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A CHARACTERIZATION OF THE SIMPLE GROUP J_4

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In [4] Janko describes the properties of a simple group of order $2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3\cdot 23\cdot 29\cdot 31\cdot 37\cdot 43$ denoted by J_4. It has exactly one conjugacy class of elements of order 3 and if π is one of them, then the centralizer of π in J_4 is isomorphic to the 6-fold cover of the Mathieu group M_{22}. We show in this paper that these properties characterize the group J_4, we prove namely the

Theorem A. *Let G be a finite group containing an element π of order 3 such that $C_G(\pi)$ is isomorphic to the 6-fold cover M_{22}. If G is not 3-normal then G is isomorphic to J_4.*

In the first section we shall list some properties of the 6-fold cover of M_{22}, which will be needed in the proof. The second section is then devoted to the proof of Theorem A. In the last section we remark that the following holds:

Theorem B *There exists no simple group G which is not 3-normal and contains an element π such that $C_G(\pi)$ is isomorphic to the triple cover of M_{22}.*

The Frattini subgroup of a group X is denoted by $D(X)$. The other notation is hopefully standard.

In the whole paper with the exception of the last section G denotes a simple group satisfying the assumptions of Theorem A and π is an element of G of order 3 such that $C_G(\pi)$ is isomorphic to the 6-fold cover of M_{22}.

1. Some known results and structure of $N_G(\langle\pi\rangle)$

We first list some well known results which will be used in the proof of our theorems.

Lemma 1.1 (Gaschütz). *Let A be an abelian normal subgroup of the group X contained in the subgroup B of X with $(|X: B|, |A|)=1$. Then if A has a complement in B, A has a complement in X.*

Proof. See [1].

Lemma 1.2 (Thompson). *If the group X admits a fixed-point-free automorphism of prime order then X is nilpotent.*

Proof. See [2; 10.2.1].

Lemma 1.3 (Thompson). *Let T_0 be a maximal subgroup of an S_2-subgroup of the group X. If X does not have a subgroup with index two then all involutions of X are conjugate to elements of T_0 in X.*

Proof. See [10. Lemma 5.38].

Lemma 1.4 (Burnside). *Let P be an S_p-subgroup of the group X and assume that $N_X(P)=C_X(P)$. Then X has a normal p-complement.*

Proof. See [2;7.4.3].

Lemma 1.5. *Let P be a p-group and let Q be a noncyclic abelian q-group of automorphisms of P, q a prime distinct from p. Then $P=\langle C_P(x) \mid 1 \neq x \in Q \rangle$.*

Proof. See [2; 5.3.16].

Lemma 1.6. *Any involution t of the group X which does not lie in the maximal normal 2-subgroup of X inverts a nontrivial element of X of odd order.*

Proof. Let t be an involution of X with $t \notin 0_2(X)$. Then there exists a conjugate t_1 of t in X such that the dihedral group $\langle t, t_1 \rangle$ is not a 2-group by [2;3.8.2]. Since the index of the cyclic subgroup $\langle t_1 t \rangle$ has index two in $\langle t, t_1 \rangle$ we see that $0(\langle t_1 t \rangle)$ is nontrivial and is inverted by t since t inverts $t_1 t$.

The following three lemmas are taken from [4; (2.1), (2.3), (2.4)].

Lemma 1.7. *Let $X \cong M_{22}$ and let T be an S_2-subgroup of X. Then T possesses precisely two distinct elementary abelian subgroups E_1 and E_2 of order 16 and they are both normal in T. We have $N_X(E_1)$ is a splitting extension of E_1 by A_6, $N_X(E_2)$ is a splitting extension of E_2 by S_5 and $N_X(E_i)$ acts transitively on $E_i^\#$, $i=1, 2$. The group X has the order $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ and exactly one conjugacy class of involutions with the representative $e \in E_1$ and we have $C_X(e)=C(e) \cap N_X(E_1)$. An S_3-subgroup P of X is elementary abelian of order 9 and we have $C_X(P)=P$ and $N_X(P)=PQ$ where Q is quaternion aud acts regularly on P. The group X has exactly one conjugacy class of elements of order 3 and if σ is one of them, then $C_X(\sigma) \cong \langle \sigma \rangle \times A_4$ and $N_X(\langle \sigma \rangle)=\langle \sigma \rangle B$ where $B \cong S_4$.*

Lemma 1.8. *Let $X \cong Aut(M_{22})$ so that $X' \cong M_{22}$ and $|X: X'|=2$. The group X possesses exactly two conjugacy classes of involutions which are contained in $X-X'$ with the representatives t_1 and t_2. If E_1 and E_2 are the only elementary abelian subgroups of rank 4 of an S_2-subgroup of X' as discribed in (1.7) then t_1 and t_2 can be chosen to lie in $C_X(E_2)=A=\langle E_2, t_2 \rangle$ which is elementary abelian of* *order* 32. *Then* $N_X(A)=AB$ *where* $B\subseteq X'$, $B\cong S_5$ *and* B *operates transitively on* $E_2^\#$ *and operates on* $A-E_2$ *in two orbits of sizes* 10 *and* 6 *represented respectively by* t_1 *and* t_2, *We have* $N_X(E_1)$ *is a splitting and faithful extension of* E_1 *by* S_6.

Lemma 1.9. *Every maximal subgroup of the simple group* M_{24} *is isomorphic to one of the following groups*:

$PSL(2, 23)$, M_{23}, $\mathrm{Aut}(M_{22})$, $\mathrm{Aut}(M_{12})$, $PSL(2, 7)$,

The holomorph of an elementary abelian group of order 16,

An extension of M_{21} *by* S_3,

A splitting and faithful extension of an elementary abelian group of order 64 *by a subgroup* Y *where* $|0_3(Y)|=3$, $Y/0_3(Y)\cong S_6$, $|Y: Y'|=2$, $Y''=Y'$ *and* $C_Y(0_3(Y))=Y'$,

A splitting and faithful extension of an elementary abelian group of order 64 *by* $S_3\times PSL(2, 7)$.

In the next lemma we list some properties of $N_G(\langle\pi\rangle)$ which can be easily deduced from (1.7) and (1.8) and are essentially proved in [4]. Throughout the paper we shall fix the notation which will be introduced in the following lemma.

