Let $G$ be the group $Z/2$. Denote by $\pi^S_{p,q}$ the equivariant stable homotopy group of Landweber [12]. In a similar way to the usual $e$-invariants we define equivariant $e$-invariants $e_G$ and $e_{G,R}$ on $\pi^S_{p,2q-1}$ by using the Adams operations in the $K_G$- and $KO_G$-theories and the equivariant Chern character. And we compute these invariants, in particular $e_{G,R}$, on the image of the equivariant $J$-homomorphism, making use of the Adams' result for $e'$. Here we study the case when $KO_G(\Sigma \mathbb{P}^{2p+1})$ is torsion-free. The torsion case is discussed by Löffler [14].

1. Definitions

Let $R^{p,q}$ denote the $R^{p+q}$ with non trivial $G$-action on the first $p$ coordinates. By $B^{p,q}$ and $S^{p,q}$ we denote the unit ball and unit sphere in $R^{p,q}$ and by $\Sigma^{p,q}$ the $B^{p,q}/S^{p,q}$. If $p$ and $q$ are even then $R^{p,q}$ is a complex $G$-module. In particular, we write $1$ and $L$ for $R^{0,2}$ and $R^{2,0}$. Then $\{1, L\}$ are basis of the complex representation ring $R(G)$ of $G$.

For the Thom class of $R^{2p,2q}$ as a complex $G$-vector bundle over a point we write $\lambda_{2p,2q}$, so that $K_G(\Sigma^{2p,2q})=R(G)\cdot \lambda_{2p,2q}$ [16]. Here let $A \cdot x$ denote the module generated by $x$ over a ring $A$. Then we have the formula

$$\psi^t(\lambda_{2p,2q}) = \rho^t(2p, 2q)\lambda_{2p,2q}, \quad \rho^t(2p, 2q) \in R(G)$$

for the $t$-th Adams operation $\psi^t$, and $\rho^t(2p, 2q)$ is computed briefly, using the result for $\psi^t$ in $K(S^{2n})$, as follows.

**Lemma 1.1.** $\rho^t(0, 2q)=t^q$, and if $p>0$ then

$$\rho^t(2p, 2q) = \begin{cases} \frac{1}{2} t^{p+q}(L+1) & \text{(t even)} \\ t^{p+q} + \frac{1}{2} t^q(t^p-1)(L-1) & \text{(t odd)} \end{cases}$$

As is easily seen, $K_G(\Sigma^{1,0})$ is isomorphic to the augmentation-ideal of $R(G)$. Identifying $K_G(\Sigma^{1,0})$ with $Z \cdot (1-L)$ it is clear that $K_G(\Sigma^{2p+1,2q})=Z \cdot$
Hence we have the following

**Corollary 1.2.** \( \psi^t \) operates on \( \tilde{K}_c(\Sigma^{p+1,2l}) \) as multiplication by 0 if \( t \) is even and by \( t^q \) if \( t \) is odd.

For \( p, q-1 \geq 0 \) suppose given a base point preserving \( G \)-map \( f: \Sigma^{p+2k,2q-1+2l} \to \Sigma^{2k,2l} \) for \( k, l \) large, which is fixed in this section. \( f \) yields a cofiber sequence

\[
\Sigma^{p+2k,2q-1+2l} \xrightarrow{f} \Sigma^{2k,2l} \xrightarrow{i} C_f \xrightarrow{j} \Sigma^{p+2k,2q+2l} \xrightarrow{0,1} \Sigma^{2k,2l+1}
\]

where \( i, j \) are the inclusion and projection maps and \( C_f \) is the mapping cone of \( f \). Applying \( \tilde{K}_c \) we obtain the following exact sequence.

\[
0 \xleftarrow{\Lambda} \tilde{K}_c(\Sigma^{2k,2l}) \xleftarrow{\theta^*} \tilde{K}_c(C_f) \xleftarrow{\mu^*} \tilde{K}_c(\Sigma^{p+2k,2q+2l}) \xrightarrow{0}
\]

Choose generators \( \xi, \eta \) of \( \tilde{K}_c(C_f) \) so that

\[
i^*(\xi) = \lambda_{2k,2l} \quad \text{and} \quad \eta = \begin{cases} j^*(\lambda_{p+2k,2q+2l}) & (p \text{ even}) \\ j^*((1-L)\lambda_{p-1+2k,2q+2l}) & (p \text{ odd}) \end{cases} .
\]

For any odd integer \( t(\neq \pm 1) \), \( \psi^t(\xi) \) must be given by the formula

\[
\psi^t(\xi) = \rho^t(2k, 2l)\xi + \begin{cases} (c(t)+d(t)(J-1))\eta & (p \text{ even}) \\ c(t)\eta & (p \text{ odd}) \end{cases} ,
\]

\( c(t), d(t) \in \mathbb{Z} \). So we set

\[
\lambda(f) = \frac{c(t)}{t^{2k+2} + t^{k+i} - i^{k+i}} \quad (p \text{ even})
\]

\[
\mu(f) = \begin{cases} 1 \left( \frac{c(t)}{t^{2k+2} + t^{k+i} - i^{k+i}} + \frac{2d(t)-c(t)}{t^{e+i} - i^l} \right) & (p \text{ even}) \\ \frac{c(t)}{t^{e+i} - i^l} & (p \text{ odd}) \end{cases} .
\]

