UNIPOTENT CHARACTERS OF $SO_{2n}^+$, $SP_{2n}$ AND $SO_{2n+1}$
OVER $F_q$ WITH SMALL $q$

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0. Introduction. Let $G$ be a special orthogonal group or symplectic group over a finite field $F_q$, $F$ the Frobenius mapping and $G^F$ the group of all $F$-stable points of $G$. G. Lusztig [7], [8] has obtained explicit formulas for the characters of the unipotent representations of $G^F$ on any regular semisimple element of $G^F$ provided that the order $q$ of the defining field $F_q$ is sufficiently large. Our purpose in this paper is to show that his formulas are valid for any $q$.

Let $W$ be the Weyl group of $G$ and $m$ an odd positive integer. For $w \in W$, let $R_w^{(m)}$ be the Deligne-Lusztig virtual representation [2], [6, 3.4] of $G^{F^m}$. By [2, 7.9], to determine the values of the character of a unipotent representation $\rho$ of $G^{F^m}$ on regular semisimple elements, it suffices to determine the inner product

$$\langle R_w^{(m)}, \rho \rangle$$

for any $w \in W$. This has been done by G. Lusztig [7], [8] for a sufficiently large $q^m$. Let $n$ be the rank of $G$ and $\Psi_n$ be the set of symbol classes (cf. [5, §3]) that parameterizes the unipotent representations (up to equivalence) of $G^F$ or $G^{F^m}$, i.e.

$$\Psi_n = \begin{cases} \Phi_n & \text{if } G = SO_{2n+1} \text{ or } Sp_{2n} \\ \Phi_n^\pm & \text{if } G = SO_{2n} \end{cases}$$

in the notations in [5, §3]. For $\Lambda \in \Psi_n$, let $\rho_\Lambda^{(1)}$ and $\rho_\Lambda^{(m)}$ be the corresponding unipotent representations of $G^F$ and $G^{F^m}$ respectively. Our main result (Theorem 4.2, (iii)) is

$$(*) \quad \langle R_w^{(m)}, \rho_\Lambda^{(m)} \rangle = \langle R_w^{(1)}, \rho_\Lambda^{(1)} \rangle$$

for any $\Lambda \in \Psi_n$ and $w \in W$ if $m$ is any sufficiently large positive integer prime to $2p$ with $p$ the characteristic of $F_q$. Hence the required character formula is obtained for any $q$.

Our proof goes as follows. Firstly, we write the Frobenius mapping $F$
as $F=jF_0$ with $F_0$ a split Frobenius mapping and $j$ an automorphism of $G$ of finite order commuting with $F_0$, and let $\sigma_0=F_0|G^{\sigma_0}$ and $\langle \sigma_0 \rangle$ be the cyclic group generated by $\sigma_0$. Let $X_w^{(m)}$ ($w \in W$) be the Deligne-Lusztig varieties [2], [6] of $G$ defined using the Frobenius mapping $F^m$. Then $G^{\sigma_0}$ and $F_0$ act naturally on $X_w^{(m)}$, hence on their $\ell$-adic cohomology spaces $H^i(X_w^{(m)})$.

Then we prove (Theorem 3.2) the relation
\[(**)
Tr((xF_0)^*, \sum_{i=0}^{\infty} (-1)^i H^i(X_w^{(m)})) = Tr((yF_0)^*, \sum_{i=0}^{\infty} (-1)^i H^i(X_w^{(1)}))
\]
for any odd integer $m$ and any $x \in G^{\sigma_0}$, where $y=N^{(m)}(x)$ and $N^{(m)}$ is the norm mapping defined by N. Kawanaka (see our definition preceding Theorem 3.2).

As a next step, we show that any unipotent representation of $G^{\sigma_0}$ is $\sigma_0$-invariant if $m$ is odd. Then by applying N. Kawanaka's result on the lifting [3], [4], we prove (Theorem 4.2) that
\[(***)
Tr(xj\sigma, \rho_\Lambda^{(m)}) = Tr(N^{(m)}(x)j, \rho_\Lambda^{(1)})
\]
for any $x \in G^{\sigma_0}$, any symbol class $\Lambda \in \Psi_n$ and any positive integer prime to $2p$, where $\rho_\Lambda^{(m)}$ and $\rho_\Lambda^{(1)}$ are the representations of the semi-direct product groups $G^{\sigma_0}\langle \sigma_0 \rangle$ and $G^{\sigma_0}\langle j \rangle$ that extend $\rho_\Lambda^{(m)}$ and $\rho_\Lambda^{(1)}$ respectively in a normalized manner. Combining polynomial equations (in $q$) obtained from (**) and (***) with a result on Frobenius eigenvalues given in [1] (resp. [8]), we get the asserted relation (*) for $G=Sp_{2n}$, $SO_{2n+1}$ (resp. $SO_{2n}^*$.)