Lemma 1.10. *The following hold in* G:

(i) *Let* $H=N_G(\langle\pi\rangle)$. *Then* $|H: H'|=2$, $H''=H'=C_H(\pi)$, $Z(H')$ *is cyclic of order* 6, $H'/Z(H')\cong M_{22}$ *and* $H/Z(H')\cong\mathrm{Aut}(M_{22})$. *Let us denote the involution in* $Z(H')$ *by* z.

(ii) *Let* T *be an* S_2*-subgroup of* H *and let* $T_0=T\cap H'$. *Then* T_0 *contains exactly two elementary abelian subgroups* E_1 *and* E_2 *of rank* 5. *These are normal in* T *and we have* $C_{H'}(E_i)=E_i\langle\pi\rangle$, $i=1, 2$ *and*

$N_{H'}(E_1)=E_1B_1$ *where* B_1 *is isomorphic to the triple cover of* A_6,

$C_H(E_1)=E_1\langle\pi\rangle$, $N_H(E_1)/C_H(E_1)\cong S_6$,

$N_{H'}(E_2)=E_2(\langle\pi\rangle\times B_2)$ *where* $B_2\cong S_5$ *and acts transitively on* $(E_2/\langle z\rangle)^\#$,

$N_H(E_2)=\langle\pi\rangle E_2^*B_2$ *where* E_2^* *is an abelian group of order* 64, B_2 *normalizes* E_2^*, $\langle z\rangle\geq D(E_2^*)$, E_2^* *is elementary abelian if and only if there exist involutions in* $H-H'$ *and if so then* E_2^* *is the only elementary abelian subgroup of* T *of order* 64. *Furthermore* $E_2^*\cap H'=E_2$ *and* $E_2^*\leq T$.

(iii) *Let* P *be an* S_3*-subgroup of* B_1. *Then* P *is an* S_3*-subgroup of* G. $P=\langle\pi, \sigma, \tau\rangle$ *is extraspecial of order* 27 *and exponent* 3, *where the generators of* P *are chosen in such a way that* $C_{E_1}(\tau)=\langle z\rangle$ *and* $C_{E_1}(\sigma)=E_0$ *is of order* 8. *We have* $N_{H'}(P)=\langle z\rangle\times PQ$ *where* Q *is quaternion and acts regularly on* $P/\langle\pi\rangle$. *There exists exactly one conjugacy class of elements of order* 3 *in* $H-\langle\pi\rangle$ *and exactly one conjugacy class of subgroups of order* 9 *in* H' *represented by* $\boldsymbol{M}=\langle\pi, \sigma\rangle$. *We have* $C_G(M)=M\times E_0$ *and* $N_{H'}(M)/C_{H'}(M)\cong S_3$.

(iv) *We have* $C_{H'}(E_0)=E_1M$.

(v) $E_1\cap E_2$ *is of order* 8 *and we have* $C_{H'}(E_1\cap E_2)=\langle\pi\rangle\times E_1E_2$ *and*

$N_{H'}(E_1 \cap E_2)=C_{H'}(E_1 \cap E_2)U$ where $U \cong S_3$ and $O_3(U)$ acts regularly on $E_1E_2/\langle z\rangle$. Furthermore $E_1 \cap E_2 \subseteq T'=\langle T' \cap E_1,\ T' \cap E_2\rangle$ and $T' \cap E_i$, $i=1, 2$, are the only elementary abelian subgroups of T' of rank four. So $E_1 \cap E_2$ is normal in $N_G(T)$. We have $C(E_1) \cap E_2^* = E_1 \cap E_2$.

Proof. $C_G(\pi)$ is isomorphic to the 6-fold cover M_{22}, i.e. $C_G(\pi)'=C_G(\pi)$, $Z(C_G(\pi))$ is cyclic of order 6 and $C_G(\pi)/Z(C_G(\pi)) \cong M_{22}$.

Let P be an S_3-subgroup of $C_G(\pi)$. Then $\langle\pi\rangle \subseteq Z(P)$ and P does not split over by $\langle\pi\rangle$ (1.1). So $D(P)=\langle\pi\rangle$ by (1.7) and hence P is an S_3-subgroup of G. Let R be an S_2-subgroup of $N(P) \cap C_G(\pi)$. Since R operates transitively on $(P/D(P))^{\#}$ by (1.7) we see that P is extraspecial of order 27 and exponent 3. Furthermore we have $R/\langle z\rangle \cong Q_8$ where z is the involution in $Z(C_G(\pi))$. So R must split over $\langle z\rangle$ and we get $N(P) \cap C_G(\pi)=\langle z\rangle \times PQ$ where $Q \cong Q_8$ and acts regularly on $P/D(P)$. In particular there exists exactly one conjugacy class of elements of order 3 in $C_G(\pi)-\langle\pi\rangle$ and hence exactly one conjugacy class of subgroups of order 9.

Since G is not 3-normal, π must be conjugate to an element in $P-\langle\pi\rangle$ and hence to π^{-1}. So $|N_G(\langle\pi\rangle): C_G(\pi)|=2$. Let $H=N_G(\langle\pi\rangle)$. Then $H'=C_G(\pi)$ and $H/Z(H')$ is isomorphic to $\mathrm{Aut}(M_{22})$ or $Z_2 \times M_{22}$ since $|\mathrm{Aut}(M_{22}): M_{22}|=2$ by (1.8). But the second case is not possible since otherwise there would exist a 2-element in $H-H'$ which operates trivially on $P/D(P)$ and inverts $D(P)$ and this is absurd. So $H/Z(H') \cong \mathrm{Aut}(M_{22})$.

Let T be an S_2-subgroup of H. Then all assertions of (ii) are proved in [4; Proposition 1 and 3] and we shall use them in the following.

Since an S_3-subgroup of $B_1 \subseteq H$ is also an S_3-subgroup of H we can assume that $P \subseteq B_1$. By the action of the non-cyclic abelian 3-group $P/D(P)$ on the 2-group E_1 we see that there is an element σ in P with $E_0=E_1 \cap C(\sigma)$ is elementary abelian of order 8 and an element τ in P with $\langle z\rangle=E_1 \cap C(\tau)$. Let $M=\langle\pi, \sigma\rangle$. Then $C_G(M)=E_0 \times M$ and $N_{H'}(M)/C_{H'}(M) \cong S_3$ by (1.7). This completes the proof of the first three assertions of the lemma.

