates regularly on $K/E_1$ and normalizes $C_{K/E_1}(x)$ we get that $K/E_1 = C_{K/E_1}(x)$ is elementary abelian of order four since $|K/E_1| \leq 8$. By the structure of the centralizer of an element of order three in $M_{22}$ we get that $C_{\bar{K}}(x)$ is four group and that $K = E_1 C_{\bar{K}}(x)$.

$\bar{S} = C(C_{\bar{K}}(x)) \cap E_1$ is non-trivial and is normalized by $x$ which operates
regularly on it. This yields that $|\tilde{S}| \geq 4$. So $\tilde{D} = C_\pi(x)\tilde{S}$ is elementary abelian of rank at least four. Since an $S_2$-subgroup of $M_2$ contains exactly two elementary abelian subgroups of rank four by (1.7) we see that $\tilde{D}$ is conjugate in $H'$ to $E_2$. But $\tilde{D}$ is normalized by $P$ whereas an $S_2$-subgroup of $H_2(E_2)$ is of order 3. This contradiction shows that $K = E_1$ and hence $C_{H'}(E_0) = E_iM$.

For the proof of (v) observe that $E_1E_2/E_1$ is a non-trivial elementary abelian 2-group of $H_2(E_0)/E_1 = \tilde{B}_1$ which is isomorphic to the triple cover of $A_6$. Since an $S_2$-subgroup of $\tilde{B}_1$ is dihedral of order 8 there exists a four group $V$ of $\tilde{B}_1$ containing $E_2$. By the structure of $\tilde{B}_1$ and by (1.1) we get that $N(V) \cap \tilde{B}_1 = \langle \pi \rangle x V$ where $\tilde{U} \cong S_3$ and operates faithfully on $V$. Let $U_0$ be an $S_2$-subgroup of the inverse image of $\tilde{U}$. By (1.6) we can assume that $U_0$ is inverted by an involution $x$ in $T_0 - E_1E_2$ such that $\langle U_0, x \rangle$ maps into $\tilde{U}$ and $\langle U_0, x \rangle \cong S_3$. $U_0$ normalizes the inverse image $V$ of $\tilde{V}$. Since $E_1E_2 \subseteq V$ and $E_1$ and $E_2$ are the only elementary abelian $2$-groups of $T_0$ hence of $V$ of rank 5 we see that $U_0$ normalizes both $E_1$ and $E_2$ and hence $E_1E_2$. This implies that $V = E_2$ and $E_1 \cap E_2$ is of order 8. Furthermore $U_0$ maps onto an $S_2$-subgroup of $N_{H_2}(E_2)/E_2$ since $B_2$ operates transitively on $(E_2/xz)^g$ by (ii) we obtain that $U_0$ operates regularly on $E_1/\langle x \rangle$. Since $T_0$ does not split over $\langle x \rangle$ we see that $\langle x \rangle$ is properly contained in $(E_1E_2)' = [E_1, E_2] \subseteq E_1 \subseteq E_2$ by (ii). Since $U_0$ operates regularly on $E_1 \cap E_2/\langle x \rangle$ we get that $(E_1E_2)' = E_1 \cap E_2$ and hence $D(E_1E_2) = E_1 \cap E_2$. Since $U_0$ acts regularly on $E_1 \cap E_2/\langle x \rangle$ we see that $E_1 \cap E_2$ is not centralized by $x$ and hence $C_{H_2}(E_1 \cap E_2) = \langle \pi \rangle \times E_1E_2$. By (iv) we get that $E_1 \cap E_2$ is not normalized by an $S_2$-subgroup of $H'$, because otherwise it would be centralized by a subgroup of order 9 and would be conjugate to $E_0$ by (iii). This implies that $U_0$ operates regularly also on $E_1/\langle x \rangle$, because otherwise an $S_2$-subgroup of $B_1$ containing $\langle \pi, U_0 \rangle$ would normalize $\langle [E_1, U_0], z \rangle = E_1 \cap E_2$. So $U_0$ acts regularly on $E_1E_2/\langle x \rangle$ and we have $C(E_1) \cap E_2 = E_1 \cap E_2$.

