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ON SEMI-FIELD PLANES OF EVEN ORDER

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1. Introduction

Let π be a non-Desarguesian semi-field plane with an autotopism group G and let $u(\pi)$ denote the number of the orbits of G on the points not incident with any side of the autotopism triangles.

In their paper [9], M.J. Kallaher and R.A. Liebler have conjectured that $u(\pi)\geq 5$ and they have proved that the conjecture is true if G is solvable and the order of π is not 2^6.

In this paper we treat semi-field planes of even order whose autotopism groups are not necessarily solvable and prove the following.

Theorem 1. *Let π be a non-Desarguesian semi-field plane of order 2^r. If r is not divisible by 4, then $u(\pi)\geq 5$.*

The proof requires the use of the Kallaher-Liebler's theorem mentioned above and the following lemma which we prove in section 3.

Lemma 2. *Let π be a non-Desarguesian semi-field plane of order 2^6 with a solvable autotopism group. Then $u(\pi)\geq 5$.*

2. Notations and preliminaries

Our notation is largely standard and taken from [3] and [6]. Let G be a permutation group on Ω. For $X\leq G$ and $\Delta\subset\Omega$, we define $F(X)=\{\alpha\in\Omega\mid\alpha^x=\alpha$ for all $x\in X\}$, $X(\Delta)=\{x\in X\mid\Delta^x=\Delta\}$, $X_\Delta=\{x\in X\mid\alpha^x=\alpha$ for all $\alpha\in\Delta\}$ and $X^\Delta=X(\Delta)/X_\Delta$, the restriction of X on Δ. When X is a collineation group of a projective plane, we denote by $F(X)$ the set of fixed points and fixed lines of X.

Lemma 2.1. *Let G be a transitive permutation group on a finite set Ω, H a stabilizer of a point of Ω and M a nonempty subset of G. Then $|F(M)|=|N_G(M)|\times|ccl_G(M)\cap H|/|H|$. Here $ccl_G(M)\cap H=\{g^{-1}Mg\mid g^{-1}Mg\subset H, g\in G\}$.*

Proof. Set $W=\{(L, \alpha)\mid L\in ccl_G(M), \alpha\in F(L)\}$ and $W_\alpha=\{L\mid L\in ccl_G(M),$

$\alpha\in F(L)\}$. By the transitivity of G, $|W_\alpha|=|W_\beta|$ holds for every α, $\beta\in\Omega$. Counting the number of elements of W in two ways, we obtain $|G: N_G(M)|\times|F(M)|=|G: H|\times|ccl_G(M)\cap H|$. Thus we have the lemma.

Lemma 2.2. *Let $PG(2,q)$ denote the Desarguesian projective plane of order q where $q=2^n$ and $n\equiv 1 \pmod 2$. Set $Y=PSL(3, q)$ and $X=\langle f\rangle Y$, where f is a field automorphism of Y of order n. Set $G=X_{P,Q,R}$ and $N=G\cap Y$, where $P=[1, 0, 0]$, $Q=[0, 1, 0]$ and $R=[0, 0, 1]$.*

(i) *Let A be a noncyclic abelian p-subgroup of G of order p^2 for a prime p. Then A is not semi-regular on the set of points contained in $PG(2, q)$-$F(A)$.*

(ii) *Let C be a cyclic subgroup of G of order q-1. Then $C\subset N$.*

Proof. Since $A\cap N\neq 1$ and $N\simeq Z_{q-1}\times Z_{q-1}$, p is an odd prime. Let T be the translation group with respect to the line g joining [1, 0, 0] and [0, 1, 0]. Deny (i) and let Ω denote the set of points in $F(A)$. Then, by Theorem 5.3.6 of [3], $T=\langle C_T(x)\,|\,1\neq x\in A\rangle$. By the semi-regularity of A, $C_T(x)$ acts on Ω for each $x\in A-\{1\}$. Hence T acts on Ω.

Let Δ denote the set of points not incident with the line g. Clearly $[0, 0, 1]\in\Delta\cap\Omega$. Since T is transitive on Δ, we have (i).

Set $D=C\cap N$ and let $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ be a generator of D. Then $C\triangleright D$ and $C/D\simeq CN/N\leq G/N\simeq Z_n$ and so $|D|\geq(q-1)/n$. Set $\langle h\rangle=C_{\langle f\rangle}(D)$ and $s=|\langle h\rangle|$. Then $n=r\times s$ for an integer r. It follows that ba^{-1}, $ca^{-1}\in GF(2^r)^\times$. Hence $|D|\leq 2^r-1$. From this, $2^r-1\geq|D|\geq(q-1)/n$. We can easily verify that $s=1$. Therefore $C_{\langle f\rangle}(D)=1$, whence $C\leq C_G(D)=NC_{\langle f\rangle}(D)=N$. Thus $C\leq N$.

In the rest of the paper we assume the following.

Hypothesis 2.3. *Let π be a non-Desarguesian semi-field plane of order 2^r coordinatized by a semi-field D with respect to the points $U_1=(0, 0)$, $U_2=(0)$, $U_3=(\infty)$ and let G be the autotopism group of π with respect to U_1, U_2, U_3. Let l_i be the line joining U_j and U_k for i, j, k with $\{i, j, k\}=\{1, 2, 3\}$ and let $\Phi(\pi)$ be the set of points of π not incident with l_1, l_2 or l_3. Let $u(\pi)$ denote the number of G-orbits on $\Phi(\pi)$. Set $K_i=G_{(U_i,l_i)}$ for $1\leq i\leq 3$ and let N_1, N_2 or N_3 be the right, middle or left nucleus, respectively.*

D may be considered as a right vector space over N_1 or N_2 and as a left vector space over N_2 or N_3. The multiplicative group $N_i^\times$ is isomorphic to K_i for each i with $1\leq i\leq 3$ (Chapter 8 of [6]). Set $\bar{l}_i=l_i-\{U_j, U_k\}$ for i, j, with $\{i, j, k\}=\{1, 2, 3\}$.

3. The proof of Lemma 2.

Throughout this section π is a projective plane satisfying the hypothesis 2.3 and the following.

Hypothesis 3.1. (i) *The order of π is 2^6.*
(ii) *Set $u=u(\pi)$. Then $u\leq 4$.*
(iii) *The autotopism group G is solvable.*

Lemma 3.2. $|K_t|=1$, 3 *or* 7 *for every* $t\in\{1, 2, 3\}$ *and* $u=3$ *or* 4.

Proof. Since π is non-Desarguesian, D is not a field. Hence, N_t is isomorphic to $GF(2)$, $GF(4)$ or $GF(8)$ for $t\in\{1, 2, 3\}$. By Theorem 8.2 of [6], $|K_t|=1$, 3 or 7.

By Corollary 4.1.1 of [9] and Hypothesis 3.1 (ii), $u=3$ or 4.