Using Lemma 1.1, Corollary 1.2 and the relation \( \psi^t \psi^t = \psi^{at} \) we can check that the values \( \{\lambda(f)\}, \{\mu(f)\} \) do not depend on the choice of an integer \( t \) where \( \{ \} \) denotes the coset in \( \mathbb{Q}/\mathbb{Z} \). As in [1, IV], §7 we see that the assignment

\[
f \mapsto \begin{cases} \{\lambda(f)\}, \{\mu(f)\} & (p \text{ even}) \\ \{\mu(f)\} & (p \text{ odd}) \end{cases}
\]

induces a group homomorphism
\[ e_G: \pi_{p,2q-1}^S \rightarrow \begin{cases} O/Z \oplus Q/Z & (p \text{ even}) \\ Q/Z & (p \text{ odd}) \end{cases} \text{ for } p, q - 1 \geq 0. \]

Regard \( e_G \) as taking values in \( \tilde{K}_G(\Sigma^{p+2k,2q+2l}) \otimes Q/Z \), namely let \( e_G[f] \) be \( \{\lambda(f)\} + \{\mu(f)\} (L-1)\lambda_{p+2k,2q+2l} \) or \( \{\mu(f)\} (1-L)\lambda_{p-1+2k,2l} \) according as \( p \) is even or odd where \([f] \) is the stable homotopy class of \( f \). Then we have easily the following

**Proposition 1.3.** \( e_G \) is natural for stable maps from \( \Sigma^{p,2q-1} \) to \( \Sigma^{r,2q-1} \).

To evaluate \( \psi^*(\xi) \) we shall next describe \( e_G \) in terms of the equivariant Chern character. Let \( ch_G \) be as in [18] and \( ch^G_2 \) denote the 2n-dimensional component of \( ch_G \) which is a homomorphism of \( K_G \) to \( H^{2n}_{cr}(G, R_G) \) in the notation of [18]. By the definition of equivariant Bredon cohomology [7] we have the following canonical isomorphisms

\[
H_\phi^{p+2k+2q+2l}(C_f, R_G) \approx H^{p+2k+2q+2l}(C_{\psi f}, Q) \\
\approx H^{p+2k+2q+2l}(S^{p+2k+2q+2l}, Q),
\]

\[
H_\phi^{2k+2l}(C_f, R_G) \approx H^{2k+2l}(C_{\psi f}, Q) \cdot (1-L),
\]

\[
\approx H^{2k+2l}(S^{2k+2l}, Q) \cdot (1-L).
\]

Here \( \psi \) and \( \phi \) are the forgetful and fixed point functors [3]. Under the identification of the above isomorphisms we may set

\[
ch^G_{p+2k+q+l}(\xi) = a(f)h^{p+2k+2q+2l}
\]

and

\[
ch^G_{q+l}(\xi) = b(f)h^{2q+2l}(1-L),
\]

\( a(f), b(f) \in Q \) (\( p \) even) where \( h^{2l} \in H^{2l}(S^{2l}, Z) \) is a canonical generator such that \( ch^G(\psi\lambda_{0,2l}) = h^{2l} \). Then we obtain

**Proposition 1.4.** If \( p \) even then

\[
\lambda(f) = a(f), \mu(f) = \frac{1}{2} \left( a(f) - \frac{b(f)}{2^{p/2+k-1}} \right)
\]

and if \( p \) is odd then

\[
\mu(f) = \frac{b(f)}{2^{(p-1)/2+k}}.
\]

Proof. Consider the following commutative diagram with the exact sequence which \( \phi f \) yields as \( f \) does.

\[
0 \leftarrow \tilde{K}_G(\Sigma^{2k,2l}) \leftarrow \tilde{K}_G(C_f) \leftarrow \tilde{K}_G(\Sigma^{p+2k,2q+2l}) \leftarrow 0
\]

\[
(*)
\]

\[
0 \leftarrow \tilde{K}_G(\Sigma^{0,2l}) \leftarrow \tilde{K}_G(C_{\psi f}) \leftarrow \tilde{K}_G(\Sigma^{0,2q+2l}) \leftarrow 0
\]
(Here \(h\)'s are the inclusions.) Choose \(\xi_1 \subset K_c(C, f)\) so that \(i\sharp(\xi_1) = \lambda_{0, 2l}\) and put \(\eta_l = j\sharp(\lambda_{0, 2l+2})\). Then we may write

\[
h^*(\xi) = 2^{k-1}(1-L)\xi_1 + x(1-L)\eta_l, \quad x \in Z
\]

for a cohomological reason and the fact that \(h^*(\lambda_{2k, 2l}) = 2^{k-1}(1-L)\lambda_{0, 2l}\). Applying \(\psi^f\) we have

\[
\psi^f(h^*\xi) = 2^{k-1}(1-L)\psi^f(\xi_1) + xt^{q+l}(1-L)\eta_l.
\]

On the other hand, applying \(h^*\) to the defining formula of \(c(t), d(t)\) we have

\[
\psi^f(h^*\xi) = 2^{k-1}t'(1-L)\xi_1 + xt'(1-L)\eta_l
\]

\[
+ \begin{cases} 
2^{p/2+k-1}(c(t) - 2d(t))(1-L)\eta_l & (p \text{ even}) \\
2^{(p-1)/2+k}c(t)(1-L)\eta_l & (p \text{ odd})
\end{cases}
\]

Combining (1) and (2) shows

\[
\psi^f(\xi_1) = t\xi_1 + \frac{xt'(1-t^q+l)}{2^{k-1}} \eta_l \left[ \begin{array}{l} 2^{p/2}(c(t) - 2d(t)) \eta_l \quad (p \text{ even}) \\
2^{(p-1)/2+k}c(t) \eta_l \quad (p \text{ odd}) \end{array} \right]
\]

Case \(p\) even. From the definition of \(ch\) it follows easily that

\[
ch^{k/2+k+q+l}(\xi_1) = ch^{k/2+k+q+l}(\psi^f\xi)
\]

and

\[
ch^{q+l}(\xi) = 2^{k-1}ch^{q+l}(\psi^f\xi_1)(1-L) + xh^{2q+2l}(1-L).
\]