So we have seen that the elementary abelian group $E_1E_2/E_1 \cap E_2$ of rank 4 is normalized by $\langle U_0, x \rangle \cong S_3$ such that $U_0$ operates regularly on it. This shows that $T_0/E_1 \cap E_2 = C_2(\langle E_1 \cap E_2 \rangle \cap (E_1E_2/E_1 \cap E_2)$ and hence that $T_0 = \langle T_0 \cap E_1, T_0 \cap E_2 \rangle$ where $T_0 \cap E_i, i = 1, 2$, is of order 16. Since $T_0 \subseteq E_1 \cap E_2$ and hence $C_{H_2}(E_1 \cap E_2) = \langle \pi \rangle \times E_1E_2$. Thus $[t, T_0] \subseteq E_1 \cap E_2$ and hence $T_0 \cap E_2$ is of order 32, $T_0$ is normal in $N_6(T)$ we get that $E_1 \cap E_2 = (T_0 \cap E_1) \cap (T_0 \cap E_2)$ is normal in $N_6(T)$.

This completes the proof of the lemma.

2. Proof of Theorem A

In this section we prove Theorem A in a sequence of lemmas. We shall
Lemma 2.1. We have $N_G(M)/C_G(M) \cong GL(2, 3)$ and $N_G(M)$ is contained in $N_G(E_0)$.

Proof. Since $G$ is not 3-normal and there exists precisely one conjugacy class of elements of order 3 in $H = \langle \pi \rangle$ represented by $\sigma$ we have that $\pi \sim \sigma$ in $G$. So there exists an element $g$ in $G$ such that $\pi = \sigma g$ and $C_G(\sigma) = M^f \leq P$. Since there exists in $H$ exactly one conjugacy class of subgroups of order 9 we can assume that $M^f = M$. So $\pi \sim \sigma$ in $N_G(M)$.

Since $M^f$ is the union of $N_B(M)$-orbits of sizes 1, 1 and 6 represented by $\pi, \pi^{-1}, \sigma$ respectively we get that $|N_G(M)/C_G(M)| = |GL(2, 3)|$ and hence $N_G(M)/C_G(M) \cong GL(2, 3)$.

Since $E_0 = 0_2(C_G(M))$ by (1.10.iii) we see that $E_0 \leq N_G(M)$.

Lemma 2.2. We have $C_G(E_1) = 0_2(C_G(E_1))/\langle \pi \rangle$ where $0_2(C_G(E_1))$ is either equal to $E_1$ or is an elementary abelian group of order $2^11$.

Proof. By (1.10.i) we have $C_H(E_1) = E_1 \langle \pi \rangle$. Burnside’s transfer theorem yields then that $C_G(E_1) = 0_2(C_G(E_1))/\langle \pi \rangle$ since $\langle \pi \rangle$ is an $S_3$-subgroup of $C_G(E_1)$ by (1.10.iii).

Let $K = 0_2(C_G(E_1))$. Since $C_H(E_1) = E_1 \langle \pi \rangle$ we see that $\pi$ operates regularly on $K/E_1$. Thus $K/E_1$ is nilpotent by (1.2). As $E_1 \leq Z(K)$ we get that $K$ is nilpotent. Furthermore $K$ is normalized by $P$ and hence we have $K = \langle C_K(x) \mid 1 \neq x \in M \rangle$, by (1.5).

We have $E_0 = C(x) \cap E_1 \leq Z(C_K(x))$ for any $x \in M - \langle \pi \rangle$. Since $N_G(M)$ is contained in $N_G(E_0)$ by (2.1) we see that $C(E_0) \cap C(x) = 0_2(C(E_0) \cap C(x))M$ for any $1 \neq x \in M$, where the maximal normal 2-subgroup of $C(E_0) \cap C(x)$ is elementary abelian of order 32 by (1.10.iv). So $C_K(x)$ is an elementary abelian 2-group of order at most 32. On the other hand $\pi$ operates regularly on $C_K(x)E_0$ for $x \in M - \langle \pi \rangle$. This implies that we have either $C_K(x) = 0_2(C(E_0) \cap C(x))$ or $C_K(x) = E_0$ for $x \in M - \langle \pi \rangle$. Since all elements of the set $\{C_K(x) \mid x \in M - \langle \pi \rangle \}$ are conjugate to each other via $\pi$ we have either

$$C_K(x) = E_0 \quad \text{for all } x \in M - \langle \pi \rangle,$$

or

$$C_K(x) = 0_2(C(E_0) \cap C(x)) \quad \text{for all } 1 \neq x \in M,$$

where $0_2(C(E_0) \cap C(x))$ is elementary abelian of order 32 for all $1 \neq x \in M$. We can assume that we are in the second case.