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1. First we need a generalization of Lusztig [6, 3.9]. Let $G$ be a connected reductive group defined over a finite field $F_q$ and $F$ the Frobenius mapping. Let $B$ be a fixed $F$-stable Borel subgroup, $T$ a fixed $F$-stable maximal torus in $B$, $U$ the unipotent radical of $B$ and $W$ the Weyl group of $G$ relative to $T$. There exists an automorphism $j$ of $G$ of finite order $\delta$ defined over $F_q$ such that $j$ stabilizes $B$, $T$ and induces the same action on $W$ as that of $F$. For a positive integer $m$, we set
\[
\sigma = F|G^{\sigma_0}, \quad F_0 = j^{-1}F, \quad \sigma_0 = j^{-1}\sigma.
\]
$\sigma$ and $\sigma_0$ generate the cyclic groups $\langle \sigma \rangle$ of order $m$ and $\langle \sigma_0 \rangle$ of order $m\delta$ respectively. We denote by $X$ the variety $G/B$ of all Borel subgroups. For our purpose we have to borrow almost all the notations in [6, 3.3–3.9] such as
\[
X_w, Y_{w,w',w''}, Z_{w,w',w''}, \quad (w, w', w'' \in W).
\]
But to specify the Frobenius mapping (either $F$ or $F^m$), we write as follows (cf. [6, 3.3–3.4]).
Theorem 1.1. For \( w, w' \in W, F_0 \) acts naturally on the variety \( G^{F_w} \backslash (X^{(m)}_w \times X^{(m)}_{w'}) \), and

(i) all the eigenvalues of \( F^*_0 \) on \( H^i_c(G^{F_w} \backslash (X^{(m)}_w \times X^{(m)}_{w'})) \) are integral powers of \( q \),

(ii) for a positive integer \( e \), the number of \( F^*_0 \)-fixed points of the quotient variety \( G^{F_w} \backslash (X^{(m)}_w \times X^{(m)}_{w'}) \) is equal to the trace of the linear transformation \( x \to t_wF_0(x)t^{-1}_w \) of \( \mathcal{B}(W, q') \).

Proof. The proof of \( [6, 3.8] \) shows that it suffices to prove the following variation of \( [6, 3.5] \):

There exists a natural isomorphism \( H^i_c(Y^{(m)}_{w, w'}) \cong H^i_c(Z^{(m)}_{w, w'}) \) for any \( i \geq 0 \) which commutes with the action of \( F^*_0 \).

But this can be proved by almost the same argument as in the proof of \( [6, 3.5] \).

Let \( \rho \) be a unipotent representation of \( G^{F_w} \). For \( w \in W \) and \( i \geq 0 \), \( H^i_c(X^{(m)}_w)_\rho \) denotes the largest subspace of \( H^i_c(X^{(m)}_w) \) on which \( G^{F_w} \) acts by a multiple of \( \rho \). We choose \( w \) and \( i \) in such a way that \( H^i_c(X^{(m)}_w)_\rho \neq 0 \). Fix a decomposition

\[
H^i_c(X^{(m)}_w)_\rho = (\bar{Q}_l \oplus \cdots \oplus \bar{Q}_l) \otimes \rho
\]

\[ \text{r-times} \]

as a \( G^{F_w} \)-module. Then the \( G^{F_w} \)-module endomorphism algebra of \( H^i_c(X^{(m)}_w)_\rho \) is identified with the matrix algebra \( M_r(\bar{Q}_l) \) of rank \( r \). Assume that \( \rho \) is \( \sigma_0 \)-invariant (up to equivalence). Then \( \rho \) is extended to an irreducible representation \( \bar{\rho} \) of the semi-direct product \( G^{F_w} \langle \sigma_0 \rangle \). There are \( m \delta \)-choices for such \( \bar{\rho} \). We fix \( \bar{\rho} \) to be one of them. We may regard \( H^i_c(X^{(m)}_w)_\rho \) as a \( G^{F_w} \langle \sigma_0 \rangle \)-module by the identification

\[
H^i_c(X^{(m)}_w)_\rho = (\bar{Q}_l \oplus \cdots \oplus \bar{Q}_l) \otimes \bar{\rho}
\]

\[ \text{r-times} \]