For the proof of (iv) let $\bar{H}'=H'/Z(H')$, which is isomorphic to M_{22}. By (1.7) we have $C_{\bar{H}}(\bar{\sigma})=\langle\bar{\sigma}\rangle x\bar{E}_0\langle\bar{\tau}\rangle$ where $\bar{E}_0\langle\bar{\tau}\rangle$ is isomorphic to A_4 and $N_{\bar{H}'}(\langle\bar{\sigma}\rangle)/\langle\bar{\sigma}\rangle$ is isomorphic to S_4. This gives that $C_{\bar{H}}(\bar{E}_0) \cap N_{\bar{H}'}(\langle\bar{\sigma}\rangle)=\bar{E}_0\langle\bar{\sigma}\rangle$ By Burnside's transfer theorem we get $C_{\bar{H}'}(\bar{E}_0)=0_{3'}(C_{\bar{H}'}(\bar{E}_0))\langle\bar{\sigma}\rangle$. By the structure of M_{22}, $\bar{K}=0_{3'}(C_{\bar{H}}(\bar{E}_0))$ is a 2-group containing $\bar{E}_1$.

Suppose that $\bar{K} \neq \bar{E}$. Then the non-trivial group $\bar{K}/\bar{E}_1$ is normalized by $\bar{P}=\langle\bar{\sigma}, \bar{\tau}\rangle$. Since P is not cyclic there is by (1.5) a non-trivial element $\bar{x}$ in $\bar{P}$ such that $C_{\bar{K}/\bar{E}_1}(\bar{x})=C_{\bar{K}}(\bar{x})\bar{E}_1/\bar{E}_1 \neq 1$. As $\bar{\sigma}$ operates regularly on $\bar{K}/\bar{E}_1$ and normalizes $C_{\bar{K}/\bar{E}_1}(\bar{x})$ we get that $\bar{K}/\bar{E}_1=C_{\bar{K}/\bar{E}_1}(\bar{x})$ is elementary abelian of order four since $|\bar{K}/\bar{E}_1| \leq 8$. By the structure of the centralizer of an element of order three in M_{22} we get that $C_{\bar{K}}(\bar{x})$ is four group and that $\bar{K}=\bar{E}_1C_{\bar{K}}(\bar{x})$.

$\bar{S}=C(C_{\bar{K}}(\bar{x})) \cap \bar{E}_1$ is non-trivial and is normalized by $\bar{x}$ which operates regularly on it. This yields that $|\bar{S}|\geq 4$. So $\bar{D}=C_{\bar{K}}(\bar{x})\bar{S}$ is elementary abelian of rank at least four. Since an S_2-subgroup of M_{22} contains exactly two elementary abelian subgroups of rank four by (1.7) we see that $\bar{D}$ is conjugate in $\bar{H}'$ to $\bar{E}_2$. But $\bar{D}$ is normalized by $\bar{P}$ whereas an S_3-subgroup of $N_{\bar{H}}(\bar{E}_2)$ is of order 3. This contradiction shows that $\bar{K}=\bar{E}_1$ and hence $C_{H'}(E_0)=E_1M$.

For the proof of (v) observe that E_1E_2/E_1 is a non-trivial elementary abelian 2-group of $N_{H'}(E_1)/E_1=\bar{B}_1$ which is isomorphic to the triple cover of A_6. Since an S_2-subgroup of $\bar{B}_1$ is dihedral of order 8 there exists a four group $\bar{V}$ of $\bar{B}_1$ containing $\bar{E}_2$. By the structure of $\bar{B}_1$ and by (1.1) we get that $N(\bar{V})\cap\bar{B}_1=\langle\bar{\pi}\rangle x\bar{V}\bar{U}$ where $\bar{U}\cong S_3$ and operates faithfully on $\bar{V}$. Let U_0 be an S_3-subgroup of the inverse image of $\bar{U}$. By (1.6) we can assume that U_0 is inverted by an involution x in $T_0-E_1E_2$ such that $\langle U_0, x\rangle$ maps into $\bar{U}$ and $\langle U_0, x\rangle\cong S_3$. U_0 normalizes the inverse image V of $\bar{V}$. Since $E_1E_2\subseteq V$ and E_1 and E_2 are the only elementary abelian 2-groups of T_0 hence of V of rank 5 we see that U_0 normalizes both E_1 and E_2 and hence E_1E_2. This implies that $\bar{V}=\bar{E}_2$ and $E_1\cap E_2$ is of order 8. Furthermore U_0 maps onto an S_3-subgroup of $N_{H'}(E_2)/E_2$ Since B_2 operates transitively on $(E_2/\langle z\rangle)^{\#}$ by (ii) we obtain that U_0 operates regularly on $E_1/\langle z\rangle$. Since T_0 does not split over $\langle z\rangle$ we see that $\langle z\rangle$ is properly contained in $(E_1E_2)'=[E_1, E_2]\subseteq E_1\subset E_2$ by (ii). Since U_0 operates regularly on $E_1\cap E_2\langle z\rangle$ we get that $(E_1E_2)'=E_1\cap E_2$ and hence $D(E_1E_2)=E_1\cap E_2$. Since U_0 acts regularly on $E_1\cap E_2/\langle z\rangle$ we see that $E_1\cap E_2$ is not centralized by x and hence $C_{H'}(E_1\cap E_2)=\langle\pi\rangle\times E_1E_2$. By (iv) we get that $E_1\cap E_2$ is not normalized by an S_3-subgroup of H', because otherwise it would be centralized by a subgroup of order 9 and would be conjugate to E_0 by (iii). This implies that U_0 operates regularly also on $E_1/\langle z\rangle$, because otherwise an S_3-subgroup of B_1 containing $\langle\pi, U_0\rangle$ would normalize $\langle[E_1, U_0], z\rangle=E_1\cap E_2$. So U_0 acts regularly on $E_1E_2/\langle z\rangle$ and we have $C(E_1)\cap E_2=E_1\cap E_2$.