Let $S$ be an $S_3$-subgroup of $N_G(M)$. Then $S$ acts transitively on $M^f$ and normalizes $E_0$. So $S$ acts transitively on the set $\{0_2(C(E_0) \cap C(x)) \mid 1 \neq x \in M \}$
and hence normalizes $K$. Since $E_1 = 0_{4}(C(E_0) \cap C_{G}(\pi)) \subseteq Z(K)$ we get that $K \subseteq Z(K)$ and hence that $K$ is elementary abelian of order $2^{11}$ since

$$K = K/E_1 = C_{K}(\sigma) \times C_{K}(\sigma \pi) \times C_{K}(\sigma \pi^{-1})$$

is of order $2^{6}$.

**Lemma 2.3.** If $0_{4}(C_{G}(E_1)) = E_1$, then $T$ is an $S_2$-subgroup of $G$.

Proof. By (2.2) and the assumption of this lemma we have $C_{G}(E_1) = E_1 \times \langle \pi \rangle$. Then $N_{G}(E_1)$ normalizes $\langle \pi \rangle$ and hence we get $N_{G}(E_1) = N_{H}(E_1)$. Thus $T$ is an $S_2$-subgroup of $N_{G}(E_1)$.

Suppose that $T$ is not an $S_2$-subgroup of $G$. Then there exists a 2-group $T' \langle x \rangle$ in $G$ with $|T'\langle x \rangle|: T_{0} = 2$. If $E'_1 \subseteq T_{0}$ we get $E'_1 = E_0$ by (1.10.ii). This contradicts the fact that $T$ is an $S_2$-subgroup of $N_{G}(E_1)$. So $E'_1 \notin T_{0}$ and thus $T - T_{0}$ contains involutions. Then $E^*_2$ is the only elementary abelian subgroup of $T$ of order $64$ by (1.10.ii) and hence $x$ normalizes $E^*_2$.

Since $x$ normalizes $T' \cap E^*_2 = T' \cap E_2$ and since $T'$ contains exactly two elementary abelian subgroups of rank four, namely $T' \cap E_1$, $i = 1, 2$, we see that $x$ also normalizes $T' \cap E_1$. Since $E_1 \cap E_2 \subseteq N_{G}(T)$ by (1.10.v) we get that $E'_1 \subseteq C_{T}(E_1 \cap E_2) \subseteq N_{G}(T)$ and hence $X = E_1 E^*_2 \cap C(T' \cap E_1) = E_1 E^*_2 (T' \cap E_1)$ is normalized by $x$. (1.10.v) gives then that $Z(X) = T' \cap E_1$ and that $E_1$ and $(T' \cap E_1) \times (E^*_2 \cap C(T' \cap E_1))$ are the only elementary abelian subgroups of $X$ of rank five. Since $E^*_2 \cap C(T' \cap E_1)$ is normalized by $x$ we get that $E'_1 = E_1$ which is a contradiction. Thus $T$ is a Sylow 2-subgroup of $G$.

**Lemma 2.4.** If $T$ is an $S_2$-subgroup of $G$ then the centralizer of the involution $x$ in $G$ is $H$.

Proof. Let $C = C_{G}(x)$ and denote the homomorphic image of any subset $X$ of $C$ in $C/\langle x \rangle$ by $\bar{X}$. Obviously $H$ is contained in $C$.