Lemma 3.3. *If G is transitive on $\bar{l}_t$ for some $t\in\{1, 2, 3\}$, then the following hold.*

(i) *$G/K_t\leq\Gamma L(1, 2^6)$ and G/K_t contains an element of order* 9.

(ii) *Let m be an arbitrary line through U_t such that $m\neq l_j$, l_k for $\{t, j, k\}=\{1, 2, 3\}$. Set $A=m\cap l_t$. Then $G_m=G_A$, $|G: G_A|=3^2\cdot 7$ and the number of G_A-orbits on m-$\{U_t, A\}$ is equal to u.*

(iii) *Let $\Delta_1, \Delta_2, \cdots, \Delta_u$ be the orbits stated in* (ii). *Set $x_s=|\Delta_s|$, $1\leq s\leq u$, and assume that $x_1\leq x_2\leq\cdots\leq x_u$. Then $|G_A|$ is divisible by x_s for every s and $6\times|K_t|$ is divisible by $|G_A|$. Furthermore $\sum_{s=1}^{u} x_s=63$.*

Proof. By Lemma 2.1 of [9], G is a transitive linear group on D. Hence it follows from a Huppert's theorem ([7]) that $G/K_t\leq\Gamma L(1, 2^6)$. If G/K_t contains no element of order 9, then its Sylow 3-subgroup is an elementary abelian 3-subgroup of order at most 9. By the structure of $\Gamma L(1, 2^6)$, G/K_t is not a transitive linear group, a contradiction. Thus G/K_t contains an element of order 9 and (i) holds.

Let m, A be as in (ii). Since G fixes U_t and l_t, we have $G_m=G_A$. Clearly $|G: G_A|=|A^G|=|\bar{l}_t|=2^6-1=3^2\cdot 7$. As any point of $\Phi(\pi)$ lies on a line of $[U_t]-\{l_j, l_k\}$, $\Phi(\pi)\cap m\,(=m-\{U_t, A\})$ is a union of u G_A-orbits, hence (ii) holds.

Since $G/K_t\leq\Gamma L(1, 2^6)$, $G_A/K_t\leq Z_6$. Hence $6\times|K_t|$ is divisible by $|G_A|$. Clearly $x_s=|\Delta_s|$ divides $|G_A|$ and $\sum_{s=1}^{u} x_s=|\Delta_1\cup\Delta_2\cup\cdots\cup\Delta_u|=|\bar{l}_t|=2^6-1=3^2\cdot 7$. Thus (iii) holds.

Lemma 3.4. *Suppose $u=4$. Then there exists $i\in\{1, 2, 3\}$ having the following properties:*

(i) *G is transitive on* $\bar{l}_i$,

(ii) K_i *is isomorphic to* Z_7 *and G has a normal Sylow* 7*-subgroup and*

(iii) $|G: G_A|=63$, G_A/K_i *is isomorphic to* Z_6 *and* $C_{G_A}(K_i)=K_i$ *for each* $A\in\bar{l}_i$.

Proof. By Lemma 6.1 of [9], there exists $i\in\{1, 2, 3\}$ such that G is transitive on $\bar{l}_i$. Assume that $K_i\neq Z_7$. Then $K_i\leq Z_3$ by Lemma 3.2. Let m, A, x_s be as in Lemma 3.3. We have $x_s \Big| 6|K_i|=6$ or 18 and $x_1+x_2+x_3+x_4=63$, hence $|K_i|=3$, $|G_A|=18$ and $(x_1, x_2, x_3, x_4)=(9, 18, 18, 18)$.

Let z be an involution in G_A. Then z is a Baer involution and so $|F(z)\cap(m-\{U_i, A\})|=7$ because $m\in F(z)$. If $F(z)\cap\Delta_s\neq\phi$, then $|\Delta_s|\leqq\frac{1}{2}|G_A|$. In particular $F(z)\cap\Delta_s=\phi$ for $s\geq 2$ and so $|F(z)\cap\Delta_1|=7$. Since $G_A/K_i\simeq Z_6$ and $z\notin K_i$, $C_{G_A}(z)\neq\langle z\rangle$. Hence an element of $C_{G_A}(z)$ of order 3 acts on $F(z)\cap\Delta_1$ and fixes at least one point on it. It follows that $|\Delta_1|\leq\frac{1}{6}|G_A|=3$, a contradiction. Therefore we have $K_i\simeq Z_7$ and so G has a normal Sylow 7-subgroup by Lemma 3.3. Thus (ii) holds.

Let $m\,(=U_iA)$, Δ_s, x_i for $t=i$ be as in Lemma 3.3 (ii). Since $G_A\geq K_i\simeq Z_7$ and K_i acts semi-regularly on m-$\{U_i, A\}$, $7\Big|\,|\Delta_s|=x_s$ for all $s\in\{1, 2, 3, 4\}$. Moreover, by Lemma 3.3, $x_1+x_2+x_3+x_4=63$. Hence $(x_1, x_2, x_3, x_4)=(7, 7, 7, 42)$ $(7, 14, 21, 21)$ or $(14, 14, 14, 21)$ and so $|G_A|=42$. Thus $G_A/K_i\simeq Z_6$ by the similar argument as in the proof of Lemma 3.3 (iii). Let y be an element of $C_{G_A}(K_i)$ and assume that the order of y is 2 or 3. Since $G_A/K_i\simeq Z_6$ and $K_i\simeq Z_7$, y is contained in the center of G_A. Hence G_A acts on $F(y)$ and therefore Δ_s is contained in $F(y)$ for each s with $|\langle y\rangle|\not| x_s$. As above, $(x_1, x_2, x_3, x_4)=(7, 7, 7, 42)$, $(7, 14, 21, 21)$ or $(14, 14, 14, 21)$ and hence $|F(y)\cap m|\geq 21+2=23$. Since $F(y)\cap\Phi(\pi)\neq\phi$, y is a planar collineation, Therefore $y=1$, a contradiction. Thus $C_{G_A}(K_i)=K_i$.

Lemma 3.5. *Suppose* $u=4$ *and let notations be as in Lemma* 3.4. *Then, for some* $s\in\{1, 2, 3\}-\{i\}$ $O(G)$ *has no orbit of length* 7 *on* l_s.

Proof. Suppose false. Let P be a Sylow 7-subgroup of G. By Lemma 3.4 (ii), $|P|=7^2$ and P is a normal subgroup of G. Let $s\in\{1, 2, 3\}-\{i\}$ and let Ω_1 be a P-orbit of length 7 on l_s. Then there exists another P-orbit of length 7, say Ω_2, on l_s because $7^2\not|\,|\bar{l}_s-\Omega_1|$.