Since \( \rho \) is \( \sigma_0 \)-invariant, \( F^*_0 \) stabilizes \( H^i_c(X^{(m)}_w)_\rho \) and \( F^*_0 \) acts on \( H^i_c(X^{(m)}_w)_\rho \) by
Let \( \rho \) be a \( \sigma_0 \)-invariant unipotent representation of \( G^{p_m} \) and \( \bar{\rho} \) be its extension to an irreducible representation of \( G^{p_m} \langle \sigma_0 \rangle \). Let \( \mu \) be any eigenvalue of the matrix \( \xi \) defined as above for some \( i \) and \( w \). Then \( \mu \) is uniquely determined by \( \bar{\rho} \) up to a multiplicative factor \( q^a \) for an integer \( a \) and does not depend on the choice of \( i \) and \( w \).

Proof. We proceed quite identically with the proof of [6, 3.9]. Let \( \bar{\rho} \) be the dual representation of \( \rho \). Obviously the representation \( \bar{\rho} \) restricted to \( G^{p_m} \) is the dual representation \( \bar{\rho} \) of \( \rho \). Take \( w' \in W, i' \geq 0 \) such that \( \bar{\rho} \) is a subrepresentation of \( H_i^{w'}(X^{p_m}) \). Fix an identification

\[ H_{i'}^{w'}(X^{p_m})_{\bar{\rho}} = (\mathcal{Q}_{\xi} \oplus \cdots \oplus \mathcal{Q}_{\xi}) \otimes \rho \]

d and write \( F_\sigma^\times = \xi' \otimes \bar{\rho}(\sigma^{-1}) \) on \( H_{i'}^{w'}(X^{p_m})_{\bar{\rho}} \) with \( \xi' \in M_r(\mathcal{Q}_{\xi}) \). First we consider the orthogonal projection from the space \( \bar{\rho} \otimes \bar{\rho} \) to the \( G^{p_m} \)-invariant subspace \( (\bar{\rho} \otimes \bar{\rho}) G^{p_m} = \mathcal{Q}_{\rho} \), which is defined by

\[ v_1 \otimes v_2 \rightarrow \left| G^{p_m} \right|^{-1} \sum_{s \in G^{p_m}} \bar{\rho}(x)v_1 \otimes \bar{\rho}(x)v_2 \]

Since \( Tr(\left| G^{p_m} \right|^{-1} \sum_{s \in G^{p_m}} \bar{\rho}(x\sigma_0) \otimes \bar{\rho}(x\sigma_0) = 1 \), the following diagram commutes.

The commutativity of this diagram in turn shows the commutativity of the following.

\[ H_{i'}^{w'}(X^{p_m})_{\bar{\rho}} \otimes H_{i'}^{w'}(X^{p_m})_{\bar{\rho}} \rightarrow (H_{i'}^{w'}(X^{p_m})_{\bar{\rho}} \otimes H_{i'}^{w'}(X^{p_m})_{\bar{\rho}}) G^{p_m} \]

Thus the induced action of \( F_\sigma^\times \) on

\[ (H_{i'}^{w'}(X^{p_m})_{\bar{\rho}} \otimes H_{i'}^{w'}(X^{p_m})_{\bar{\rho}}) G^{p_m} \approx (\mathcal{Q}_{\xi} \oplus \cdots \oplus \mathcal{Q}_{\xi}) \otimes (\mathcal{Q}_{\xi} \oplus \cdots \oplus \mathcal{Q}_{\xi}) \]

\[ r\text{-times} \]

[Diagram]

[Diagram]
is identified with $\xi \otimes \xi'$. Now, the canonical inclusion

$$(H^i_\epsilon(X^{(m)}_w)_\rho \otimes H^i_\epsilon'(X^{(m)}_w)_\rho)^{G^{F^m}} \to H^i_\epsilon(X^{(m)}_w \times X^{(m)}_w))$$

commutes with the action of $F^\#$. Therefore, Theorem 1.1 shows that all the eigenvalues of $\xi \otimes \xi'$ have the form $q^a$ for some integer $a$. Since another choice of $i$ and $w$ yields the same result, the required statement follows.