Then $T$ is an $S_2$-subgroup of $C$ by our assumption and $\bar{T}$ is isomorphic to an $S_2$-subgroup of $\text{Aut}(M_{32})$. Since $H \subseteq C$ all involutions in $\bar{T}$ are conjugate to $\bar{\sigma} \in Z(\bar{T})$ in $\bar{C}$ and all involutions in $\bar{T} - \bar{T}_0$ are conjugate to involutions in $E^*_2 - E_2$ in $\bar{C}$ where $E^*_2 = C_{T}(E_2)$ is the only elementary abelian subgroup of $\bar{T}$ of order 32 by (1.7) and (1.8). Furthermore we have $N_{H}(E^*_2) = E^*_2 \bar{B}_2$ where $\bar{B}_2 = S_5$ and $(E^*_2)^\sigma$ splits into $\bar{B}_2$-orbits of sizes 15, 6 and 10 represented respectively by $\bar{t}$, $\bar{t}_1$ and $\bar{t}_2$ where $\bar{t}_1$ and $\bar{t}_2$ are in $E^*_2 - E_2$ by (1.8).

If $\bar{C}$ has no subgroups of index two then $\bar{t}_i, i = 1, 2$, must be conjugate to an element of $\bar{T}_0$ hence to $\bar{\sigma}$ in $\bar{C}$ by (1.3), Thompson's transfer lemma. But this conjugation must take place in $N_{\bar{C}}(E^*_2)$ since $E^*_2$ is the only elementary abelian subgroup of $\bar{T}$ of rank 5. So we get by the above paragraph that all involutions of $E^*_2$ are conjugate to each other in $N_{\bar{C}}(E^*_2)$. In particular 31 divides the order of the group $N_{\bar{C}}(E^*_2)/C_{\bar{C}}(E^*_2)$. 
Let $N = N_\overline{E}(\overline{\tau})/C_\tau(\overline{E}^\tau)$. Then $N$ is isomorphic to a subgroup of $GL(5, 2)$, has dihedral $S_7$-subgroups of order 8 and contains a subgroup $B_2$ which is isomorphic to $S_5$. So $N/0(N)$ is either isomorphic to $A_7$ or to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$ where $q$ is an odd prime power by [3].

Assume first that $31 | 0(N)$. Let $\tilde{S}$ be an $S_{31}$-subgroup of $0(N)$. Since $31^2$ does not divide the order of $GL(5, 2)$ and since $N$ is isomorphic to a subgroup of $GL(5, 2)$ we see that $\tilde{S}$ is cyclic of order 31. By Frattini's argument we get that $N_{31}(\tilde{S})$ covers $N/0(N)$ and hence that $N_{31}(\tilde{S})/N_{31}(\tilde{S})$ contains a subgroup isomorphic to $S_5$ by the above paragraph. Since $\text{Aut}(\tilde{S})$ is cyclic we conclude that $\tilde{S}$ is centralized by an element $\mathfrak{a}$ of order 31. But $C(\mathfrak{a}) \cap E^\tau_2$ is nontrivial and is normalized by $\tilde{S}$. But this is not possible since $\tilde{S}$ operates regularly on $E^\tau_2$. Thus $31 | 0(N)$ and hence $31 | 0(N)$.

So $N/0(N)$ is isomorphic to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$. Since $N$ is isomorphic to a subgroup of $GL(5, 2)$ and $|GL(5, 2)| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ this is possible only if $q = 31$. But $|PSL(2, 31)| = 2^6 \cdot 3 \cdot 5 \cdot 31$ whereas $|N|_2 = 8$. This contradiction shows that $C$ contains a subgroup $C_0$ with index 2.

We have $1 \neq C_0 \cap \tilde{T}$. Thus $Z(\tilde{T}) = \langle \mathfrak{e} \rangle$ is contained in $C_0$. Since all involutions of $\tilde{T}_0$ are conjugate to $\mathfrak{e}$ in $H' \subseteq C$ and since $\tilde{T}_0$ is generated by its involutions we get $\tilde{T}_0 \subseteq C_0$. In particular $\tilde{T}_0$ is an $S_5$-subgroup of $C_0$ and $H' \subseteq C_0$.

Let now $\tilde{Y}$ be a minimal normal subgroup of $C_0$. Since $0(\tilde{Y})$ is characteristic in $\tilde{Y}$ we get either $0(\tilde{Y}) = \tilde{Y}$ or $0(\tilde{Y}) = 1$.