So we have seen that the elementary abelian group $E_1E_2/E_1\cap E_2$ of rank 4 is normalized by $\langle U_0, x\rangle\cong S_3$ such that U_0 operates regularly on it. This shows that $T_0'/E_1\cap E_2=C(x(E_1\cap E_2)\cap(E_1E_2/E_1\cap E_2)$ and hence that $T_0'=\langle T_0'\cap E_1, T_0'\cap E_2\rangle$ where $T_0'\cap E_i$, $i=1, 2$, is of order 16. Since $T/E_1\cong Z_2xD_8$ we see that there exists an element t in $(T-T_0)\cap E_2^*$ such that $T/E_1=\langle tE_1\rangle x(T_0/E_1)$. Thus $[t, T_0]\subseteq E_1\cap E_2^*=E_1\cap E_2$. This implies that $T'=T_0'$. Since T_0 contains exactly two elementary abelian subgroups of order 32, T' is no abelian. This yields that $T'\cap E_i$, $i=1, 2$, are the only elementary abelian subgroups of T' of order 16. Since T' is normal in $N_G(T)$ we get that $E_1\cap E_2=(T'\cap E_1)\cap(T'\cap E_2)$ is normal in $N_G(T)$.

This completes the proof of the lemma.

2. Proof of Theorem A

In this section we prove Theorem A in a sequence of lemmas. We shall use the notation introduced in (1.10).

Lemma 2.1. *We have $N_G(M)/C_G(M) \cong GL(2, 3)$ and $N_G(M)$ is contained in $N_G(E_0)$.*

Proof. Since G is not 3-normal and there exists precisely one conjugacy class of elements of order 3 in $H-\langle\pi\rangle$ represented by σ we have that $\pi\sim\sigma$ in G. So there exists an element g in G such that $\sigma^g=\pi$ and $C_P(\sigma)^g=M^g\subseteq P$. Since there exists in H' exactly one conjugacy class of subgroups of order 9 we can assume that $M^g=M$. So $\pi\sim\sigma$ in $N_G(M)$.

Since $M^\sharp$ is the union of $N_{H'}(M)$-orbits of sizes 1, 1 and 6 represented by π, π^{-1}, σ respectively we get that $|N_G(M)/C_G(M)|=|GL(2, 3)|$ and hence $N_G(M)/C_G(M)\cong GL(2, 3)$.

Since $E_0=0_2(C_G(M))$ by (1.10.iii) we see that $E_0\triangleleft N_G(M)$.

Lemma 2.2. *We have $C_G(E_1)=0_2(C_G(E_1))\langle\pi\rangle$ where $0_2(C_G(E_1))$ is either equal to E_1 or is an elementary abelian group of order 2^{11}.*

Proof. By (1.10.ii) we have $C_H(E_1)=E_1\langle\pi\rangle$. Burnside's transfer theorem yields then that $C_G(E_1)=0_{3'}(C_G(E_1))\langle\pi\rangle$ since $\langle\pi\rangle$ is an S_3-subgroup of $C_G(E_1)$ by (1.10.iii).

Let $K=0_{3'}(C_G(E_1))$. Since $C_H(E_1)=E_1\langle\pi\rangle$ we see that π operates regularly on K/E_1. Thus K/E_1 is nilpotent by (1.2). As $E_1\subseteq Z(K)$ we get that K is nilpotent. Furthermore K is normalized by P and hence we have $K=\langle C_K(x)\,|\,1\neq x\in M\rangle$, by (1.5).

We have $E_0=C(x)\cap E_1\subseteq Z(C_K(x))$ for any $x\in M-\langle\pi\rangle$. Since $N_G(M)$ is contained in $N_G(E_0)$ by (2.1) we see that $C(E_0)\cap C_G(x)=0_2(C(E_0)\cap C_G(x))M$ for any $1\neq x\in M$, where the maximal normal 2-subgroup of $C(E_0)\cap C_G(x)$ is elementary abelian of order 32 by (1.10.iv). So $C_K(x)$ is an elementary abelian 2-group of order at most 32. On the other hand π operates regularly on $C_K(x)E_0$ for $x\in M-\langle\pi\rangle$. This implies that we have either $C_K(x)=0_2(C(E_0)\cap C_G(x))$ or $C_K(x)=E_0$ for $x\in M-\langle\pi\rangle$. Since all elements of the set $\{C_K(x)\,|\,x\in M-\langle\pi\rangle\}$ are conjugate to each other via τ we have either

$$C_K(x) = E_0 \qquad \text{for all } x\in M-\langle\pi\rangle, \text{ i.e. } K=E_1,$$

or

$$C_K(x) = 0_2(C(E_0)\cap C_G(x)) \qquad \text{for all } 1\neq x\in M, \text{ i.e.}$$
$$K = \langle 0_2(C(E_0)\cap C_G(x))\,|\,1\neq x\in M\rangle$$

where $0_2(C(E_0)\cap C_G(x))$ is elementary abelian of order 32 for all $1\neq x\in M$. We can assume that we are in the second case.

Let S be an S_2-subgroup of $N_G(M)$. Then S acts transitively on $M^\sharp$ and normalizes E_0. So S acts transitively on the set $\{0_2(C(E_0)\cap C_G(x))\,|\,1\neq x\in M\}$

and hence normalizes K. Since $E_1=0_2(C(E_0)\cap C_G(\pi))\subseteq Z(K)$ we get that $K\subseteq Z(K)$ and hence that K is elementary abelian of order 2^{11} since

$$\bar{K}=K/E_1=C_{\bar{K}}(\sigma)\times C_{\bar{K}}(\sigma\pi)\times C_{\bar{K}}(\sigma\pi^{-1})$$

is of order 2^6.

Lemma 2.3. *If $0_2(C_G(E_1))=E_1$, then T is an S_2-subgroup of G.*

Proof. By (2.2) and the assumption of this lemma we have $C_G(E_1)=E_1\times\langle\pi\rangle$. Then $N_G(E_1)$ normalizes $\langle\pi\rangle$ and hence we get $N_G(E_1)=N_H(E_1)$. Thus T is an S_2-subgroup of $N_G(E_1)$.