Let Q be a Sylow 3-subgroup of $O(G)$. By Lemmas 3.3 and 3.4, $K_i\simeq Z_7$ and a Sylow 3-subgroup of G/K_i is isomorphic to that of a Sylow 3-subgroup of $\Gamma L(1, 2^6)$. Hence $Q=\langle a, b\,|\,a^9=b^3=1,\ b^{-1}ab=a^4\rangle$ for suitable a, b in Q. We note that $Q'=[Q, Q]=\langle a^3\rangle$.

Since $|\Omega_1|=|\Omega_2|=7<9$, a^3 acts trivially on $\Omega_1\cup\Omega_2$, hence $|F(a^3)\cap l_s\geq 2+|\Omega_1|+|\Omega_2|=16$. As $s\,(\in\{1, 2, 3\}-\{i\})$ is arbitrary, a^3 is planar and moreover we have $F(a^3)=\pi$, by Theorem 3.7 of [6], which implies that $a^3=1$. This is a contradiction. Thus we have the lemma.

Lemma 3.6. $u=3$.

Proof. Assume that $u\neq 3$. Then, by Lemma 3.2, $u=4$ and we can apply Lemmas 3.4 and 3.5. Let notations be as in them.

Let P be a Sylow 7-subgroup of G and Γ the set of P-orbits on $\bar{l}_s$. Set $H=O(G)$. Since P is a normal subgroup of H by Lemma 3.4 (ii), H induces a permutation group on Γ. Since $P\geq K_i$ and K_i is semi-regular on $\bar{l}_s$, every P-orbit in Γ has length 7 or 7^2. If an orbit in Γ has length 7^2, Γ contains exactly two P-orbits of length 7, which are also H-orbits of length 7, contrary to Lemma 3.5. Therefore each P-orbit in Γ has length 7 and so $|\Gamma|=9$.

If H acts transitively on Γ, G is transitive on $\bar{l}_s$ and therefore $G/K_s\leq \Gamma L(1, 2^6)$ by Lemma 3.3 (i). It follows that $\Gamma L(1, 2^6)\geq G_A K_s/K_s\simeq G_A/G_A \cap K_s\simeq G_A$. Therefore an involution in G_A centralizes a Sylow 7-subgroup of G_A by the structure of $\Gamma L(1, 2^6)$, contrary to Lemma 3.4 (iii). Hence H is not transitive on Γ.

Let Q be a Sylow 3-subgroup of H. Then $|Q|=27$ and $[Q, Q]=Q'\simeq Z_3$ as in the proof of Lemma 3.5. Since $H=PQ$, $\Gamma^H=\Gamma^Q$. On the other hand H is not transitive on Γ. Hence Q^Γ is abelian and therefore Q' acts trivially on Γ. We note that $G/C_G(P)\leq Z_6$ or $G/C_G(P)\leq GL(2, 7)$ according as $P\simeq Z_{49}$ or $Z_7\times Z_7$, respectively. Hence Q' is contained in $C_G(P)$. Since Q' acts trivially on Γ and each orbit $\Delta\in\Gamma$ is of length 7, $F(Q')\cap\Delta\neq\phi$. Therefore $Q'\leq K_s$ because $[P, Q']=1$. In particular Q' is semi-regular on $\bar{l}_j$, where $\{j\}=\{1, 2, 3\}-\{i, s\}$. Hence QK_i is transitive on $\bar{l}_j$. By Lemma 3.3 (i), $G/K_j \simeq\Gamma L(1, 2^6)$ and $K_j\simeq Z_7$. Let z be an involution in G_A. Then $[z, P]\leq K_i \cap K_j=1$ and so $z\in C_{G_A}(K_i)$, contrary to Lemma 3.4 (iii). This we have $u=3$.

Lemma 3.7. *Assume that there exists a line l through U_i with $l\neq l_j, l_k$, where $\{i, j, k\}=\{1, 2, 3\}$, such that G_l acts transitively on $l-\{U_i, l\cap l_i\}$. Then the following hold.*

(i) *G_l is transitive on $\bar{l}_t$ for $t=j, k$.*

(ii) *G has two or three orbits on $\bar{l}_i$.*

Proof. Let $A_1, A_2\in\bar{l}_j$ and set $B_1=U_jA_1\cap l$ and $B_2=U_jA_2\cap l$. By assumption, there exists an element $x\in G_l$ such that $B_1^x=B_2$. Since $U_jA_2\cap l=B_2 =B_1^x=U_jA_1^x\cap l$ and $A_2, A_1^x\in\bar{l}_j$, it follows that $A_1^x=A_2$. Hence G_l is transitive on $\bar{l}_j$. Similarly G_l is transitive on $\bar{l}_k$. Thus (i) holds.

Let d be the number of G-orbits on $\bar{l}_i$. Clearly d is at most 3. If $d=1$, G acts transitively on $\Phi(\pi)$, contrary to $u=3$. Thus (ii) holds.

Lemma 3.8. *Let l be the line satisfying the assumption in Lemma 3.7. If $7^2\,\big|\,|G|$ and $7^3\nmid|G|$, then $K_i\simeq Z_3$ and $|G|\,\big|\,2\cdot 3^2\cdot 7^2$.*

Proof. By Lemmas 3.2, 3.3 (i) and 3.7 (i), K_j and K_k are isomorphic to Z_7; otherwise $7^2 \not| |G|$. Set $A=l\cap l_i$. Then $G_l=G_A$ and so $G_l/K_i=G_A/K_i$. Since $G/K_i\leq GL(6,2)$, G_l/K_i is isomorphic to a subgroup of L, where

$$L=\left\{\begin{bmatrix} 1 & a_2\cdots a_6 \\ 0 & \\ \vdots & M \\ 0 & \end{bmatrix} \middle| a_2, \cdots, a_6\in GF(2), M\in GL(5,2)\right\}.$$

Since $L/O_2(L)\simeq GL(5, 2)$, a Sylow 3-subgroup of L is an elementary abelian group of order 9. On the other hand, by Lemmas 3.3 and 3.7 (i), G_l contains an element of order 9. Therefore $K_i\simeq Z_3$.