Suppose that $0(\tilde{Y}) = 1$. Then $\tilde{T}_0 \cap \tilde{Y}$ is nontrivial and hence $\tilde{T}_0 \subseteq \tilde{Y}$ as above. Thus $H' \subseteq \tilde{Y}$. So $\tilde{Y}$ is a direct product of isomorphic, non-abelian simple groups. Since $\tilde{P}$ is an $S_5$-subgroup of $\tilde{Y}$ and $Z(\tilde{P})$ is cyclic we see that $\tilde{Y}$ is simple. $\tilde{T}_0$ is an $S_5$-subgroup of the simple group $\tilde{Y}$ and is isomorphic to an $S_5$-subgroup of $M_{22}$. So we get by [6; Corollary 1.3] that $\tilde{Y}$ is isomorphic to one of the following groups: $M_{22}, M_{23}, McI, PSL(4, q), q \equiv 3 \pmod{8}, PSU(4, 1), q \equiv 5 \pmod{8}$. An $S_5$-subgroup of $McI$ is of order $3^6$, $M_{22}$ and $M_{23}$ have abelian $S_5$-subgroups, and $PSL(4, q)$ and $PSU(4, q)$ have $S_3$-subgroups which are not isomorphic to $P$ by [6; Lemma 2.1 and 2.2]. This contradiction shows that $0(\tilde{Y}) = \tilde{Y}$.

If $\tilde{Y} \cap H' = 1$ then $\mathfrak{p}$ acts regularly on $\tilde{Y}$ and hence $\tilde{Y}$ is nilpotent by (12). We have $\tilde{Y} = \langle C_\tau(x) | x \in \tilde{M} \rangle$ by (1.5). Since $C_\tau(x)$ is isomorphic to a subgroup of $H'$ for any $x \in \tilde{M}$ we get that $\pi(\tilde{Y}) \subseteq \{5, 7, 11\}$ and $C_\tau(x)$ is cyclic of prime order or 1 by (1.7). Since $\pi$ acts regularly on $C_\tau(x)$ we get that $C_\tau(x)$ is of order 7 for $x \in \tilde{M} - \langle \mathfrak{p} \rangle$. Since $\tilde{P}$ operates nontrivially on $\tilde{M}$ and normalizes $Z(\tilde{Y}) = \langle C_\tau(x) | x \in \tilde{M} \rangle = \tilde{Y}$. Thus $\tilde{Y}$ is elementary abelian of order $7^3$. Since $|GL(3, 7)| = 2^6 \cdot 3^4 \cdot 7^3 \cdot 19$ we get that $H'$ cannot operate faithfully on $\tilde{Y}$, i.e. $H'$ centralizes $\tilde{Y}$. But this is not possible. Thus $\tilde{Y} \cap H' = 1$. Since $\tilde{Y}$ is of odd order and $\tilde{Y} \cap H'$ is normal in $H'$ we get
that $\bar{Y} \cap H' = \langle \pi \rangle$ and hence by (1.4) $\bar{Y} = 0_3(\bar{Y}) \langle \pi \rangle$ since $H' = N(\langle \pi \rangle) \cap C_o$. Since $\bar{Y}$ is a minimal normal subgroup of $C_o$ we obtain $0_3(\bar{Y}) = 1$ and hence $\langle \pi \rangle = C_o$. This yields that $C_o = H'$ and thus $C_G(z) = H$.

**Lemma 2.5.** $0_2(C_G(E_1))$ is elementary abelian of order $2^{11}$.

**Proof.** Assume that $0_2(C_G(E_1))$ is not of order $2^{11}$. Then we get by (2.2) that $0_2(C_G(E_1)) = E_1$ and hence by (2.3) and (2.4) that $T$ is an $S_2$-subgroup of $G$ and $C_G(z) = H$.

Let $F = C_G(E_0)$ and $F = F/E_0$. $M$ is an $S_2$-subgroup of $F$ by (1.10,iii). We show first that $0_3(F) = E_0$.

Let $K = 0_3(F)$. Then $K$ is a characteristic subgroup of $F$ and hence normal in $N_G(E_0)$. Furthermore we have by (1.5) that $K = \langle C_E(x) | 1 \pm x \in M \rangle$. Since $N_G(M) \subseteq N_G(E_0)$ by (2.1) and $N_G(M)$ operates transitively on $M^4$ we see that $N_G(M)$ operates transitively on the set $\{ C_E(x) | 1 \pm x \in M \}$. Since $E_0 \subseteq Z(C_E(x))$ for any $x \in M$ we get by (1.10,iv) as in the proof of (2.2) that $K/E_0$ is an elementary abelian group of order $2^8$ if $K = E_0$. But this is not possible since $T$ is an $S_2$-subgroup of $G$. So $K = E_0$ and hence $0_3(F) = 1$.