Hence we get

\[
ch^{k/2+k+q+l}(\psi^f\xi) = a(f)h^{k+2k+2q+2l} \quad \text{and} \quad ch^{q+l}(\psi^f\xi_1) = \frac{b(f)-x}{2^{k-1}} h^{2q+2l}.
\]

Therefore [1, IV], Proposition 7.5 for \(\psi f\) and \(\phi f\) leads to the equalities

\[
a(f) = \frac{c(f)}{t^{k/2+k+q+l}-t^{k+1}} \quad \text{and} \quad \frac{b(f)}{2^{k/2+k-1}} = \frac{c(t) - 2d(t)}{t^{k+1} - t^l}.
\]

Case \(p\) odd. Similar to the proof of the above case.

q.e.d.

2. \((0, 2q-1)\)-stem

Let \(\pi : \Sigma_2^{2k, 2q-1+2l} \to \Sigma_2^{2k, 2q-1+2l}/\Sigma_0^{2q-1+2l}\) be the canonical projection map for \(k, l \text{ large}\). Let \(\lambda^\delta_{\phi, q}\) denote the equivariant stable homotopy group introduced in \([12]\). Then we have by \([12]\) a split short exact sequence

\[
0 \to \lambda^\delta_{0, 2q-1} \overset{\pi^*}{\to} \lambda^\delta_{0, 2q-1} \overset{\phi}{\to} \lambda^\delta_{2q-1} \to 0
\]
where $\pi^*$ is the homomorphism induced by $\pi$ and $\theta$ denotes a left inverse of $\phi$ as in [4], §5.

By the definition we can easily describe the values of $e_\phi$ on $\text{Im} \theta$ in terms of the complex $e$-invariant $e_\phi$ in [1, IV]. So we consider $e_\phi$ on $\text{Im} \pi^*$ in this section.

Suppose given a base point preserving $G$-map $\tilde{f} : \Sigma^{2q,2q-1+2l}/\Sigma^{0,2q-1+2l} \to \Sigma^{2k,2l}$, so that $f$ and $\tilde{f} \pi$ define elements $[f]$ and $[\tilde{f} \pi]$ of $\lambda_{\delta,2q-1}^e$ and $\pi_{\delta,2q-1}^e$ respectively. We consider $\tilde{f} \pi$ as $f$ in §1.

Since $\Sigma^{i,j}/\Sigma^{0,i}$ is equivariantly homeomorphic to $\Sigma^{0,i+1}\Sigma^{i,0}$ ([12], Lemma 4.1), we have $\tilde{K}_0(\Sigma^{i,j}/\Sigma^{0,i}) \approx K^{-i-j}(\Sigma^{i,j})$ [16] where $RP^n$ is the real $n$-dimensional projective space. Let $\eta_n$ be the complexification of a canonical real line bundle over $RP^n$ and put $\eta_n = 1 - \eta_n$. We now recall [6] that

$$\tilde{K}_0'(RP^{2k}) = \mathbb{Z}/2^* \star \tilde{\eta}_{2k}, \ K'(RP^{2k}) = 0$$

$$\tilde{K}_0'(RP^{2k+1}) = \mathbb{Z}/2^* \star \tilde{\eta}_{2k+1}, \ K'(RP^{2k+1}) \approx \mathbb{Z}.$$  

Then we can identify

$$\tilde{K}_0(\Sigma^{2k,2q-1+2l}/\Sigma^{0,2q-1+2l}) = \mathbb{Z} \oplus \mathbb{Z}/2^{k-1} \star (\psi_{\lambda_{0,2q+2l}}) \eta_{2k-1}.$$  

Consider $f^* : \tilde{K}_0(\Sigma^{2k,2l}) \to \tilde{K}_0(\Sigma^{2k,2q-1+2l}/\Sigma^{0,2q-1+2l})$. Because $[\tilde{f}] \in \lambda_{\delta,2q-1}$ for $q \geq 1$ is of finite order ([12], Theorem 2.4 and Corollary 6.3) we may put

$$f^*(\lambda_{2k,2l}) = [b(\tilde{f})] (\psi_{\lambda_{0,2q+2l}}) \eta_{2k-1}, \ b(\tilde{f}) \in \mathbb{Z}$$

where $[\ ]$ denotes the coset in $\mathbb{Z}/2^{k-1}$.

**Lemma 2.1.** $b(\tilde{f}) = -b(f(\pi)) \mod 2^{k-1}$

where $b(f(\pi))$ is as in §1.

**Proof.** Observe the following commutative diagram involving (* in §1.

\[
\begin{array}{cccccccc}
\tilde{K}_0(\Sigma^{2k,2q-1+2l}/\Sigma^{0,2q-1+2l}) & \leftarrow & \tilde{K}_0(\Sigma^{2k,2l}) & \leftarrow & \tilde{K}_0(\Sigma^{2k,2q+2l}/\Sigma^{0,2q+2l}) \\
0 & \leftarrow & \tilde{K}_0(\Sigma^{2k,2l}) & \downarrow & \tilde{K}_0(\Sigma^{2k,2q+2l}) & \leftarrow & 0 \\
0 & \leftarrow & \tilde{K}_0(\Sigma^{0,2l}) & \downarrow & \tilde{K}_0(\Sigma^{2k,2q+2l}) & \leftarrow & 0 \\
\delta & \downarrow & \tilde{K}_b(\Sigma^{2k,2q+2l}/\Sigma^{0,2q+2l}) & \downarrow & \tilde{K}_b(\Sigma^{2k,2q+2l}/\Sigma^{0,2q+2l}) & \downarrow & \delta
\end{array}
\]