For a subgroup X of G, $\bar{X}$ denotes the homomorphic image of X in G/K_i. Since $\bar{K}_j\neq\bar{K}_k$ and $\bar{G}\leq GL(6, 2)$, $\bar{K}_j\times\bar{K}_k$ is a Sylow 7-subgroup of $\bar{G}$ ans so $\bar{K}_j\times\bar{K}_k$ has two subgroups $\langle\bar{a}\rangle$ and $\langle\bar{b}\rangle$ of order 7 which fix nonzero vectors on $\bar{l}_i$. Set $H=O(G)$. By Lemmas 3.3 (i) and 3.7 (i), $G/K_t\leq\Gamma L(1, 2^6)$ for $t\in\{j, k\}$, so that $|G: H|\leq 2$. Since $\bar{G}\lhd\bar{K}_t$ for $t\in\{j, k\}$, $\bar{H}$ normalizes $\bar{K}_j$, $\bar{K}_k$, $\langle\bar{a}\rangle$ and $\langle\bar{b}\rangle$. As K_t acts semi-regularly on $\bar{l}_i$, we have $\bar{K}_t\neq\langle\bar{a}\rangle$, $\langle\bar{b}\rangle$ for $t\in\{j, k\}$. Without loss of generality, we can assume that $\langle\bar{a}\bar{b}\rangle=\bar{K}_j$. Let $\bar{g}\in\bar{H}$. Then $\bar{g}^{-1}\bar{a}\bar{g}=\bar{a}^p$ and $\bar{g}^{-1}\bar{b}\bar{g}=\bar{b}^q$ for some p, q with $1\leq p, q\leq 6$, so we have $\bar{g}^{-1}\bar{a}\bar{b}\bar{g}=\bar{a}^p\bar{b}^q\in\bar{K}_j=\langle\bar{a}\bar{b}\rangle$. Hence $p=q$. From this, $\bar{H}/C_{\bar{H}}(\langle\bar{a}\rangle\times\langle\bar{b}\rangle)\leq O(\mathrm{Aut}(Z_7))\simeq Z_3$. Since $C_{GL(6,2)}(\langle\bar{a}\rangle\times\langle\bar{b}\rangle)=\langle\bar{a}\rangle\times\langle\bar{b}\rangle$, we have $|\bar{H}| \,\big|\, 3|\langle\bar{a}\rangle\times\langle\bar{b}\rangle|=3\cdot 7^2$ and therefore $|H| \,\big|\, 3^2\cdot 7^2$. Thus we obtain $|G| \,\big|\, 2\cdot 3^2\cdot 7^2$.

Lemma 3.9. *Let $i\in\{1, 2, 3\}$ and set $\{i, j, k\}=\{1, 2, 3\}$. Then the following hold.*

(i) *For every line $m\in[U_i]-\{l_j, l_k\}$, G_m has three orbits on $m-\{U_i, m\cap l_i\}$.*

(ii) *G acts transitively on $\bar{l}_i$ and $G/K_i\leq\Gamma L(1, 2^6)$.*

Proof. Deny (i). Then, since $u=u(\pi)=3$, there exists a line $l\in[U_i]$ satisfying the assumption of Lemma 3.7. Let $\{\Omega_1, \Omega_2, \cdots, \Omega_p\}$ be the set of G-orbits on $\bar{l}_i$ and set $b_s=|\Omega_s|$ for $1\leq s\leq p$. By Lemma 3.7 (ii), $p=2$ or 3.

Assume $p=3$. Set $b=\max\{b_1, b_2, b_3\}$, $b=|\Omega_v|$ and let $A\in\Omega_v$. Since $u=3$, G_A is transitive on $m-\{U_i, A\}$, where $m=AU_i$. Therefore $63 \mid |G_A|$. Hence $63b \,\big|\, |G|$ because $|G|=b|G_A|$. By Lemmas 3.2, 3.3 (i) and 3.7 (i), we have $|G| \,\big|\, 2\cdot 3^4\cdot 7^2$ and so $b \,\big|\, 2\cdot 3^2\cdot 7$. Since $3b\geq b_1+b_2+b_3=63$, it follows that $21\leq b<63$, hence $b=21$ or 42 and $3^3\cdot 7^2 \,\big|\, |G|$, contrary to Lemma 3.8. Thus $p\neq 3$.

Assume $p=2$. Let $A\in\Omega_1$, $B\cup\Omega_2$ and set $g=AU_i$, $h=BU_i$. Since $u=3$, without loss of generality we may assume that G_A is transitive on $g-\{U_i, A\}$ and that G_B has two orbits on $h-\{U_i, B\}$, say Γ_1, Γ_2. Similarly as in the last paragraph we obtain the following:

$$b_1, b_2 \,\Big|\, |G|,\ |G| \,\Big|\, 2\cdot 3^4\cdot 7^2,\ b_1+b_2=63\,.$$

Hence $\{b_1, b_2\}=\{21, 42\}$, $\{14, 49\}$ or $\{9, 54\}$. We note that $|G: G_g|=|G: G_A|=b_1$, $|G: G_h|=|G: G_B|=b_2$ and $63 \mid |G_A|$.

If $\{b_1, b_1\}=\{21, 42\}$, $|G|=|G_A|b_1$ and $21 \mid b_1$. Hence $3^3 \cdot 7^2 \mid |G|$, contrary to Lemma 3.8.

If $\{b_1, b_2\}=\{14, 49\}$, $|G: G_A|=14$ because $7^3 \nmid |G|$. Hence $|G: G_h| =49$. By Lemma 3.8, $|G| \mid 2 \cdot 3^2 \cdot 7^2$. Therefore $|G_h| \mid 18$. Since $h-\{U_i, B\}$ is a union of G_h-orbits Γ_1, Γ_2, we have $|\Gamma_1|+|\Gamma_2|=63$ and $|\Gamma_1|, |\Gamma_2| \mid 18$. This is a contradiction.

If $\{b_1, b_2\}=\{9, 54\}$, we have $|G: G_A|=9$ as $3^5 \nmid |G|$. Hence $3^4 \mid |G|$ and so $7^2 \nmid |G|$ by Lemma 3.8. Therefore $|G|=2 \cdot 3^4 \cdot 7$. From this, $|G_h| =|G_B|=21$. Hence $|\Gamma_1|, |\Gamma_2| \mid 21$. However, $|\Gamma_1|+|\Gamma_2|=63$, a contradiction. Thus we have (i), and (ii) follows immediately from (i).

By Lemma 3.9, we can apply Lemma 3.3 for every $t \in \{1, 2, 3\}$ and obtain the following.

Lemma 3.10. *Let notations be as in Lemma* 3.3. *Then the following hold.*
(i) $3^2 \cdot 7 \mid |G|$, $|G| \mid 2 \cdot 3^4 \cdot 7^2$ *and* $3^4 \cdot 7^2 \nmid |G|$.
(ii) 3^3, $7^2 \nmid x_s$ *for all* $s \in \{1, 2, 3\}$.

Proof. By Lemmas 3.2 and 3.3 (i) (ii), we have (i). By Lemma 3.3 (ii) (iii), $|G_A|=|G|/63$ and $x_s \mid |G_A|$. Hence $x_s \mid 2 \cdot 3^2 \cdot 7$. Thus we have (ii).

Lemma 3.11. *Let notations be as in Lemma* 3.3 *and assume that* $21 \mid x_2$. *Then the following hold.*

(i) $K_1 \simeq K_2 \simeq K_3 \simeq Z_7$ *and* G/K_t *is isomorphic to a subgroup of* $\Gamma L(1, 2^6)$ *of index at most* 2 *for each* $t \in \{1, 2, 3\}$.