We have $N_{F_0}(M) = M$ where $M$ is a 2-group which acts regularly on $Q$ by (2.1). Since $N_{F_0}(M)$ is normalized by an element $a$ of order 3 contained in $N_G(E_0)$ we can assume by Frattini’s argument that $a$ normalizes $Q$. By the structure of $\text{Aut}(M) = GL(2, 3)$ we see that $Q$ is not of order 4 because otherwise it would centralize $Q$. So $Q$ is either isomorphic to the quaternion group or cyclic of order two. In the second case we get $F = 0_3(F)N_{F_0}(M)$ by [9, II]. Since $0_3(F) = 1$ this implies that $F = N_{F_0}(M)$ which is not possible since $E_0 \subseteq F$. So we have $N_{F_0}(M) = M$ where $Q$ is quaternion and acts regularly on $M$.

Let $\bar{Y}$ be a minimal normal subgroup of $F$. Since $0_3(F) = 1$ we have $\bar{M} \cap \bar{Y} = 1$. Since $\bar{Q}$ operates transitively on $M^4$ we obtain $\bar{M} \subseteq \bar{Y}$. As $\bar{M}$ not normal in $F$, $\bar{Y}$ is not solvable. Furthermore $\bar{Y}$ is the unique minimal normal subgroup of $F$. Thus $\bar{Y}$ is normal in $N_G(E_0)/E_0$. So there exists an element $a$ of order 3 in $N_G(E_0)/E_0$ which normalizes $N_Y(M)$. The argument we used above to show that an $S_2$-subgroup of $N_{F_0}(M)$ is quaternion applies also to this situation and we get that $N_Y(M) = N_{F_0}(M)$. Since $\bar{Q}$ is quaternion we see that $\bar{Y}$ must be simple. By Frattini’s argument we get furthermore that $\bar{Y} = \bar{F}$.

We have $C_{F_0}(\pi) = E_0 \bar{M} \cong Z_3 \times A_4$ and all elements of $\bar{M}^4$ are conjugate to $\pi$ in $F$. So [7] gives that $F$ is isomorphic to one of the following groups: $PSL(3, 7)$, $PSU(3, 5^2)$, $M_{23}$, $M_{23}$, $HS$, $PSL(5, 2)$, $PSp(4, 4)$, $M_{31}$, $R$, $J_2$. The last three of these groups have $S_3$-subgroups of order 27 but $F$ has an $S_3$-subgroup of order 9. $PSL(5, 2)$, $PSp(4, 4)$, $M_{23}$, $M_{23}$, $HS$ have 2-subgroups of order $\geq 2^7$. But $T$ is an $S_2$-subgroup of $G$ and is of order $2^9$. We have $19 \mid |PSL(3, 7)|$ and $5^3 \mid |PSU(3, 5^2)|$ but $F \subseteq C_G(z) = H$ and $|H| = 2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$. This is a
contradiction.

This contradiction completes the proof of the lemma.

**Lemma 2.6.** $G$ is isomorphic to $J_4$.

Proof. By (2.2) and (2.5) we have $C_G(E_1) = K \langle \pi \rangle$ where $K$ is elementary abelian of order $2^{11}$ and is normal in $C_G(E_1)$. Let $N = N_G(K)$. Then we have

(i) $C_G(K) = K$ since $C_G(K) \subseteq C_G(E_1) = K \langle \pi \rangle$.

(ii) $N_G(E_1) \subseteq N$ where $N_G(E_1)/E_1$ is isomorphic to the triple cover of $A_6$ and $N_G(E_1)/E_1 \langle \pi \rangle = S_6$ by (1.10.ii).

(iii) $C_N(M) = MxE_0$ and $N_M(M) = N_G(M)$ as we have seen in the proof of (2.2).