**Definition 1.3.** Let $\rho, \mu$ be as in Theorem 1.2. We define $\mu_\rho$ by

$$1 \leq |\mu_\rho| < q, \quad \mu_\rho = q^a$$

for some integer $a$.

**Corollary 1.4.** For $w \in W$, there exists a unique polynomial $f_{\rho,w}(X)$ such that

(i) $Tr((xF^\rho_\delta)^t, \sum_{i=0}^\infty (-1)^i H^i_\epsilon(X^{(m)}_w)_\rho) = f_{\rho,w}(q^t) Tr((x\sigma_\delta)^{-1}, \rho)$

for any $x \in G^{F^m}$ and positive integer $e$,

(ii) $f_{\rho,w}(1) = \langle \rho_w^{(m)}, R^{(m)}_w \rangle$,

where $R^{(m)}_w$ denotes the virtual $G^{F^m}$-module $\sum_{i=0}^\infty (-1)^i H^i_\epsilon(X^{(m)}_w)$.

Since $j^t=1$, $F^m_{0^{m^5}}=F^{m^5}$. Let $\lambda_\rho$ be the normalized eigenvalue of $(F^{m^5})^*$ associated with $\rho$, i.e. $\lambda_\rho$ is equal to an eigenvalue of $(F^{m^5})^*$ (acting on $H^i_\epsilon(X^{(m)}_w)_\rho$ for some $i$ and $w$) up to a multiplicative factor $q^{m^5a}$ for some integer $a$, and satisfies

$$1 \leq |\lambda_\rho| < q^{m^5}$$

By [6, 3.9], $\lambda_\rho$ is uniquely determined by $\rho$. Let $\bar{\rho}$, $\mu_\bar{\rho}$ be as in Definition 1.3. Obviously $\mu^{m^5}_\bar{\rho} = \lambda_\rho$. There are $m^5\delta$-extensions $\bar{\rho}$ for the fixed $\sigma_0$-invariant $\rho$ and there are $m^5\delta$-constants $\mu$ such that $\mu^{m^5}_\bar{\rho} = \lambda_\rho$. Therefore we have

**Lemma 1.5.** Let $\rho$ be a $\sigma_0$-invariant unipotent representation of $G^{F^m}$. Then the mapping $\bar{\rho} \mapsto \mu_\bar{\rho}$ induces the bijection

$$\{\bar{\rho} \in (G^{F^m}\langle \sigma_0 \rangle) \cap \bar{\rho} | G^{F^m} = \rho \} \to \{\mu ; \mu^{m^5} = \lambda_\rho\}$$

where $(G^{F^m}\langle \sigma_0 \rangle) \cap \bar{\rho}$ denotes the set of irreducible representations of $G^{F^m}\langle \sigma_0 \rangle$ (up to equivalence).

2. Henceforth we assume that the positive integer $m$ is prime to the order $\delta$ of $j$. Let $S$ be the set of simple reflections of $W$ associated with the Borel subgroup $B$. For $I \subseteq S$, let $P_I$ be the corresponding standard parabolic subgroup and $L_I$ its standard Levi subgroup. Let $I_0$ be an $F$-stable subset of $S$. Let $\rho_0$ be a unipotent cuspidal representation of $L^{F^m}_I$. Let $\rho$ be a unipotent representation of $G^{F^m}$. If $\rho$ appears in the induced representation of $G^{F^m}$ from
the representation \( \rho_0 \) inflated to \( P^F_0 \), then we call \( \rho \) a unipotent representation of \( G^F \) in the series of \( \rho_0 \). Now, we assume that \( \rho_0 \) is \( \sigma_0 \)-invariant, and we fix a representation \( \bar{\rho}_0 \) of the semi-direct product \( L^F_0 \langle \sigma_0 \rangle \) that extends \( \rho_0 \). Let \( J \) be any \( F \)-stable subset of \( S \) containing \( I_0 \). We further assume that any unipotent representation \( \rho \) of \( L^F_j \) in the series of \( \rho_0 \) is \( \sigma_0 \)-invariant (for any \( j \)). By [2, 8.2], the eigenvalues of \( (F^F)^* \) associated with \( \rho \) and \( \rho_0 \) coincide with each other (up to a multiplicative factor \( q^{\text{mult}} \) for some integer \( a \)). Therefore we may fix a representation \( \bar{\rho} \) of \( L^F_j \langle \sigma_0 \rangle \) extending \( \rho \) by the condition

\[
\mu_{\bar{\rho}} = \mu_{\rho_0}
\]

(cf. Lemma 1.5).