(ii) *Let* Q *be a Sylow* 3-*subgroup of* G. *Then* $|Q|=3^3$ *and* $Q=\langle a, b \mid a^9=b^3=1, b^{-1}ab=a^4\rangle$ *for suitable* a, b *in* Q. *Moreover, for any element* v *of order* 3 *in* $Q-Z(Q)$, $F(v)$ *is a subplane of order* 4.

Proof. By Lemma 3.3, x_3 divides $|G_A|$ and $|G: G_A|=3^2 \cdot 7$, so that $3^3 \cdot 7^2 \mid |G|$. It follows from Lemma 3.10 (i) that $|G|=3^3 \cdot 7^2$ or $2 \cdot 3^3 \cdot 7^2$. Therefore (i) holds.

By (i), the order of a Sylow 3-subgroup Q of G is 3^3. Hence Q is of the form stated in (ii) by the structure of $\Gamma L(1, 2^6)$. We note that Q has exactly two conjugacy classes of subgroups of order 3. Let $v \in Q-Z(Q)$ such that $\langle v\rangle \simeq Z_3$. Then, as an element in $\Gamma L(1, 2^6)$, v fixes three nonzero elements, that is, $|F(v) \cap \bar{l}_t|=3$ for all $t \in \{1, 2, 3\}$. Hence $F(v)$ is a subplane of order 4.

Lemma 3.12. *Let notations be as in Lemma* 3.3. *Then* $(x_1, x_2, x_3)=(7, 14, 42)$, $K_t \simeq Z_7$ *and* $G/K_t \simeq \Gamma L(1, 2^6)$ *for each* $t \in \{1, 2, 3\}$.

Proof. By Lemmas 3.3 (iii) and 3.10, we have $x_1 \leq x_2 \leq x_3$, $x_1+x_2+x_3=63$

and $3^3, 7^2 \not| x_s, x_s \Big| |G| \Big| 2\cdot 3^4\cdot 7^2$ for $s\in\{1, 2, 3\}$. Hence $(x_1, x_2, x_3)=(21, 21, 21)$ or $(7, 14, 42)$. On the other hand $K_1\simeq K_2\simeq K_3\simeq Z_7$ by Lemma 3.11 (i).

Assume that $(x_1, x_2, x_3)=(21, 21, 21)$. Let Δ_s be as defined in Lemma 3.3. Let $P_s\in\Delta_s$ and let Φ_s be the G-orbit containing P_s for $s\in\{-1, 2, 3\}$. Clearly $|\Phi_s|=63x_s$. Let v be the element as defined in Lemma 3.11 (ii) and let $P\in F(v)\cap\Phi$. Then $P\in\Phi_s$ for some $s\in\{1, 2, 3\}$. Therefore $|G_P|=|G|/|\Phi_s| \Big| 2\cdot 3^3\cdot 7^2/63\cdot 21=2$, contrary to $v\in G_P$. Thus $(x_1, x_2, x_3)=(7, 14, 42)$ and so $G/K_t\simeq\Gamma L(1, 2^6)$ for all $t\in\{1, 2, 3\}$.

Lemma 3.13. *Let Δ_1 be as in Lemma 3.3. Then the following hold.*

(i) *Let $P\in\Delta_1$. Then $G_P=\langle x\rangle\simeq Z_6$ and a Sylow 7-subgroup of G acts on $F(x^3)\cap\bar{l}_t$ for all $t\in\{1, 2, 3\}$.*

(ii) *$F(x^c)$ is a subplane of π of order 2^c for $c=2, 3$.*

Proof. Similarly as in the proof of Lemma 3.12, we obtain $|G_P|=|G|/|P^G|=2\cdot 3^3\cdot 7^2/63\cdot 7=6$. Since $G_P\leq G_A$, $G_P\cap K_t=1$ and $G_A/K_t\simeq Z_6$, we have $G_P\simeq G_PK_t/K_t\leq Z_6$. Hence $G_P\simeq Z_6$. Set $\langle x\rangle=G_P$. Clearly x^3 is an involution in G_P and so by the property of $\Gamma L(1, 2^6)$, x^3 centralizes the Sylow 7-subgroup of G/K_t for all $t\in\{1, 2, 3\}$. Let S be the Sylow 7-subgroup of G. Then $[x^3, S]\leq\bigcap_{t=1}^{3}K_t=1$ and therefore S centralizes x^3. Hence S acts on $F(x^3)\cap\bar{l}_t$ for all $t\in\{1, 2, 3\}$. Thus (i) holds.

By Theorem 4.3 of [6], $F(x^3)$ is a subplane of π of order 2^3 and by Lemmas 3.11 (ii) and 3.12, $F(x^2)$ is a subplane of order 2^2.

If we coordinatize π by choosing $(0, 0)$ as U_1, (0) as U_2, (∞) as U_3, $(1, 1)$ as P which was defined in Lemma 3.13, then we get a semi-field F. In general, F is not always isomorphic to D and since π is non-Desarguesian, F is not a field. Thus π is a semi-field plane coordinatized by F and it also satisfies Hypothesis 2.3.

Lemma 3.14. *Set $F_1=\{d\,|\,d\in F, (d, 0)\in F(x^3)\}$, $F_2=\{d\,|\,d\in F, (0, d)\in F(x^3)\}$ and $F_3=\{d\,|\,d\in F, (d, 0)\in F(x^2)\}$. Then $F_1=F_2\simeq GF(8)$ and $F_3\simeq GF(4)$.*

Proof. Since $F(x)$ contains $(0, 0)$, (0), (∞) and $(1, 1)$, it also contains (1). By Lemma 3.13 and the definition of the coordinatization of π, we have the lemma.

Lemma 3.15. *Let N_1, N_2 or N_3 be the right, middle or left nucleus, respectively. Then $N_1=N_2=N_3\simeq GF(8)$.*

Proof. By Lemmas 3.12, we have $N_t\simeq GF(8)$ for all $t\in\{1, 2, 3\}$. Furthermore, the multiplicative group $N_t^{\times}=\{d\,|\,(d, 0)\in(1, 0)^{K_t}\}$ for $t=1, 2$ and $N_3^{\times}=\{d\,|\,(0, d)\in(0, 1)^{K_3}\}$ by the proof of Theorems 7.9 and 8.2 of [6]. Since

K_1 and K_2 are semi-regular on $\bar{l}_3$ and $(1, 0)\in F(x^3)$, it follows from Lemma 3.13 that $N_1=N_2=F_1$. Similarly $N_3=F_2$. By Lemma 3.14, we have $N_1=N_2=N_3\simeq GF(8)$.

Lemma 3.16. *Set $N=N_1=N_2=N_3$ and $F_3^{\times}=\langle\theta\rangle$.*

(i) *N does not contain θ and F is a right and left vector space over N with a basis $\{1, \theta\}$.*

(ii) *For any $\xi\in F$, $(\xi\theta)\theta=\xi(\theta^2)$.*

Proof. (i) follows immediately from Lemmas 3.14 and 3.15.