We first show that $0_3(N) = K$. Since $0_3(N)$ is normalized by $M$ we get by (1.5) that $0_3(N) = \langle C(x) \cap 0_3(N) \mid x \in M \rangle$. We have $C_K(\pi) = E_1$ and $C(\pi) \cap 0_3(N)/C_K(\pi) \subseteq 0_3(N)/E_1 = 1$. Thus $C(\pi) \cap 0_3(N) = C_K(\pi)$. By (iii) this yields that $C_K(\pi) = C(\pi) \cap 0_3(N)$ for all $x \in M$ and hence $0_3(N) = K$.

Let $\bar{N} = N/K$ and let $\bar{Y}$ be a minimal normal subgroup of $\bar{N}$. Since $0_3(\bar{N}) = 1$ we see that $\bar{P} \cap \bar{Y} = 1$. Thus $Z(\bar{P})$ is contained in $\bar{Y}$ which implies that $M \subseteq \bar{Y}$ by (iii). By (ii) we see that $\bar{Y}$ is not solvable. Since $C_3(\bar{\pi})$ is isomorphic to the triple cover of $A_6$ by (ii) and $\langle \pi \rangle \subseteq C_3(\bar{\pi}) \subseteq C_3(N_\pi) = 1$. Thus $C(\pi) \cap 0_3(N) = C_K(\pi)$. In particular $\bar{Y}$ is simple since $\bar{P}$ is an $S_3$-subgroup of $\bar{Y}$ and $Z(\bar{P})$ is cyclic. Since $C_3(\bar{\pi}) \subseteq \bar{Y}$ we get $N(\bar{M}) \cap C_3(\bar{\pi}) = N_\pi(M)$ where $N(\bar{M}) \cap C_3(\bar{\pi})/\bar{M}$ is isomorphic to $S_3$ by (1.10.iii). Since $N_\pi(\bar{M})/\bar{M}$ is isomorphic to $G/L(2,3)$ by (iii) and (2.1) and since $N_\pi(\bar{M}) \subseteq N_\pi(\bar{M})$ we get by the structure of $G/L(2,3)$ that $N_\pi(\bar{M}) \leq N_\pi(\bar{M})$. So we have seen that $\bar{Y}$ is a simple group containing an element $\pi$ of order 3 such that $C_3(\pi) \langle \pi \rangle$ is isomorphic to $A_6 \cong PSL(2,9)$ and an elementary abelian subgroup $\bar{M}$ of order 9 all identity elements of which are conjugate to $\pi$ in $\bar{Y}$. So [7] gives that $\bar{Y}$ is isomorphic to $M_{24}$ or $R$ or $J_2$. But $J_2$ is 3-normal by [5] and $R$ cannot operate faithfully on an elementary abelian 2-group of order $2^{11}$ since $29 \not| |R|$ and $29 \not| (2^2 - 1)$ for $1 \leq k \leq 11$. So $\bar{Y} \cong M_{24}$. On the other hand $\bar{P}$ is an $S_3$-subgroup the normal subgroup $\bar{Y}$ of $\bar{N}$ and hence $N_\pi(\bar{P})$ covers $N_\pi(\bar{Y})$. Since $N_\pi(\bar{P}) \subseteq N_\pi(Z(\bar{P}))$ and $N_\pi(\bar{P})/\bar{P}$ is a 2-group we get that $N_\pi(\bar{Y})$ is a 2-group. Since $\text{Aut}(M_{24}) = M_{24}$ we obtain then that $\bar{N} = \bar{Y}$, for otherwise every element in $N_\pi \bar{Y}$ would induce a nontrivial outer automorphism of $M_{24}$ by the structure of $N_\pi(\bar{P})$.

Now we can apply [8; Theorem A] and obtain that $K$ splits into two $N$-classes of involutions the sizes of which are either 759 and 1288 or 1771 and 276. Since $z \in K$ is centralized by an $S_3$-subgroup of $N$ the number of conjugates of $z$ in $N$ is either 1288 = $2^3 \cdot 7 \cdot 23$ or 1771 = $7 \cdot 11 \cdot 23$. In the first case we have $|C_N(z)/K| = 2^2 \cdot 3^3 \cdot 5 \cdot 11$. By (1.9) we get then that $C_N(z)/K \cong \text{Aut}(M_{12})$. We have $(C_N(z)/K)' \cong M_{12}$ and $N_G(E_1)/K/K$ is contained in $(C_N(z)/K)'$. This implies that $M_{12}$ contains an element of order 3 which centralizes a dihedral
group of order 8. But $M_{12}$ has exactly two classes of involutions the centralizers of which in $M_{12}$ are isomorphic to a faithful extension of $Q_8 \times Q_8$ by $S_3$ or to $Z_2 \times S_3$. So there exists no dihedral subgroup of $M_{12}$ of order 8 which is centralized by an element of order 3. This contradiction shows that $K$ splits into two $\mathbb{N}$-orbits of sizes 1771 and 276.