Suppose that T is not an S_2-subgroup of G. Then there exists a 2-group $T\langle x\rangle$ in G with $|T\langle x\rangle : T|=2$. If $E_1^x\subseteq T_0$ we get $E_1^x=E_1$ by (1.10.ii). This contradicts the fact that T is an S_2-subgroup of $N_G(E_1)$. So $E_1^x\nsubseteq T_0$ and thus $T-T_0$ contains involutions. Then E_2^* is the only elementary abelian subgroup of T of order 64 by (1.10.ii) and hence x normalizes E_2^*.

Since x normalizes $T'\cap E_2^*=T'\cap E_2$ and since T' contains exactly two elementary abelian subgroups of rank four, namely $T'\cap E_i$, $i=1, 2$, we see that x also normalizes $T'\cap E_1$. Since $E_1\cap E_2\lhd N_G(T)$ by (1.10.v) we get that $E_1E_2^*=C_T(E_1\cap E_2)\lhd N_G(T)$ and hence $X=E_1E_2^*\cap C(T'\cap E_1)=E_1C_{E_2^*}(T'\cap E_1)$ is normalized by x. (1.10.v) gives then that $Z(X)=T'\cap E_1$ and that E_1 and $(T'\cap E_1)\times(E_2^*\cap C(T'\cap E_1))$ are the only elementary abelian subgroups of X of rank five. Since $E_2^*\cap C(T'\cap E_1)$ is normalized by x we get that $E_1^x=E_1$ which is a contradiction. Thus T is a Sylow 2-subgroup of G.

Lemma 2.4. *If T is an S_2-subgroup of G then the centralizer of the involution z in G is H.*

Proof. Let $C=C_G(z)$ and denote the homomorphic image of any subset X of C in $C/\langle z\rangle$ by $\bar{X}$. Obviously H is contained in C.

Then T is an S_2-subgroup of C by our assumption and $\bar{T}$ is isomorphic to an S_2-subgroup of Aut(M_{22}). Since $H\subseteq C$ all involutions in $\bar{T}_0$ are conjugate to $\bar{e}\in Z(\bar{T})$ in $\bar{C}$ and all involutions in $\bar{T}-\bar{T}_0$ are conjugate to involutions in $\bar{E}_2^*-\bar{E}_2$ in $\bar{C}$ where $\bar{E}_2^*=C_{\bar{T}}(\bar{E}_2)$ is the only elementary abelian subgroup of $\bar{T}$ of order 32 by (1.7) and (1.8). Furthermore we have $N_{\bar{H}}(\bar{E}_2^*)=\bar{E}_2^*\bar{B}_2$ where $\bar{B}_2\simeq S_5$ and $(\bar{E}_2^*)^\sharp$ splits into $\bar{B}_2$-orbits of sizes 15, 6 and 10 represented respectively by $\bar{e}$, $\bar{t}_1$ and $\bar{t}_2$ where $\bar{t}_1$ and $\bar{t}_2$ are in $\bar{E}_2^*-\bar{E}_2$ by (1.8).

If $\bar{C}$ has no subgroups of index two then $\bar{t}_i$, $i=1, 2$, must be conjugate to an element of $\bar{T}_0$ hence to $\bar{e}$ in $\bar{C}$ by (1.3), Thompson's transfer lemma. But this conjugation must take place in $N_{\bar{C}}(\bar{E}_2^*)$ since $\bar{E}_2^*$ is the only elementary abelian subgroup of $\bar{T}$ of rank 5. So we get by the above paragraph that all involutions of $\bar{E}_2^*$ are conjugate to each other in $N_{\bar{C}}(\bar{E}_2^*)$. In particular 31 divides the order of the group $N_{\bar{C}}(\bar{E}_2^*)/C_{\bar{C}}(\bar{E}_2^*)$.

Let $\tilde{N}=N_{\bar{C}}(\bar{E}_2^*)/C_{\bar{C}}(\bar{E}_2^*)$. Then $\tilde{N}$ is isomorphic to a subgroup of $GL(5, 2)$, has dihedral S_2-subgroups of order 8 and contains a subgroup $\tilde{B}_2$ which is isomorphic to S_5. So $\tilde{N}/0(\tilde{N})$ is either isomorphic to A_7 or to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$ where q is an odd prime power by [3].

Assume first that $31 \,\Big|\, |0(\tilde{N})|$. Let $\tilde{S}$ be an S_{31}-subgroup of $0(\tilde{N})$. Since 31^2 does not divide the order of $GL(5, 2)$ and since $\tilde{N}$ is isomorphic to a subgroup of $GL(5, 2)$ we see that $\tilde{S}$ is cyclic of order 31. By Frattini's argument we get that $N_{\tilde{N}}(\tilde{S})$ covers $\tilde{N}/0(\tilde{N})$ and hence that $N_{\tilde{N}}(\tilde{S})/N_{0(\tilde{N})}(\tilde{S})$ contains a subgroup isomorphic to S_5 by the above paragraph. Since $\mathrm{Aut}(\tilde{S})$ is cyclic we conclude that $\tilde{S}$ is centralized by an element $\tilde{a}$ of order 5. But $C(\tilde{a})\cap \bar{E}_2^*$ is nontrivial and is normalized by $\tilde{S}$. But this is not possible since $\tilde{S}$ operates regularly on $\bar{E}_2^*$. Thus $31 \not| \, |0(N)|$ and hence $31 \,\Big|\, |N/0(N)|$.

So $\tilde{N}/0(\tilde{N})$ is isomorphic to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$. Since $\tilde{N}$ is isomorphic to a subgroup of $GL(5, 2)$ and $|GL(5, 2)|=2^{10}\cdot 3^2\cdot 5\cdot 7\cdot 31$ this is possible only if $q=31$. But $|PSL(2, 31)|=2^5\cdot 3\cdot 5\cdot 31$ whereas $|\tilde{N}|_2=8$. This contradiction shows that $\bar{C}$ contains a subgroup $\bar{C}_0$ with index 2.

