We have defined a new class of rings in [3] which we call self mini-injective rings and we have noted that there exist artinian rings in the new class which are not quasi-Frobenius rings (briefly QF-rings).

We shall show in this note that if a ring $R$ is an algebra over a field with finite dimension, then a self mini-injective algebra is a QF-algebra.

Throughout this note we assume a ring $R$ contains the identity and every module is a unitary right $R$-module. We shall refer for the definitions of mini-injectives and the extending property, etc. to [3].

Let $K$ be a field and $R$ a $K$-algebra with finite dimension over $K$.

**Theorem 1** (cf. [3], Theorems 13 and 14). Let $R$ be as above. Then the following conditions are equivalent.

1) $R$ is self mini-injective as a right $R$-module.
2) $R$ is self mini-injective as a left $R$-module.
3) Every projective right $R$-module has the extending property of direct decomposition of the socle.
4) Every projective left $R$-module has the extending property of direct decomposition of the socle.
5) $R$ is a QF-algebra.

Proof. $R$ is self-injective as a left or right $R$-module if and only if $R$ is a QF-algebra by [2]. In this case $R$ is self-injective as both a right and left $R$-module by [1]. It is clear from [3], Theorem 3 and Proposition 8 that 1), 2) are equivalent to 3), 4), respectively. Hence, we may assume $R$ is a basic algebra by [4] and [6].

1) $\rightarrow$ 5). Let $R=\sum_{i=1}^{n} \oplus e_i R$ be the standard decomposition, namely $\{e_i\}$ is a set of mutually orthogonal primitive idempotents and $e_i R \cong e_i' R$ if $i \neq i'$. Since $R$ is right self mini-injective, $R$ is right QF-2 by [3], Proposition 8 and $S(e_i R) \cong S(e_i R)$ for $i \neq i'$ by [3], Theorem 5, where $S(\ )$ means the socle. Now $e_i R$ is uniform as a right $R$-module and so the injective envelope $E(e_i R)$ of $e_i R$ is indecomposable. We put $M^\ast = \text{Hom}_K(M, K)$ for a $K$-module $M$. Then $E(e_i R)^\ast$
is indecomposable and projective as a left $R$-module. Hence, $E(e_i|R)^* \cong R e_i$, and $E(e_i|R) \cong (Re_i)^*$. From the fact $E(e_i|R) \cong E(e_j|R)$ for $i \neq j$, a mapping $\pi: i \mapsto i'$ is a permutation on $\{1, 2, \cdots, n\}$. Accordingly, \[ \sum_{i=1}^n [E(e_i|R): K] = \sum_{i=1}^n [Re_{i'}: K] = [R: K]. \] Therefore, $E(R) = \sum_{i=1}^n E(e_i|R) = R$.

The remaining part is clear.

In the above proof we have used only the facts that $R$ is right QF-2 and $S(e_i|R) \cong S(e_i'R)$ if $e_i \neq e_i'$, where $e$ and $e'$ are primitive idempotents, where $J$ is the Jacobson radical of $R$.

**Theorem 2.** Let $R$ be a $K$-algebra as above. If $R$ is right QF-2 and $S(e_i|R) \cong S(e_i'R)$ if $e_i \neq e_i'$ then $R$ is QF, where $e$ and $e'$ are primitive idempotents, where $J$ is the Jacobson radical of $R$.

**Corollary.** Let $R$ be the $K$-algebra as above. We assume $R/J$ is a simple algebra. Then $R$ is a QF-algebra if and only if $R$ is a right QF-2 algebra.

We note that the above facts are not true for right and left artinian rings (see [3], Example 2).

Next we shall consider a characterization of a right artinian and self mini-injective ring.

**Theorem 3.** Let $R$ be a right artinian ring. Then the following conditions are equivalent.

1) $R$ is self mini-injective as a right $R$-module.
2) $R$ satisfies
   i) if $e_i R \cong e_i R$, any minimal right ideal in $e_i R$ is not isomorphic to one in $e_2 R$.
   ii) there exists a minimal right ideal $I$ contained in $e_1 J$ such that $\text{End}_R(I) = \{a \in e_1 Re_1 | aI \subseteq I\}$, i.e. $\text{End}_R(I)$ is extended to $\text{End}_R(e_i R)$ and $S(e_i R) = e_i Re_1 I$ for each $e_i$, where the $e_i$ is primitive idempotent and $S(\cdot)$ is socle and $R = R/J$.