where the right-hand sequence is the exact sequence for a pair $(\Sigma^{2k,2q+2l}, \Sigma^{0,2q+2l})$. Clearly $C_{\phi(f,2l)} \approx \Sigma^{0,2q+2l}/\Sigma^{0,2l}$, hence we can verify that $f^*(\lambda_{2k,2l}) = -\delta j^* h^*(\xi)$ where $\xi$ is as in §1. Hence the canonical identification such that $\tilde{K}_0(\Sigma^{0,2q+2l}) = \tilde{K}(S^{2q+2l}) \otimes R(G) = H^{2q+2l}(S^{2q+2l}, Z) \otimes R(G)$ leads to the desired assertion. q.e.d.
Let $BG$ denote the real infinite dimensional projective space. There is an integer $c(n)$ such that $c(n)\eta_{2k-1}$ becomes trivial (see, e.g. [9], p. 219). So we have an equivariant homeomorphism $\Sigma^{c(n),0} S^n, 0 \cong \Sigma^{c(n)} S^n, 0$. This homeomorphism, the equivariant suspension theorem and the Spanier-Whitehead duality theorem yield an isomorphism

$$\chi_{\delta, n}^{S} \rightarrow \pi_{n}^{S}(BG_{+}),$$

denoted by $I$, as follows. Let $\tau$ be the tangent bundle of $RP^{2k-1}$ and $\nu$ be a normal bundle of $RP^{2k-1}$ for an embedding of $RP^{2k-1}$ in $R^{2m-1}$ for $m$ suitably large. Note that the Thom complex $T(\nu)$ of $\nu$ is a $(2m-1)$-dual of $RP^{2k-1}$ [5], and $\tau \oplus 1 \approx 2k\eta_{2k-1}$ so that $S^{2m} T((sc-k)\eta_{2k-1}) \approx S^{2c} T(\nu)$ for $sc > k$ where $\eta_{2k-1}$ denotes the underlying real vector bundle of $\eta_{2k-1}$ and $c = c(k)$ is as above.

Then we have the following isomorphisms.

$$\chi_{\delta, n}^{S} = \lim_{k, i} \left[ \Sigma^{2k, n+2l}/\Sigma^{0, n+2l}, \Sigma^{2k, 2l}\right]^{G}$$

by definition [12]

$$\approx \lim_{k, i} \left[ \Sigma^{2k, n+2l+1}/\Sigma^{2k, 2l+1} S^{2k, 0}, \Sigma^{2k, 2l}\right]^{G}$$

by [3], Theo. 11.9

$$\approx \lim_{k, i} \left[ \Sigma^{2k, n+2l+1}/\Sigma^{2k, 2l+1} S^{2k, 0}, \Sigma^{2k, 2l}\right]^{G}$$

for some $c$

$$\approx \lim_{k, i} \left[ S^{n+2l+1}/\Sigma^{2c} T((sc-k)\eta_{2k-1}), S^{2l}\right]$$

by [19], Cor. (7.10)

$$\approx \lim_{k} \left[ S^{n+2l+1}/T(\nu), S^{2l}\right]$$

by definition [12]

$$\approx \lim \left[ S^{n+2l+1}/T(\nu), S^{2l}\right]$$

by [19], Cor. (7.10)

On the other hand, the geometrical interpretation of $I$ by Landweber [12] shows that the composite $\varphi \varpi^{*} I^{-1}$: $\pi_{n}^{S}(BG_{+}) \rightarrow \pi_{n}^{S}$ agrees with the $Z/2$-transfer. So we write $t = \varphi \varpi^{*} I^{-1}$ as usual.

Following the homotopical construction of $I$ we see that $I[\tilde{f}]$ is represented by a stable map $g: S^{2k-1} \rightarrow RP^{2k-1}$. Let $\tilde{g}: S^{2k-1} \rightarrow RP^{2k-1}$ be the composite $g$ and the canonical projection from $RP^{2k-1}$ to $RP^{2k-1}$ and let

$$\alpha_{t} \in \pi_{2k-1}^{S}(BG)$$

denote the stable homotopy class induced by $\tilde{g}$. Then we have

**Proposition 2.2.**

$$\left\{ b(\tilde{f}) \right\} = e_{c}(\alpha_{t})$$

where $e_{c}$ is as in [1, IV].
We prepare a lemma for a proof of Proposition 2.2. We recall the following universal coefficient sequence for a finite CW-complex $X$ [2]

$$0 \to \text{Ext} (\tilde{K}^q(X), Z) \to K_i(X) \to \text{Hom}(K^l(X), Z) \to 0$$

where $k$ is a map induced by the Kronecker product. Here we denote by $\iota$ the injection map. Furthermore we have a natural homomorphism

$$\text{Hom}(\tilde{K}^q(X), Q/Z) \to \text{Ext}(\tilde{K}^q(X), Z),$$

which we denote by $\Delta$. In particular, for $X = \mathbb{RP}^{2k}$, $\iota$ and $\Delta$ are isomorphisms.