Set $\xi=a+b\theta$ for $a, b\in N$. Then $(\xi\theta)\theta=((a+b\theta)\theta)\theta=(a\theta+(b\theta)\theta)\theta=(a\theta)\theta+((b\theta)\theta)\theta=a\theta^2+(b\theta^2)\theta=a\theta^2+b\theta^3$ because $a, b\in N=N_3$ and $\langle\theta\rangle=F_3^{\times}$. Hence $(\xi\theta)\theta=a\theta^2+(b\theta)\theta^2=\xi(\theta^2)$. Thus (ii) holds.

Lemma 3.17. *$\theta\in N$.*

Proof. Let $\xi, \eta\in F$ and set $\xi=a+b\theta$, $\eta=c+d\theta$ for $a, b, c, d\in N$. Then, $(\xi\eta)\theta=((a+b\theta)(c+d\theta))\theta=(ac)\theta+((b\theta)c)\theta+(a(d\theta))\theta+((b\theta)(d\theta))\theta$. Similarly $\xi(\eta\theta)=a(c\theta)+(b\theta)(c\theta)+a((d\theta)\theta)+(b\theta)((d\theta)\theta)$. Since $a\in N=N_3$ and $c\in N=N_2$, we have $(ac)\theta=a(c\theta)$, $((b\theta)c)\theta=(b\theta)(c\theta)$ and $(a(d\theta))\theta=a((d\theta)\theta)$. Since $d\in N=N_2$, $((b\theta)(d\theta))\theta=(((b\theta)d)\theta)\theta$ and by Lemma 3.16, $((b\theta)d)\theta)\theta=((b\theta)d)\theta^2$, so that $((b\theta)(d\theta))\theta=((b\theta)d)\theta^2=(b\theta)(d\theta^2)=(b\theta)((d\theta)\theta)$ as $d\in N=N_2=N_3$. Hence $(\xi\eta)\theta=\xi(\eta\theta)$ and so $\theta\in N_3=N$.

Proof of Lemma 2.

By Lemmas 3.16 (i) and 3.17, we obtain a contradiction and so the lemma holds.

4. The proof of Theorem 1

Throughout this section π is a semi-field plane satisfying Hypothesis 2.3 and the following.

Hypothesis 4.1. $r\not\equiv 0 \pmod 4$ *and* $u(\pi)\leq 4$.

Lemma 4.2. (i) *G is not solvable.*

(ii) $u(\pi)=2$, 3 or 4.

(iii) *There exists $i\in\{1, 2, 3\}$ such that G is transitive on $\bar{l}_i$.*

Proof. By Theorem of [8], Theorem 6.3 of [9] and the lemma proved in §3, we have (i).

It follows from Kallaher's theorem [8] that $u(\pi)\neq 1$ and so (ii) holds.

If $u(\pi)=2$ or 3, we have (iii) by a similar argument as in the proof of Lemma 3.7. If $u=4$, we can apply Lemma 6.1 of [9] and (iii) follows.

Lemma 4.3. *Let S be a Sylow 2-subgroup of G and set $\pi_0=F(S)$, $H=G(\pi_0)$, $\bar{G}=G/O(G)$. Then the following hold.*

(i) *$S\neq 1$ and S is semi-regular on $\pi-\pi_0$.*

(ii) *π_0 is a Baer subplane of π.*

(iii) *$\bar{G}'\simeq PSL(2, q)$ for some even q. Moreover $H=O(G)N_G(S)$ and $|G:H|=q+1$.*

Proof. By the Feit-Thompson theorem [2] and Lemma 4.1 (i), the order of G is even and so $S\neq 1$. Let z be an involution in the center of S. Then $F(z)$ is a Bear subplane of order $2^{r/2}$ and $S^{F(z)}$ is a collineation group. By Hypothesis 4.1, $2^{r/4}$ is not an integer. Therefore $S^{F(z)}=1$. Hence (i) and (ii) hold.

By Lemma 4.2 (ii), $G\neq H$ and clearly $H\geq S$. Hence H is a strongly embedded subgroup of G. By a Bender's theorem [1] and by Corollary 3.2 of [4], (iii) holds.

Lemma 4.4. *Set $\Delta=\pi_0\cap \bar{l}_i$ and $\Gamma=\{\Delta^g\,|\,g\in G\}$. Then the following hold.*

(i) *$\bar{l}_i=\bigcup_{\Delta^x\in\Gamma}\Delta^x$ and $\Delta^x\cap\Delta^y=\phi$ for distinct Δ^x and Δ^y in Γ.*

(ii) *Set $N=O(G)$. Then $G(\Delta)=H\geq N=G_\Gamma$ and G is doubly transitive on Γ.*

Proof. By Lemma 4.3 (iii), $H=N\cdot N_G(S)$. Since $G(\pi_0)\leq G(\Delta)$ and H is a maximal subgroup of G, we have $H=G(\Delta)$. Hence G is doubly transitive on Γ (See [1] §3). Since N is a normal subgroup of G and $N\leq G(\Delta)$, N is contained in G_Γ and so $N=G_\Gamma$ by Lemma 4.3 (iv). Thus (ii) holds.

Clearly $\Delta^g\subset\bar{l}_i$ for all $g\in G$, hence $\bar{l}_i=\bigcup_{\Delta^g\in\Gamma}\Delta^g$ by Lemma 4.2 (iii). Suppose $\Delta^x\neq\Delta^y$ and $\Delta^x\cap\Delta^y\neq\phi$ and set $g=xy^{-1}$. Then $\Delta^g\neq\Delta$ and $\Delta^g\cap\Delta\neq\phi$. By Lemma 4.3 (i), S and S^g fix $\Delta^g\cap\Delta$ pointwise. By (ii), $G=\langle N, S, S^g, G(\Delta)\cap G(\Delta^g)\rangle$. Hence G fixes $\Delta^g\cap\Delta$ as a set, contrary to Lemma 4.2 (iii). Thus (i) holds.

Lemma 4.5. *$q^2=2^r$ and $|\Delta|=q-1$, $|\Gamma|=q+1$.*

Proof. By Lemmas 4.3 (iii) and 4.4 (ii), $|\Gamma|=|G:H|=q+1$ and by Lemma 4.4 (i) $|\Gamma|=|\bar{l}_i|/|\Delta|=(2^r-1)/|\Delta|$. On the other hand $|\Delta|=2^{r/2}-1$ since π_0 is a Baer subplane of π. Hence $q^2=2^r$ and $|\Delta|=q-1$.