We have $1\neq\bar{C}_0\cap\bar{T}$. Thus $Z(\bar{T}')=\langle\bar{e}\rangle$ is contained in $\bar{C}_0$. Since all involutions of $\bar{T}_0$ are conjugate to $\bar{e}$ in $\bar{H}'\subseteq\bar{C}$ and since $\bar{T}_0$ is generated by its involutions we get $\bar{T}_0\subseteq\bar{C}_0$. In particular $\bar{T}_0$ is an S_2-subgroup of $\bar{C}_0$ and $\bar{H}'\subseteq\bar{C}_0$.

Let now $\bar{Y}$ be a minimal normal subgroup of $\bar{C}_0$. Since $0(\bar{Y})$ is characteristic in $\bar{Y}$ we get either $0(\bar{Y})=\bar{Y}$ or $0(\bar{Y})=1$.

Suppose that $0(\bar{Y})=1$. Then $\bar{T}_0\cap\bar{Y}$ is nontrivial and hence $\bar{T}_0\leq\bar{Y}$ as above. Thus $\bar{H}'\subseteq\bar{Y}$. So $\bar{Y}$ is a direct product of isomorphic, non-abelian simple groups. Since $\bar{P}$ is an S_3-subgroup of $\bar{Y}$ and $Z(\bar{P})$ is cyclic we see that $\bar{Y}$ is simple. $\bar{T}_0$ is an S_2-subgroup of the simple group $\bar{Y}$ and is isomorphic to an S_2-subgroup of M_{22}. So we get by [6; Corollary 1.3] that $\bar{Y}$ is isomorphic to one of the following groups: M_{22}, M_{23}, McL, $PSL(4, q)$, $q\equiv 3 \pmod 8$, $PSU(4, 1)$, $q\equiv 5 \pmod 8$. An S_3-subgroup of McL is of order 3^6, M_{22} and M_{23} have abelian S_3-subgroups, and $PSL(4, q)$ and $PSU(4,q)$ have S_3-subgroups which are not isomorphic to P by [6; Lemma 2.1 and 2.2]. This contradiction shows that $0(\bar{Y})=\bar{Y}$.

If $\bar{Y}\cap\bar{H}'=1$ then $\bar{\pi}$ acts regularly on $\bar{Y}$ and hence $\bar{Y}$ is nilpotent by (12). We have $\bar{Y}=\langle C_{\bar{Y}}(\bar{x})\,|\,1\neq\bar{x}\in\bar{M}\rangle$ by (1.5). Since $C_{\bar{Y}}(\bar{x})$ is isomorphic to a subgroup of $\bar{H}'$ for any $\bar{x}\in\bar{M}$ we get that $\pi(\bar{Y})\subseteq\{5, 7, 11\}$ and $C_{\bar{Y}}(\bar{x})$ is cyclic of prime order or 1 by (1.7). Since $\bar{\pi}$ acts regularly on $C_{\bar{Y}}(\bar{x})$ we get that $C_{\bar{Y}}(x)$ is of order 7 for $\bar{x}\in\bar{M}-\langle\bar{\pi}\rangle$. Since $\bar{P}$ operates nontrivially on $\bar{M}$ and normalizes $Z(\bar{Y})\neq 1$ we get by (1.5) $Z(\bar{Y})=\langle C_{Z(\bar{Y})}(\bar{x})\,|\,1\neq\bar{x}\in\bar{M}\rangle=\bar{Y}$. Thus $\bar{Y}$ is elementary abelian of order 7^3. Since $|GL(3, 7)|=2^6\cdot 3^4\cdot 7^3\cdot 19$ we get that $\bar{H}'$ cannot operate faithfully on $\bar{Y}$, i.e. $\bar{H}'$ centralizes $\bar{Y}$. But this is not possible. Thus $\bar{Y}\cap\bar{H}'\neq 1$. Since $\bar{Y}$ is of odd order and $\bar{Y}\cap\bar{H}'$ is normal in $\bar{H}'$ we get that $\bar{Y}\cap\bar{H}'=\langle\bar{\pi}\rangle$ and hence by (1.4) $\bar{Y}=0_{3'}(\bar{Y})\langle\bar{\pi}\rangle$ since $\bar{H}'=N(\langle\bar{\pi}\rangle)\cap\bar{C}_0$. Since $\bar{Y}$ is a minimal normal subgroup of $\bar{C}_0$ we obtain $0_3(\bar{Y})=1$ and hence $\langle\bar{\pi}\rangle\lhd\bar{C}_0$. This yields that $\bar{C}_0=\bar{H}'$ and thus $C_G(z)=H$.

Lemma 2.5. *$0_2(C_G(E_1))$ is elementary abelian of order 2^{11}.*

Proof. Assume that $0_2(C_G(E_1))$ is not of order 2^{11}. Then we get by (2.2) that $0_2(C_G(E_1))=E_1$ and hence by (2.3) and (2.4) that T is an S_2-subgroup of G and $C_G(z)=H$.

Let $F=C_G(E_0)$ and $\bar{F}=F/E_0$. M is an S_3-subgroup of F by (1.10.iii). We show first that $0_{3'}(F)=E_0$.

Let $K=0_{3'}(F)$. Then K is a characteristic subgroup of F and hence normal in $N_G(E_0)$. Furthermore we have by (1.5) that $K=\langle C_K(x)\,|\,1\neq x\in M\rangle$. Since $N_G(M)\subseteq N_G(F_0)$ by (2.1) and $N_G(M)$ operates transitively on $M^\#$ we see that $N_G(M)$ operates transitively on the set $\{C_K(x)\,|\,1\neq x\in M\}$. Since $E_0\subseteq Z(C_K(x))$ for any $x\in M$ we get by (1.10.iv) as in the proof of (2.2) that K/E_0 is an elementary abelian group of order 2^8 if $K\neq E_0$. But this is not possible since T is an S_2-subgroup of G. So $K=E_0$ and hence $0_{3'}(\bar{F})=1$.