**Proof.** 1) $\rightarrow$ 2). It is clear from [3], Theorem 5. 2) $\rightarrow$ 1). The second part of ii) implies that each minimal right ideal $I'$ in $e_2 R$ is isomorphic to $I$. We assume $I \cong e_2 R$ and $I = xR$, $I' = x'R$. Then we may assume $x e_2 = x$ and $x' e_2 = x'$. We obtain from ii) that $x' = x' e_2 = \sum y_i x r_i$, $y_i \in e_1 R e_1, r_i \in R e_2$. Now $x r_i e_2 = x e_2 r_i = x e_2 r_i$. Since a mapping $xz \mapsto x e_2 r_i z$ is an $R$-endomorphism of $I$, there exists an element $a_i$ in $e_i Re_1$ with $a_i x = x e_2 r_i$ from ii). Hence, $x' = \sum y_i a_i x = \sum y_i a_i x$. We quote the proof of [3], Proposition 9. Since $\bar{b} = 0$, $x = \bar{b}^{-1} x'$. Put $g(x' x) = z x$; $z \in R$. Let $f$ be any element in $\text{Hom}_R(I, I')$. Then $gf \in \text{End}_R(I)$. Hence, there exists $a$ in $e_1 Re_1$ such that $gf(x) = ax$ by ii). Therefore, $f(x) = g^{-1}(ax) = b ax$ and $f$ is extended to an element in $\text{End}_R(e_i R)$. We know similarly that $\text{End}_R(I') = \bar{b}^{-1} \text{End}_R(I) \bar{b} = \{e \in e_1 Re_1 | e I' \subseteq I'\}$.
Hence, \( I' \) satisfies ii). Let \( h \in \text{Hom}_R(I', R) \) and \( R = \bigoplus_{i=1}^\infty e_i R \). Let \( \pi_i : R \to e_i R \) be the projection. If \( \pi_i h = h \neq 0 \) for \( i = 1, 2, \ldots, t \) and \( h_j = 0 \) for \( j > t \). Since \( e_i R \approx e_t R \) for \( i \leq t \), there exists \( c_i \in e_t R e_i \) and \( d_i \in e_i R e_i \) such that \( c_id_i = e_i \) and \( d_ice_i = e_i \). Using \( \langle f, \rangle \) and \( c_i \), we know as above that any element in \( \text{Hom}_R(I', h(I')) \) is extended to an element in \( \text{Hom}_R(e_i R, e_i R) \) for \( i \leq t \). Take \( p_i \in R \) such that \( p_i x' = h_i(x') \). Then \( h(x') = \sum h_i(x') = (\sum p_i) x' \). Hence, \( R \) is right self mini-injective by [3], Theorem 2.

**Remark.** The above three conditions in Theorem 3, 2) are independent.

**Corollary 1.** Let \( R \) be a right artinian and right self mini-injective. Then \( R \) is a right QF-2 if and only if \( \text{End}_R(I) = e_i R e_i \) in ii) of Theorem 3.

**Corollary 2.** Let \( R \) be a right artinian ring and \( e \) a primitive idempotent. We assume that i) \( R \) is right QF-2, ii) any monomorphism of \( eRe \) into itself as a division ring is isomorphic for each \( e \) (e.g. algebraic extension of the prime field) and iii) \( S(eR) \approx S(e'R) \) if \( eR \approx e'R \). Then \( R \) is right self mini-injective.

Proof. We may assume \( R \) is basic. Since \( S(eR) \supseteq eJ^k \neq 0 (eJ^{k+1} = 0) \), \( S(eR) = eJ^k \) by i). Put \( S(eR) = uR \). \( eJ^k u = eJ^{k+1} = 0 \) and so \( uR \) is a left \( eR \)-module. We assume \( uR \approx e'R \). Since \( R \) is basic, \( e'R = e'R \). Hence, \( uRe'R = uR \). Let \( x \) be in \( eR \). Then \( xu = u\bar{y} \); \( \bar{y} \in e'R e' \). It is clear that the mapping \( x \to \bar{y} \) gives us a monomorphism of the division ring \( eRe \) into \( e'R e' \) as a division ring. Repeating this procedure, we can find a chain \( e, e', \ldots, e^{(s)} \) of primitive idempotents. We know from iii) that \( e^{(s)} = e \) for some \( s \) (cf. [3], the proof of Proposition 8). Hence, \( eRe = ue'R e' \) by ii). Therefore, \( R \) is right self mini-injective by Theorem 3.

We do not know any example of a right QF-2 and right self mini-injective ring which is not QF.

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**References**


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