Denote by $p$ the collapsing map $\mathbb{RP}^{2k-1} \to \mathbb{RP}^{2k-1}/\mathbb{RP}^{2k-2}$ and identify $\mathbb{RP}^{2k-1}$ with $S^{2k-1}$. Then, clearly $p^*: K_0(S^{2k}) \to K_0(\mathbb{RP}^{2k-1})$ is an isomorphism and hence by using the universal coefficient sequence we see that $p^*: K_0(\mathbb{RP}^{2k-1}) \to K_0(S^{2k})$ is an epimorphism. Therefore, if we put $z' = p^*(s_{0,2k}) \in K_1(\mathbb{RP}^{2k-1})$ then we have an element $z \in K_1(\mathbb{RP}^{2k-1})$ such that $p^* z$ is a dual element of $s_{0,2k}$, i.e. $\langle z', z \rangle = 1$, which is a fundamental class of $\mathbb{RP}^{2k-1}$ ([19], p. 217). By [19], Corollary (7.8) we have an isomorphism

$$P = z \cap: \tilde{K}^q(\mathbb{RP}^{2k-1}) \to K_1(\mathbb{RP}^{2k-1}).$$

Consider the composite

$$\tilde{K}^q(\mathbb{RP}^{2k-1}) \xrightarrow{P} K_1(\mathbb{RP}^{2k-1}) \xrightarrow{i_*} K_1(\mathbb{RP}^{2k}) \xrightarrow{(t\Delta)^{-1}} \text{Hom}(\tilde{K}^q(\mathbb{RP}^{2k}), Q/Z)$$

where $i: \mathbb{RP}^{2k-1} \subset \mathbb{RP}^{2k}$ is the inclusion map. Then

**Lemma 2.3.** $((t\Delta)^{-1} i_*)^* P \tilde{\eta}_{2k-1} = -\{1 \over 2^{k-1}\}.$

Proof. Let $\gamma^*$ be the co-Hopf bundle on the complex $(k-1)$-dimensional projective $CP^{k-1}$ and $\gamma$ be its dual. By $D$ and $S$ we denote the total spaces of the unit disk and unit sphere bundles of $\gamma^* \otimes \gamma^*$ with respect to some metric. Then $D \cong CP^{k-1}$ clearly and $S \cong RP^{2k-1}$ (see [10], IV.1.14. Example). We identify $S$ with $\mathbb{RP}^{2k-1}$. Because, if we put $\tilde{\gamma} = 1 - \gamma$ then $K^*(D) \cong Z[\tilde{\gamma}](\tilde{\gamma}^k)$ and $i^* \tilde{\gamma} = \tilde{\eta}_{2k-1}$, we have a short exact sequence

$$0 \to K^l(S) \to K^q(D, S) \to K^q(D) \to K^q(S) \to 0$$

where $\delta$ is a coboundary homomorphism and $i, j$ are the inclusion maps. As is well known, $j^* \lambda = -\tilde{\gamma}^* + 2\tilde{\gamma}^*$ where $\tilde{\gamma}^* = 1 - \gamma^*$ and $\lambda$ is the Thom class of $\gamma^* \otimes \gamma^*$. Hence $K^*(D, S) \cong \bigoplus_{i=0}^{k-1} Z \cdot \lambda^i \tilde{\gamma}^i$. Moreover, by an observation for $\tilde{\gamma}^{k-1}$ in [6], p. 100 we have
\[ \delta^{-1}\lambda k_{k-1} = z'. \]

Put \( z_1' = \delta z' \) and denote by \( z_1 \) a dual element of \( z_1' \) so that we may suppose that \( \partial z_1 = z \) where \( \partial \) is the boundary homomorphism. Similarly \( P_0 = x_1 \cap : K_0(D) \to K_0(D, S) \) is then an isomorphism and the diagram

\[
\begin{array}{ccc}
K_0(D) & \xrightarrow{i^*} & K_0(S) \\
\downarrow P_0 & & \downarrow P \\
K_0(D, S) & \xrightarrow{\partial} & K_1(S)
\end{array}
\]

commutes.

A routine computation shows that \( \lambda k_{k-2} \in K_0(D, S) \) is a dual element of \( P_1 \gamma \), i.e.,

\[ \langle \lambda k_{k-2}, P_1 \gamma \rangle = 1. \]

Let put \( M = D \times S^{2k-1} \) and \( i_1: S \subset M \) be an embedding given by \( i_1(x) = (i(x), p(x)) x \in S \). Then we get a short exact sequence

\[ 0 \to K^*(M, S) \xrightarrow{j_1^*} \tilde{K}^*(M) \xrightarrow{i_1^*} \tilde{K}^*(S) \to 0, \]

which is a free resolution of \( \tilde{K}^*(S) \), where \( j_1 \) is the inclusion map. Hence we see that

\[ \tilde{K}_0^0(M) = \bigoplus_{i=1}^{2k-1} Z \cdot q^* \gamma^i \text{ and } K_0^0(M, S) = \bigoplus_{i=0}^{k-2} Z \cdot q^* k_0^i, \]

where \( q \) is the projection map of \( M \) to \( D \).

Here we adopt the above resolution as a free resolution in the proof of [2], Theorem 3.1 for \( K_1(S) \). Define \( f \in \text{Hom}(K_0^0(M, S), Z) \) by

\[ f(q^i \lambda k_i^j) = \begin{cases} 1 & \text{if } i = k-2 \\ 0 & \text{otherwise.} \end{cases} \]

Then

\[ \text{Hom}(q^*, 1)f = \langle \gamma, P_1 \gamma \rangle. \]

This implies that because \( \text{Coker Hom}(j_1^*, 1) = \text{Ext}(\tilde{K}_0^0(S), Z) \),

\[ \iota[f] = P \eta_{2k-1}, \]

where \([f]\) denotes the equivalence class of \( f \) in \( \text{Coker Hom}(j_1^*, 1) \).

By the definition of \( \Delta \) it is verified that

\[ (\Delta^{-1}[f]) \eta_{2k-1} = -\left\{ \frac{1}{2k-1} \right\}. \]
Hence,

$$(\iota \Delta)^{-1}(P\tilde{\eta}_{2k-1})\tilde{\eta}_{2k-1} = -\left\{ \frac{1}{2^{k-1}} \right\}.$$ 

This proves the lemma because $i'^* \iota \Delta = \iota \Delta \text{Hom}(i'^*, 1)$.