We have $N_{\bar{F}}(\bar{M})=\bar{M}\bar{Q}$ where $\bar{Q}$ is a 2-group which acts regularly on $\bar{M}$ by (2.1). Since $N_{\bar{F}}(\bar{M})$ is normalized by an element $\bar{a}$ of order 3 contained in $N_G(E_0)/E_0$ we can assume by Frattini's argument that $\bar{a}$ normalizes $\bar{Q}$. By the structure of $\mathrm{Aut}(M)\simeq GL(2,3)$ we see that $\bar{Q}$ is not of order 4 because otherwise $\bar{a}$ would centralize $\bar{Q}$. So $\bar{Q}$ is either isomorphic to the quaternion group is cyclic of order two. In the second case we get $\bar{F}=0_{3'}(\bar{F})N_{\bar{F}}(\bar{M})$ by [9, II]. Since $0_{3'}(\bar{F})=1$ this implies that $\bar{F}=N_{\bar{F}}(\bar{M})$ which is not poosible since $\bar{E}_1\subseteq\bar{F}$. So we have $N_{\bar{F}}(\bar{M})=\bar{M}\bar{Q}$ where $\bar{Q}$ is quaternion and acts regularly on $\bar{M}$.

Let $\bar{Y}$ be a minimal normal subgroup of $\bar{F}$. Since $0_{3'}(\bar{F})=1$ we have $\bar{M}\cap\bar{Y}\neq1$. Since $\bar{Q}$ operates transitively on $M^\#$ we obtain $\bar{M}\subseteq\bar{Y}$. As $\bar{M}$ not normal in $\bar{F}$, $\bar{Y}$ is not solvable. Furthermore $\bar{Y}$ is the unique minimal normal subgroup of $\bar{F}$. Thus $\bar{Y}$ is normal in $N_G(E_0)/E_0$. So there exists an element $\bar{a}$ of order 3 in $N_G(E_0)/E_0$ which normalizes $N_{\bar{Y}}(M)$. The argument we used above to show that an S_2-subgroup of $N_{\bar{F}}(\bar{M})$ is quaternion applies also to this situation and we get that $N_{\bar{Y}}(M)=N_{\bar{F}}(M)$. Since $\bar{Q}$ is quaternion we see that $\bar{Y}$ must be simple. By Frattini's argument we get furthermore that $\bar{Y}=\bar{F}$.

We have $C_{\bar{F}}(\bar{\pi})=\bar{E}_1\bar{M}\simeq Z_3xA_4$ and all elements of $\bar{M}^\#$ are conjugate to $\bar{\pi}$ in $\bar{F}$. So [7] gives that $\bar{F}$ is isomorphic to one of the following groups: $PSL(3,7)$, $PSU(3,5^2)$, M_{22}, M_{23}, HS, $PSL(5,2)$, $PSp(4,4)$, M_{24}, R, J_2. The last three of these groups have S_3-subgroups of order 27 but $\bar{F}$ has an S_3-subgroup of order 9. $PSL(5,2)$, $PSp(4,4)$, M_{22}, M_{23}, HS have 2-subgroups of order $\geq2^7$. But T is an S_2-subgroup of G and is of order 2^9. We have $19\,\big|\,|PSL(3,7)|$ and $5^3\,\big|\,|PSU(3,5^2)|$ but $F\subseteq C_G(z)=H$ and $|H|=2^9\cdot3^3\cdot5\cdot7\cdot11$. This is a contradiction.

This contradiction completes the proof of the lemma.

Lemma 2.6. *G is isomorphic to J_4.*

Proof. By (2.2) and (2.5) we have $C_G(E_1)=K\langle\pi\rangle$ where K is elementary abelian of order 2^{11} and is normal in $C_G(E_1)$. Let $N=N_G(K)$. Then we have

(i) $C_G(K)=K$ since $C_G(K)\subseteq C_G(E_1)=K\langle\pi\rangle$.

(ii) $N_G(E_1)\subseteq N$ where $N_{H'}(E_1)/E_1$ is isomorphic to the triple cover of A_6 and $N_H(E_1)/E_1\langle\pi\rangle\cong S_6$ by (1.10.ii).

(iii) $C_N(M)=MxE_0$ and $N_N(M)=N_G(M)$ as we have seen in the proof of (2.2).

We first show that $0_{3'}(N)=K$. Since $0_{3'}(N)$ is normalized by M we get by (1.5) that $0_{3'}(N)=\langle C(x)\cap 0_{3'}(N)\,|\,1\neq x\in M\rangle$. We have $C_K(\pi)=E_1$ and $C(\pi)\cap 0_{3'}(N)/C_K(\pi)\subseteq 0_{3'}(N_{H'}(E_1)/E_1)=1$. Thus $C(\pi)\cap 0_{3'}(N)=C_K(\pi)$. By (iii) this yields that $C_K(x)=C(x)\cap 0_{3'}(N)$ for all $1\neq x\in M$ and hence $0_{3'}(N)=K$.

Let $\bar{N}=N/K$ and let $\bar{Y}$ be a minimal noımal subgroup of $\bar{N}$. Since $0_{3'}(\bar{N})=1$ we see that $\bar{P}\cap\bar{Y}\neq 1$. Thus $Z(\bar{P})$ is contained in $\bar{Y}$ which implies that $\bar{M}\subseteq\bar{Y}$ by (iii). By (ii) we see that $\bar{Y}$ is not solvable. Since $C_{\bar{N}}(\bar{\pi})$ is isomorphic to the triple cover of A_6 by (ii) and $\langle\bar{\pi}\rangle\subsetneqq C_{\bar{Y}}(\bar{\pi})\trianglelefteq C_{\bar{N}}(\bar{\pi})$ we get that $C_{\bar{Y}}(\bar{\pi})=C_{\bar{N}}(\bar{\pi})$. In particular $\bar{Y}$ is simple since $\bar{P}$ is an S_3-subgroup of $\bar{Y}$ and $Z(\bar{P})$ is cyclic. Since $C_{\bar{N}}(\bar{\pi})\subseteq\bar{Y}$ we get $N(\bar{M})\cap C_{\bar{N}}(\bar{\pi})\subseteq N_Y(M)$ where $N(\bar{M})\cap C_{\bar{N}}(\bar{\pi})/\bar{M}$ is isomorphic to S_3 by (1.10.iii). Since $N_{\bar{N}}(\bar{M})/\bar{M}$ is isomorphic to $GL(2,3)$ by (iii) and (2.1) and since $N_{\bar{Y}}(\bar{M})\trianglelefteq N_{\bar{N}}(\bar{M})$ we get by the sructure of $GL(2,3)$ that $N_{\bar{Y}}(\bar{M})\trianglelefteq N_{\bar{N}}(\bar{M})$. So we have seen that $\bar{Y}$ is a simple group containing an element $\bar{\pi}$ of order 3 such that $C_{\bar{Y}}(\bar{\pi})/\langle\bar{\pi}\rangle$ is isomorphic to $A_6\cong PSL(2,9)$ and an elementary abelian subgroup $\bar{M}$ of order 9 all identity elements of which are conjugate to $\bar{\pi}$ in $\bar{Y}$. So [7] gives that $\bar{Y}$ is isomorphic to M_{24} or R or J_2. But J_2 is 3-normal by [5] and R cannot operate faithfully on an elementary abelian 2-group of order 2^{11} since $29\,\big|\,|R|$ and $29\not|(2^k-1)$ for $1\leq k\leq 11$. So $\bar{Y}\cong M_{24}$. On the other hand $\bar{P}$ is an S_3-subgroup the normal subgroup $\bar{Y}$ of $\bar{N}$ and hence $N_{\bar{N}}(\bar{P})$ covers $\bar{N}/\bar{Y}$. Since $N_{\bar{N}}(\bar{P})\subseteq N_{\bar{N}}(Z(\bar{P}))$ and $N_{\bar{H}}(\bar{P})/\bar{P}$ is a 2-group we get that $\bar{N}/\bar{Y}$ is a 2-group. Since $\mathrm{Aut}(M_{24})=M_{24}$ we obtain then that $\bar{N}=\bar{Y}$, for otherwise every element in $\bar{N}-\bar{Y}$ would induce a nontrivial outer automorphism of M_{24} by the structure of $N_{\bar{N}}(P)$.