Lemma 4.6. *$\pi_0(=F(S))$ is a Desarguesian projective plane of order q and the number of $N_G(S)$-orbits on $\Phi(\pi)\cap\pi_0$ is one or three.*

Proof. Let Λ be a G-orbit on $\Phi(\pi)$ and supopse $\Lambda\cap\pi_0\neq\phi$. Let $P\in\Lambda\cap\pi_0$. Then $G_P\geq S$. Hence $|\Lambda|=|G:G_P|\equiv 1 \pmod 2$ and moreover $N_G(S)$ is transitive on $\Lambda\cap\pi_0$ by Theorem 3.5 of [11]. Since $|\Phi|\equiv 1 \pmod 2$

and $u=u(\pi)\leq 4$, the number of G-orbits Λ on Φ such that $\Lambda\cap\pi_0\neq\phi$ is one or three. Hence the number of $N_G(S)$-orbits on $\pi_0\cap\Phi$ is one or three.

Since the order of π_0 is $2^{r/2}$ and $2^{r/4}$ is not an integer, the autotopism group of π_0 is of odd order. By Theorem 6.3 of [9] and Theorem of [8], π_0 is a Desarguesian plane of order q.

By Lemma 4.3, $|G: G(\pi_0)|=q+1$. We set $\{\pi_0{}^g\,|\,g\in G\}=\{\pi_0, \pi_1, \cdots, \pi_q\}$. Then the following lemma holds.

Lemma 4.7. *Set* $N=O(G)$. *Then*

(i) N_{π_s} *acts faithfully on* π_t *and* $|N_{\pi_s}|\,\Big|\,(q-1)^2(r/2)$ *for all* s, t $(s\neq t)$ *and*

(ii) N_{π_t} *is a normal subgroup of* N *and* $[N_{\pi_s}, N_{\pi_t}]=1$ *for all* s, t $(s\neq t)$.

Proof. By Lemma 4.4 (ii), N acts on π_t and so N_{π_t} is a normal subgroup of N. By Lemma 4.3 (ii), π_s and π_t are Baer subplanes of π, so that $N_{\pi_s}\cap N_{\pi_t}=1$. Hence N_{π_s} acts faithfully on π_t and $[N_{\pi_s}, N_{\pi_t}]\leq N_{\pi_s}\cap N_{\pi_t}=1$. Moreover $|N_{\pi_s}|\,\Big|\,(q-1)^2(r/2)$ since π_t is a Desarguesian plane of order q.

Lemma 4.8. *Assume* $N_{\pi_0}\neq 1$ *and let* P *be a minimal normal subgroup of* N_{π_0} *and let* p *be a prime dividing the order of* P. *Then a Sylow* p-*subgroup of* N_{π_0} *is cyclic and* P *is a normal subgroup of* N. *Moreover* P *is isomorphic to* Z_p.

Proof. Let Q be a Sylow p-subgroup of N_{π_0}. Since $F(Q)=\pi_0$, Q is semi-regular on $\pi_t-\pi_0$ for $t\neq 0$. By Lemma 2.2 (i) and Theorem 5.4.10 of [3], Q is cyclic. Hence, by Lemma 4.7 (ii), we have the lemma.

Lemma 4.9. *Let* P *be as in Lemma* 4.8. *Then the following hold.*

(i) *Set* $L=\langle P^g\,|\,g\in G\rangle$. *Then* L *is a normal subgroup of* G *and is an elementary abelian* p-*group.*

(ii) $p\not|\, r$ *and* $|L|\leq p^3$.

Proof. (i) follows immediately from Lemma 4.8. Clearly $L\leq N$. Set $X=N_{\pi_0}$. Since $X\cap L=P$ and $L/P\simeq LX/X\leq N/X\simeq N^{\pi_0}$, $|L/P|$ is at most p^3. Moreover $|L/P|\leq p^2$ if $p\not|\, r$. Therefore it suffices to show $p\not|\, r$. Assume $p\,\Big|\, r$. Since H normalizes X, P is a normal subgroup of H and so L contains at least $q+1$ subgroups of order p. Hence $q+1\leq(p^4-1)/(p-1)=p^3+p^2+p+1$. On the other hand $p\,\Big|\,r/2$ and $q=2^{r/2}$, so that $(r/2)^3+(r/2)^2+r/2+1\geq 2^{r/2}+1$. From this $r=6$ or 10 and $p=r/2$. But $p\not|\, q-1$ for $r=6$ or 10. Therefore, $|L/P|\leq p$ and so $q+1\leq(p^2-1)/(p-1)=p+1\leq 6$, a contradiction. Thus $p\not|\, r/2$.

Lemma 4.10. $N_{\pi_0}=1$.

Proof. Assume $N_{\pi_0}\neq 1$ and let P, L be as in Lemma 4.8, 4.9, respectively.

If $|C_G(L)|$ is even, all Sylow 2-subgroups of G are contained in $C_G(L)$ by Lemma 4.3 (iii). Hence $\langle S^g | g\in G\rangle$ acts on $F(P)$ $(=\pi_0)$, which is contrary to $G(\pi_0)=H$. Therefore $|C_G(L)|$ is odd. In particular S is isomorphic to a subgroup of $G/C_G(L)$.

By Lemmas 4.3 and 4.9, $(G/C_G(L))'\leq SL(3, p)$ and $|G/C_G(L)\colon (G/C_G(L))'|$ is odd. Hence S is isomorphic to a subgroup of $SL(3, p)$. Since a Sylow 2-subgroup of $SL(3, p)$ is semi-dihedral or wreathed, S is an elementary abelian group of order 4 and so $q=2^2$. Hence $r=4\equiv 0 \pmod 4$, a contradiction.

Lemma 4.11. *Let $G^{(\infty)}$ denote the last term of the derived series of G. Set $M=G^{(\infty)}$. Then $M\simeq PSL(2, q)$.*

Proof. Let X be a subgroup of G generated by all Sylow 2-subgroups of G. By Lemma 4.3 (iii), $X\leq M$ and $|M/X|$ is odd. It follows from the Feit-Thompson theorem that $M=M^{(\infty)}\leq X$ and hence $X=M$. By Lemmas 4.4 (ii) and 4.10, $[S, N]\leq N\cap G_{\pi_0}=N_{\pi_0}=1$, so that N centralizes X $(=M)$ and $M\cap N=Z(M)$, $M/Z(M)\simeq PSL(2, q)$. By a property of $PSL(2, q)$, $M\simeq PSL(2, q)$.

Lemma 4.12. (i) *Let $t\in\{1, 2, 3\}$, $P\in\bar{l}_t$ and let X be a subgroup of G_P. Then $|F(X)\cap l_t|=2^a+1$ for an integer $a\geq 1$.*

(ii) *M is transitive on $\bar{l}_i$ and $|M_P|=q$ for $P\in\bar{l}_i$. Here i is the integer defined in Lemma 4.2 (iii).*

Proof. Let A be the full collineation group of π and set $T_1=A_{(U_3,l_2)}$, $T_2=A_{(U_3,l_1)}$, $T_3=A_{(U_2,l_1)}$. Since U_3 is a translation point and l_1 is a translation line, $T_1\simeq T_2\simeq T_3\simeq E_{q^2}$ and XT_t is a transitive linear group on l_t. Since $(XT_t)_P=X$, we have (i) by Lemma 2.1.