Proof of Proposition 2.2. We may suppose that $\nu$ is a complex vector bundle, since the stable tangent bundle of $\mathbb{R}P^{2k-1}$ has a complex structure.

Observing the construction of $I$ we have the following commutative diagram.

$$
\begin{array}{cccc}
\tilde{K}_0(\Sigma^{0,2i+2l}S^{2i,0}) & \overset{f^*}{\longrightarrow} & \tilde{K}_0(\Sigma^{2i,2l}) \\
I_0 & \downarrow & \\
\tilde{K}_0(\Sigma^{2i,2l}) & = & \tilde{K}_0(S^{2i}) \otimes R(G) \\
\downarrow & & \\
\tilde{K}_0(S^{2i+2l-2m}T(\nu)) & \overset{\tilde{g}^*}{\longrightarrow} & \tilde{K}_0(S^{2i}) \\
D_2 & \downarrow & D_3 \\
K_1(\mathbb{R}P^{2i}) & \overset{i'^*}{\longrightarrow} & K_1(\mathbb{R}P^{2k-1}) & \overset{\tilde{g}^*}{\longrightarrow} & K_1(S^{2i-1})
\end{array}
$$

Here $D_2, D_3$ are the duality isomorphisms as in [19], Corollary (7.10), and $I_0, I_1$ are isomorphisms given by $I_0((\psi \lambda_{0,2i+2l})\tilde{\eta}_{2k-1}) = (\psi \lambda_{0,2i+2l-2m})\lambda_{2i} \tilde{\eta}_{2k-1}, I_1(\lambda_{0,2i}) = \lambda_{2i,2l}$ where $\lambda_\nu$ denotes the Thom class of $\nu$.

By [19], Corollaries (7.8) and (7.10) we have

$$D_2 I_0((\psi \lambda_{0,2i+2l})\tilde{\eta}_{2k-1}) = P\tilde{\eta}_{2k-1},$$

which is pointed out by Dyer in [8]. By Lemma 2.3 we therefore have

$$((\iota \Delta)^{-1}(i'^* \tilde{g} ) \beta) \tilde{\eta}_{2k} = -\left\{ \frac{b(\tilde{f})}{2^{k-1}} \right\}$$

where $\beta = D_3(\psi \lambda_{0,2i})$.

Identifying $K_1(\mathbb{R}P^{2i})$ with $\text{Hom}(\tilde{K}_0(\mathbb{R}P^{2i}), Q/Z)$ through the isomorphism $\iota \Delta$, we may write

$$(h \alpha_1)\tilde{\eta}_{2k} = -\left\{ \frac{b(\tilde{f})}{2^{k-1}} \right\}$$

in terms of the Hurewicz homomorphism $h: \pi_{2k-1}^{S^1}(BG) \rightarrow K_1(BG)$. Hence by [11], Theorem 2.1 we obtain

$$(CH^i(\alpha_1))\tilde{\eta}_{2k} = -\left\{ \frac{b(\tilde{f})}{2^{k-1}} \right\}$$

where $CH^i$ is the functional Chern character. By the naturality of $CH^i$ we get
Therefore

\[ \{ b(f) \} = e_c t(\alpha_1). \]

q.e.d.

Consequently we get the following

**Theorem 2.4.** For \( \alpha \in \pi^s_{6,2q-1} \) \((q \geq 1), \)

\[ e_c(\alpha) = \begin{cases} (e_c(\psi \alpha), 0) & \text{for } \alpha \in \text{Im } \theta \\ (e_c(\psi \alpha), \frac{1}{2} (e_c(\psi \alpha) + e_c t(\alpha_1) + \varepsilon)) & \text{for } \alpha \in \text{Im } \pi^* \end{cases} \]

(\( \varepsilon = 0, 1 \)) where \( \alpha_1 \) denotes the first factor of \( I\pi^{*\,-1}(\alpha) \) under the identification \( \pi^s_{2q-1}(BG_+) = \pi^s_{2q-1}(BG) \oplus \pi^s_{2q-1}. \)

Proof. As to the first factors this is clear from the definitions of \( e_c \) and \( e_c. \) As to the second this follows in addition from Proposition 1.4, Lemma 2.1 and Proposition 2.2. q.e.d.

**3. Images of the \( S^1 \)-transfer**

Let \( t: \pi^s(BS^1) \rightarrow \pi^s_{n+1}(BG_+) \) denote the \( S^1 \)-transfer, where \( BS^1 \) is the complex infinite dimensional projective space.

**Proposition 3.1.** Let \( \alpha \in \text{Im } \{ \pi^*: \lambda_{0,4q-1} \rightarrow \pi^s_{0,4q-1} \} \) \((q \geq 1) \) and \( I\pi^{*\,-1}(\alpha) \) \( \in \text{Im } t. \) Then

\[ e_c t(\alpha_1) = (1 - 2^e) e_c(\psi \alpha) \]

where \( \alpha_1 \) is as in Theorem 2.4.

Proof. Consider the isomorphisms

\[ \lambda_{0,4q-1} \cong I \pi^s_{4q-1}(BG_+) = \pi^s_{4q-1}(BG) \oplus \pi^s_{4q-1}. \]

We may write \( I\pi^{*\,-1}(\alpha) = (\alpha_1, \alpha_2). \) Applying \( t \) we have

\[ \psi \alpha = t\alpha_1 + 2\alpha_2, \]

Since \( t = \psi \pi^{*\,-1} \) and \( t \) operates on \( \pi^s_{4q-1} \) as multiplication by 2. From [13], Theorem 3.4 it follows that

\[ e_c(\alpha_2) = 2^{\varepsilon\,1} e_c(\psi \alpha). \]

Therefore we get the proposition.
The following theorem follows immediately from Theorem 2.4 and Proposition 3.1.