Now we can apply [8; Theorem A] and obtain that K splits into two N-classes of involutions the sizes of which are either 759 and 1288 or 1771 and 276. Since $z\in K$ is centralized by an S_3-subgroup of N the number of conjugates of z in N is either $1288=2^3\cdot 7\cdot 23$ or $1771=7\cdot 11\cdot 23$. In the first case we have $|C_N(z)/K|=2^4\cdot 3^3\cdot 5\cdot 11$. By (1.9) we get then that $C_N(z)/\boldsymbol{K}\cong\mathrm{Aut}(M_{12})$. We have $(C_N/(z)/K)'\cong M_{12}$ and $N_{H'}(E_1)K/K$ is contained in $(C_N(z)/K)'$. This implies that M_{12} contains an element of order 3 which centralizes a dihedral group of order 8. But M_{12} has exactly two classes of involutions the centralizers of which in M_{12} are isomorphic to a faithful extension of Q_8*Q_8 by S_3 or to $Z_2\times S_5$. So there exists no dihedral subgroup of M_{12} of order 8 which is centralized by an element of order 3. This contradiction shows that K splits into two N-orbits of sizes 1771 and 276.

So z lies in the center of an S_2-subgroup of N. We shall show that $0(C_G(z))=W$ is trivial. Since $H\subseteq C_G(z)$ and $W\cap H\subseteq 0(H)=\langle\pi\rangle$ we have either $W\cap H=1$ or $W\cap H=\langle\pi\rangle$. In the second case we get by (1.4) that $W=0_{3'}(W)\langle\pi\rangle$ and hence $C_G(z)=WH$ by the Frattini's argument. But this is not possible since $2^{21}\,\big|\,|C_G(z)|$. So $W\cap H=1$. Then W is nilpotent by (1.2) and we have $W=\langle C_W(x)\,|\,1\neq x\in M\rangle$ by (1.5). Since G has exactly one conjugacy class of elements of order 3, $C_W(x)$ is conjugate to a subgroup of H. Since π operates regularly on $C_W(x)$ for any $x\in M^{\sharp}$ we get that $C_W(x)$ is cyclic of order 7 or 1. Since P normalizes W and acts nontrivially on $M-\langle\pi\rangle$ we get that $Z(W)=W$ is elementary abelian of order 7^3 or 1. In any case H' centralizes W. This implies that $W=1$.

So we can apply [8; Theorem B] and see that either $|G|=|M(24)'|$ or $G\cong J_4$. But the first case is not possible since $3^{16}\,\big|\,|M(24)'|$. So G is isomorphic to J_4. This completes the proof of the lemma and the proof of Theorem A.

3. Proof of Theorem B

A slight modification of the proof of Theorem A gives Theorem B. We shall only indicate where differences are to be made.

Let G be a simple group which is not 3-normal and contains an element π such that $C_G(\pi)$ is isomorphic to the triple cover of M_{22}. Then Lemma (1.10) is valid for G where H is to be replaced by $H/\langle z\rangle$. We shall use the same notation as in the second section which was introduced in (1.10) with their corresponding new meanings. Then we have

Lemma 3.1. *We have $N_G(M)/C_G(M)\cong GL(2,3)$ and $N_G(M)$ is contained in $N_G/(E_0)$ where $E_0=0_2(C_G(M))$ is a four group.*

Proof. The same as in (2.1).

Lemma 3.2. *We have $C_G(E_1)=0_2(C_G(E_1))\langle\pi\rangle$ where either $0_2(C_G(E_1))=E_1$ or $0_2(C_G(E_1))$ is elementary abelian of order 2^{10}.*

Proof. The same as in (2.2).

Lemma 3.3. *If $0_2(C_G(E_1))=E_1$ then T is an S_2-subgroup of G.*

Proof. The same as in (2.3).

Lemma 3.4. *T is not an S_2-subgroup of G and hence $0_2(C_G(E_1))$ is of order* 2^{10}.

Proof. The argument we have used in (2.4) to show that $C_G(z)$ contains a subgroup C_0 with index two applies also to this case and yields that G has a subgroup with index two. But this is a contradiction since G is simple.

Conclusion 3.5. *G does not exist.*

Proof. Otherwise we get as in (2.6) that $N_G(K)/K$ is isomorphic to J_2 or M_{24} or R, where $K=0_2(C_G(E_1))$ is elementary abelian of order 2^{10}. But M_{24} and R cannot operate faithfully on a 2-group of order 2^{10}. Since J_2 is 3-normal by [5] we obtain a contradiction since we can see that $N_G(K)$ is not 3-normal as in (2.6).

This completes the proof of Theorem B.

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