So $z$ lies in the center of an $S_3$-subgroup of $\mathbb{N}$. We shall show that $0(C_G(z)) = W$ is trivial. Since $H \subseteq C_G(z)$ and $W \cap H \leq 0(H) = \langle \pi \rangle$ we have either $W \cap H = 1$ or $W \cap H = \langle \pi \rangle$. In the second case we get by (1.4) that $W = 0_3(W)\langle \pi \rangle$ and hence $C_G(z) = WH$ by the Frattini's argument. But this is not possible since $2^7 | |C_G(z)|$. So $W \cap H = 1$. Then $W$ is nilpotent by (1.2) and we have $W = \langle C_w(x) | 1 \neq x \in M \rangle$ by (1.5). Since $G$ has exactly one conjugacy class of elements of order 3, $C_w(x)$ is conjugate to a subgroup of $H$. Since $\pi$ operates regularly on $C_w(x)$ for any $x \in M$ we get that $C_w(x)$ is cyclic of order 7 or 1. Since $P$ normalizes $W$ and acts nontrivially on $M - \langle \pi \rangle$ we get that $Z(W) = W$ is elementary abelian of order 7 or 1. In any case $H'$ centralizes $W$. This implies that $W = 1$.

So we can apply [8; Theorem B] and see that either $|G| = |M(24)'|$ or $G \cong J_4$. But the first case is not possible since $3^6 | |M(24)'|$. So $G$ is isomorphic to $J_4$. This completes the proof of the lemma and the proof of Theorem A.

3. Proof of Theorem B

A slight modification of the proof of Theorem A gives Theorem B. We shall only indicate where differences are to be made.

Let $G$ be a simple group which is not 3-normal and contains an element $\pi$ such that $C_G(\pi)$ is isomorphic to the triple cover of $M_{12}$. Then Lemma (1.10) is valid for $G$ where $H$ is to be replaced by $H/\langle \pi \rangle$. We shall use the same notation as in the second section which was introduced in (1.10) with their corresponding new meanings. Then we have

**Lemma 3.1.** We have $N_G(M)/C_G(M) \cong GL(2, 3)$ and $N_G(M)$ is contained in $N_G(E_0)$ where $E_0 = 0_2(C_G(M))$ is a four group.

*Proof.* The same as in (2.1).

**Lemma 3.2.** We have $C_G(E_1) = 0_3(C_G(E_1))\langle \pi \rangle$ where either $0_3(C_G(E_1)) = E_1$ or $0_3(C_G(E_1))$ is elementary abelian of order $2^6$.

*Proof.* The same as in (2.2).

**Lemma 3.3.** If $0_2(C_G(E_1)) = E_1$ then $T$ is an $S_3$-subgroup of $G$.

*Proof.* The same as in (2.3).
Lemma 3.4. \( T \) is not an \( S_2 \)-subgroup of \( G \) and hence \( 0_2(C_G(E_1)) \) is of order \( 2^{10} \).

Proof. The argument we have used in (2.4) to show that \( C_G(e) \) contains a subgroup \( C_9 \) with index two applies also to this case and yields that \( G \) has a subgroup with index two. But this is a contradiction since \( G \) is simple.

Conclusion 3.5. \( G \) does not exist.

Proof. Otherwise we get as in (2.6) that \( N_G(K)/K \) is isomorphic to \( J_2 \) or \( M_{24} \) or \( R \), where \( K=0_2(C_G(E_1)) \) is elementary abelian of order \( 2^{10} \). But \( M_{24} \) and \( R \) cannot operate faithfully on a 2-group of order \( 2^{10} \). Since \( J_2 \) is 3-normal by [5] we obtain a contradiction since we can see that \( N_G(K) \) is not 3-normal as in (2.6).

This completes the proof of Theorem B.

References

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