**Theorem 3.2.** For $\alpha \in \pi_4, \, \gamma = 1$ as in Proposition 3.1 we have

$$e_0(\alpha) = (e_{c}(\psi \alpha), (1-2^{q-2})e_{c}(\psi \alpha) + \frac{r}{2}), \quad (\varepsilon = 0, 1).$$

Let $J_0: \tilde{KO}_0^{-1}(\Sigma^0, \gamma = 1) \to \pi_4, \gamma = 1$ be the equivariant $J$-homomorphism [14, 17]. Set $\alpha = J_0(H\nu) \in \pi_4, \gamma = 1$ where $\nu$ is a canonical generator of $\tilde{KO}_0^{-1}(S^{4q-1})$ and $H = R^{1,0}$. Then $\alpha \in \text{Im} \, \pi^*$ because $\phi(\alpha) = 0$.

**Lemma 3.3.** Let $\alpha$ be as above. Then $I\pi^* - I(\alpha)$ or $2I\pi^* - I(\alpha) \in \text{Im} \, \tilde{f}$ according as $q$ is odd or even.

**Proof.** We consider the $S^1$-homotopy theory. Replace $R^{1,0}$ by the standard complex 1-dimensional non trivial representation $V$ of $S^1$ in the $Z/2$-homotopy theory. Then by the same argument as in [12] we have the $S^1$-homotopy groups $\pi^*_n, S^1, \lambda^*_n, S^1$ and an exact sequence $\lambda^*_n, S^1 \to \pi^*_n, S^1 \to \pi^*_n$. Moreover, we have an isomorphism $\lambda^*_n, S^1 \cong \pi^*_n, S^1$. Clearly the diagram

$$\begin{array}{ccc}
\lambda^*_n, S^1 & \to & \pi^*_n, S^1 \\
\phi \downarrow & & \downarrow \phi \\
0 & \to & \pi^*_n, S^1 \\
\end{array}$$

commutes where $r$ denotes the restriction of $S^1$-actions. Identifying the left-hand groups with the cobordism groups canonically, $r$ agrees with the $S^1$-transfer $\tilde{f}$.

Analogously for $S^1$-actions we can define the equivariant $J$-map $J_V$ as follows. Denote by $U(kV+l)$ the unitary group of $kV \oplus C^l$ with the induced action and by $U_V$ the infinite unitary group obtained by taking a limit with respect to canonical inclusions of $U(kV+l)$'s. Then we have a map $J_V$ from the equivariant homotopy group $[S^*, U_V]^{S^1}$ to $\pi^*_n, S^1$ as usual.

Now a generator $\mu$ of $K^{-1}(S^{4q-1})$, viewed as a map from $S^{4q-1}$ to an unitary group, comes from $[S^{4q-1}, U_V]^{S^1}$ and so $V\mu$ does. Generally an equivariant map from $S^{4q-1}$ to $U_V$ defines an element of $K^{-1}(S^{4q-1})$. So we have a map $[S^{4q-1}, U_V]^{S^1} \to K^{-1}(S^{4q-1})$.

Because $J_V(V\mu) = 0$, using the same notation for $V\mu$ in $[S^{4q-1}, U_V]^{S^1}$, there exists $x \in \lambda^*_q, S^1$ such that $\pi^*x = J_V(V\mu)$. From the above discussion it follows that $r(J_V(V\mu)) = x$ or $2x$, so that $r(x) = \pi^*-1(x)$ or $2\pi^*-1(x)$, according as $q$ is odd or even.

Let $J_0$ be the real $J$-homomorphism. By [1, IV], Theorem 7.16 we may write
where \( m(2q), e'_R \) are as in [1, II). Then we have

**Theorem 3.4.** For \( \alpha = J_0(Hv) \in \pi_{5, 4q - 1}^S \) (\( q \geq 1 \)),

\[
e_G(\alpha) = \begin{cases} \left( \frac{2a_q}{m(2q)}, 2(1 - 2^{q-1}) \frac{a_q}{m(2q)} + \frac{e}{2} \right) & (q \text{ odd}) \\ \left( \frac{a_q}{m(2q)}, (1 - 2^{q-1}) \frac{a_q}{m(2q)} + \frac{e}{4} + \frac{e'}{2} \right) & (q \text{ even}) \end{cases}
\]

\((e, e' = 0, 1)\) as rational numbers mod 1 and the order of each factor of \( e_G(\alpha) \) is \( \frac{m(2q)}{2} \) or \( m(2q) \) according as \( q \) is odd or even.

**Proof.** The first claim follows from Theorem 3.2, Lemma 3.3 and [1, IV], Proposition 7.14. The second follows from [1, II], Lemma (2.12) and the equality \( \nu_2(m(2q)) = 3 + \nu_2(q) \) ([1, II], p. 139) immediately. \( \text{e.d.q.} \)

4. Real \( \mathbb{Z}/2 \)-invariants

We take a base point preserving \( G \)-map \( f: \Sigma^{p+5q, 2q-1+8i} \rightarrow \Sigma^{8+8i} \) as a representative of elements of \( \pi_{5q, 2q-1}^S \) for \( p, q - 1 \geq 0 \). Then the parallel argument to \( e_G \), using the Adams operation in the \( K_0 \)-theory [12] and Table of [14], yields the following equivariant \( e \)-invariants.

\[
\begin{align*}
(1) & \quad e_{G, R}: \pi_{5q+4, 8q+4q-1}^S \rightarrow \left\{ (Q/Z)^2 \bigg| \begin{array}{c} (Q/Z) \\ (Q/Z) \end{array} \right\} \quad (i = 0) \\
(2) & \quad e_{G, R}: \pi_{5q+4, 8q+4q+1}^S \rightarrow Q/Z \quad (i = 1, 2, 3)
\end{align*}
\]

for \( i, 8 = 0, 1 \).