Let $\{\Delta_1, \cdots, \Delta_m\}$ be the set of M-orbits on $\bar{l}_i$. Since G is transitive on $\bar{l}_i$ and $G\triangleright M$, $|\Delta_1|=\cdots=|\Delta_m|\equiv 1 \pmod 2$. Let $P\in\Delta_1$ and set $M_P=CS$ with $C\leq Z_{q-1}$ and $|N_M(S)\colon M_P|=k$. As $M\simeq PSL(2, q)$, $k \mid q-1$ and $F(M_P)\cap\Delta_v \neq\phi$ for each $v\in\{1, \cdots, m\}$.

Assume $C\neq 1$. Then $|N_M(C)|=2(q-1)$ as $M\simeq PSL(2, q)$. By Lemma 2.1, $|F(C)\cap\bar{l}_i|=m\times\dfrac{2(q-1)\times|S|}{|M_P|}=2mk$ and applying (i), we have $2mk=2^a-1$ for an integer $a\geq 1$, a contradiction. Thus $C=1$ and $|M_P|=q$. Therefore $|P^M|=|M\colon M_P|=q^2-1$ and (ii) follows.

Lemma 4.13. *Let $j\in\{1, 2, 3\}-\{i\}$ and $P\in\bar{l}_j$. Then $q \mid |M_P|$.*

Proof. By Lemma 4.3 (i), it suffices to consider the case that $|M_P|\equiv 1 \pmod 2$. As $M\simeq PSL(2, q)$, $M\leq Z_{q\pm 1}$. Since $|\bar{l}_j|=q^2-1\geq|P^M|=|M\colon M_P|$ and $P^M\cap F(S)=\phi$, we have $M_P\simeq Z_{q+1}$ and $|l_j-P^M|=|F(S)\cap l_j|=q+1$. Hence $F(S)\cap l_j=F(M)\cap l_j$. Therefore $|F(M_P)|=q+1+\dfrac{2(q+1)\times 1}{|M_p|}=q+3$ by Lemmma 2.1. Applying Lemma 4.12 (i), $q+3=2^a+1$ for an integer $a\geq 1$. This is a contradiction.

Lemma 4.14. *M is transitive on $\bar{l}_j$ and M_P is a Sylow 2-subgroup of M for each $j\in\{1, 2, 3\}$ and $P\in\bar{l}_j$.*

Proof. By Lemma 4.12 (ii), we may assume $j\in\{1, 2, 3\}-\{i\}$. First we argue that $F(M)\cap\bar{l}_j=\phi$. Set $\Delta=F(M)\cap\bar{l}_j$ and assume $\Delta\neq\phi$. Let π_0 be as defined in Lemma 4.6 and set $N_M(S)=DS$ with $D\simeq Z_{q-1}$. By Lemma 4.12 (ii), $D^{\pi_0}\simeq D$ and $\pi_0\cap F(D)\supset\Delta$. Since π_0 is a Desarguesian plane of order q, $F(D)\supset\pi_0\cap l_j$ by Lemma 2.2 (ii). Therefore, by Lemmas 2.1 and 4.13, $|F(D)\cap\bar{l}_j|=|\Delta|+2(q-1-|\Delta|)=2(q-1)-|\Delta|$. Applying Lemma 4.12 (i), $|\Delta|=2^a-1$ and $2(q-1)-|\Delta|=2^b-1$ for integers $a, b\geq 1$, hence $2q=2^a+2^b$. However, as $|\Delta|<|\pi_0\cap\bar{l}_j|=q-1<|F(D)\cap\bar{l}_j|=2^b-1$, we have $2^a<q<2^b$. This is a contradiction. Thus $F(M)\cap\bar{l}_j=\phi$.

Let $\{\Delta_1, \cdots, \Delta_m\}$ be the set of M-orbits on $\bar{l}_j$. By Lemma 4.3, $|\Delta_t|\mid q^2-1$ for each t. Assume $|M_P|\neq q$ for some $P\in\bar{l}_j$. We may assume $P\in\Delta_1$ and $M_P\rhd S$. Set $M_P=CS$ with $1\neq C\leq Z_{q-1}$. By a similar argument as in the last paragraph $F(S)\cap l_j\subset F(C)\cap l_j$ and so $F(C)\cap\Delta_t\neq\phi$ for each t. Hence $|F(C)\cap\Delta_t|=2\times|F(S)\cap\Delta_t|$ by Lemma 2.1. Hence $|F(C)\cap\bar{l}_j|=2\times|F(S)\cap\bar{l}_j|=2(q-1)$ and so $|F(C)\cap l_j|=2q$, contrary to Lemma 4.12 (i). Thus $|M_P|=q$ and M is transitive on $\bar{l}_j$.

Let X be the full collineation group of π and set $A=X_{(l_1,l_1)}$, $B=X_{(U_3,U_3)}$ and $T=AB$. Since U_3 is a translation point and l_1 is a translation line, A and B are elementary abelian normal 2-subgroups of X of order q^4. Hence T is a normal 2-subgroup of X.

Lemma 4.15. (i) *T is a nonabelian normal 2-subgroup of X.*
(ii) *$C_T(x)=1$ for any element $x(\neq 1)$ of M of odd order.*

Proof. If T is abelian, $T_P=1$ for $P\notin l_1$ because A acts transitively on the set of points not incident with the line l_1. Hence $|T|=|T: T_P|=q^4+q^2+1-(q^2+1)=q^4$ and so $T=A=B$, a contradiction. Thus (i) holds.

Assume $C_T(x)\neq 1$ and let t be an involution in $C_T(x)$. Then, there exist element $t_1\in A$ and $t_2\in B$ such that $t=t_1t_2$. By Lemma 4.14, $F(X)=\{U_1, U_2, U_3, l_1, l_2, l_3\}$ and so t acts on $\{U_1, U_2, U_3\}$. Since $F(X)=\{U_3, l_1\}$, it follows that $(U_2)^t\in l_1$ and $(U_3)^t=U_3$. Hence we have $(U_1)^t=U_1$ and $(U_2)^t=U_2$ and so $F(t_2)=F(t_1t)\supset\{U_2, U_3\}$. Therefore $t_2\in X_{(U_3,l_1)}\leq X_{(l_1,l_1)}$, which implies $t\in X_{(l_1,l_1)}$. However, as $(U_1)^t=U_1$, this is a contradiction. Thus $C_T(x)=1$.

Proof of Theorem 1.

Since $3\mid |M|=|PSL(2, q)|$, there exists an element $x\in M$ of order 3.

By Lemma 4.15 (ii), $C_T(x)=1$. Applying Theorem 8.2 of [5] to the group MT, T is an abelian 2-group, which is contrary to Lemma 4.15 (i). Thus we have the theorem.

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