**Theorem 4.1.** For \( \alpha = J_0(Hv), \alpha = J_0(Hv) \in \pi_{5, 4q - 1}^S \) (\( q \geq 1 \)),

\[
e_{G, R}(\alpha) = \left( \frac{a_q}{m(2q)}, 0 \right)
\]

\[
e_{G, R}(\alpha) = \left( \frac{a_q}{m(2q)}, (1 - 2^{q-1}) \frac{a_q}{m(2q)} + \frac{e}{4} + \frac{e'}{2} \right)
\]

\((e, e' = 0, 1)\) as rational numbers mod 1 and the order of the second factor of \( e_{G, R}(\alpha) \) is \( m(2q) \).

**Proof.** As to the first factors of the equalities this follows immediately from the definitions of \( e_{G, R} \) and \( e'_R \). As to the second this follows in addition from Theorem 3.4 and the fact that \( e_G = e_{G, R} \) or \( 2e_{G, R} \) according as \( q \) is even or odd. The proof of the last claim is similar to that of Theorem 3.4. \( \text{q.e.d.} \)
Finally we shall consider $e_{G,R}$ on $\text{Im } J_G$ for $\pi_{p,q-1}^\delta$ ($p \geq 1$). Let $\chi, \rho$ be as in [3] and $\eta$ be the homomorphism induced by the element of [4], (8.1). Observe $\chi, \rho$ and $\eta$ on the groups $\widetilde{K}O_G(\Sigma^{p,2q-1})$ (see [15], §2), then since $e_{G,R}J_G$ commutes with $\chi, \rho$ and $\eta$ (by an analogue of Proposition 1.3), we can compute $e_{G,R}$ of (1) on $\text{Im } J_G$ inductively by using Theorem 4.1. For $e_{G,R}$ of (2), considering $\psi e_{G,R}$ we get readily $e_{G,R}$ on $\text{Im } J_G$. Specifically we have

**Theorem 4.2.** Let $\nu_1 \in \widetilde{K}O_G(\Sigma^{3p+4\xi,5q+4\delta-1}) (8p+4q>0)$, $\nu_2 \in \widetilde{K}O_G(\Sigma^{3p+4\xi+1,8q+4\delta-1})(1 \leq i \leq 3)$ and $\nu_3 \in \widetilde{K}O_G(\Sigma^{3p+4\xi+2,3q+4\delta+1})$ be generators as modules over the real representation ring of $G$ respectively and set $\alpha_i = J_G(\nu_i) (1 \leq k \leq 3)$. Then as rational numbers mod 1

$$e_{G,R}(\alpha_1) = \left( \frac{a_{2q+2\xi+3}}{m(4p+4q+2\xi+2\delta)} \right) \left( \frac{a_{2q+2\xi+3}}{m(4p+4q+2\xi+2\delta)} \right) \left( \frac{a_{2q+2\xi+3}}{m(4p+4q+2\xi+2\delta)} \right)$$

$$- (1 - 2^{4q+2\delta-1}) \left( \frac{a_{2q+2\xi+3}}{m(4q+2\delta)} \right) \left( \frac{\varepsilon}{4} - \frac{\varepsilon'}{2} + \varepsilon'' \right),$$

$$e_{G,R}(\alpha_2) = (1 - 2^{4q+2\delta-1}) \left( \frac{a_{2q+2\xi+3}}{m(4q+2\delta)} \right) + \frac{\varepsilon}{4} + \frac{\varepsilon'}{2},$$

$$e_{G,R}(\alpha_3) = \frac{a_{2q+2\xi+3+1}}{m(4p+4q+2\xi+2\delta+2)} + \varepsilon$$

$(\varepsilon, \varepsilon', \varepsilon''=0, 1)$ up to sign and

order $e_{G,R}(\alpha_1) = \frac{m(4p+4q+2\xi+2\delta)m(4q+2\delta)}{2^d}$,

order $e_{G,R}(\alpha_2) = m(4q+2\delta)$,

order $e_{G,R}(\alpha_3) = m(4p+4q+2\xi+2\delta+2)$

where

$$d = \left( \frac{m(4p+4q+2\xi+2\delta)}{2^{(2p+2q+2\xi+3)+3}}, \frac{m(4q+2\delta)}{2^{2(2q+2\xi+3)+3}} \right)$$

and $\kappa$ is the following integer:

- $\nu_2(2q+\xi)+2$ if $\xi = 0$ and $\nu_2(2q+\xi) \leq \nu_2(p+q+\xi)$,
- $\nu_2(2q+\xi)+3$ if $\xi = 0$ and $\nu_2(2q+\xi) = \nu_2(p+q+\xi)+1$,
- $\nu_2(p+q+\xi)+3$ if $\xi = 0$ and $\nu_2(2q+\xi) \geq \nu_2(p+q+\xi)+2$,
- $3$ if $\xi = 1$ and $\delta = 1$,
- $2$ if $\xi = 1$ and $\delta = 0$.

Here let $\nu_2(s)$ denote the exponent to which 2 occurs in $s$.

By Theorems 4.1, 4.2 and the results of [15] we have

**Corollary 4.3.** For $\pi_p^\delta$, in [15], Theorems 3.1, 3.2 and 3.3,
\[ \text{Im} J_G \overset{i}{\hookrightarrow} \pi_{p,q}^S e_{G,R} \xrightarrow{\text{Im}} \text{Im} e_{G,R} \]

provides a direct sum splitting.

References


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