**Lemma 2.1.** Let the assumptions be as above. Let \( J \) be an \( F \)-stable subset of \( S \) such that \( I_0 \subseteq J \subseteq S \). Let \( \rho \) be a unipotent representation of \( L^F_j \) in the series of \( \rho_0 \). Assume that

\[
\text{Ind}_{P^F_j}^{G^F} \rho = \sum_{1 \leq i \leq r} m_i \rho_i
\]

with each \( \rho_i \) a unipotent representation of \( G^F \) in the series of \( \rho_0 \) and \( m_i \) a positive integer. Then

\[
\text{Ind}_{P^F_j}^{G^F} \langle \sigma_0 \rangle \bar{\rho} = \sum_{1 \leq i \leq r} m_i \bar{\rho}_i
\]

Proof. There are two methods in extending a unipotent representation of \( G^F \) in the series of \( \rho_0 \) to a representation of \( G^F \langle \sigma_0 \rangle \) in normalized manners:

One is by using the eigenvalues of the Frobenius mapping \( F^F \) (the method which we have adopted here). The other is simply inducing the action of \( \sigma_0 \) on the representation \( \bar{\rho}_0 \).

To prove our lemma it suffices to show that these two methods yield the same extension for any \( \rho_i \) (or \( \rho \)). But this is apparent from the proof of [2, 8.2].

3. Let \( H \) be a finite group and \( \alpha \) an automorphism of \( H \). For \( h_1, h_2 \in H \), we define the equivalence relation \( \tilde{\alpha} \) by

\[
h_1 \tilde{\alpha} h_2 \Leftrightarrow h_1 = h^{-1} h_2^a h \quad \text{for some} \ h \in H.
\]

For \( x \in G^F \), write \( x = a^{-1} r_0 a \) with \( a \in G \) and put \( y = F^a a^{-1} \). Then \( x \to y \) defines the bijection

\[
G^F \to G^F / \bar{F}^m
\]

which will be denoted by \( n_{F^m / F_0} \). Quite analogously to Lemma 1.2.1 of [1], we obtain
Lemma 3.1. For any $x \in G^{\pi m}$ and $w \in W$,

$$\text{Tr}((xF_0)^*, \sum_{i \in \mathbb{Z}} (-1)^i H_i^i(X^w))$$

$$= \left( (1 + q)^{-d \# \{ h \in G^F_0; h^{-1}n_{F_0}(x)^{-1}w h \in \mathcal{W}B \} \right),$$

where $d = \dim(U \cap wUw^{-1})$, and $\mathcal{W}$ is an $F_0$-stable representative of $w$ in the normalizer $N_G(T)$ of $T$ in $G$.

Assume $m \equiv 1 \mod 2$. Then we may define the mapping

$$N^{(\pi)} = n_{F_0}^{1/2n_0} \circ n_{F_0}^{x \in F_0} : G^{\pi m} | \tilde{P}_0 \to G^F | \tilde{P}_0$$

Thus by the relation in the lemma combined with that relation with $m = 1$, we obtain

Theorem 3.2. Assume $m \equiv 1 \mod 2$. For any $x \in G^{\pi m}$ and $w \in W$,

$$\text{Tr}((xF_0)^*, \sum_{i \in \mathbb{Z}} (-1)^i H_i^i(X^w))$$

$$= \text{Tr}((N^{(\pi)}(x)F_0)^*, \sum_{i \in \mathbb{Z}} (-1)^i H_i^i(X^w)).$$

4. We preserve the notations used until now. Assume $G = SO^+_{2n}$, $Sp_{2n}$ or $SO_{2n+1}$. In some cases, $G$ is also denoted by $G_n$ to specify $n$. If $G \neq SO_{2n}$, we take $j$ to be of length, and if $G = SO_{2n}$, we take $j$ to be of order 2. Let $\mathcal{W}$ be the semi-direct product $G \langle j \rangle$. If $m \equiv 1 \mod 2$, then $G^{\pi m} \langle \sigma \rangle = G^{\pi m} \langle \sigma_0 \rangle$.

First we need

Lemma 4.1. Assume $m \equiv 1 \mod 2$. Then all the unipotent representations of $G^{\pi m}$ (resp. $G^F$) are $\sigma_0$-invariant.

Proof. For an $F$-stable closed subgroup $H$ of $G$, we denote by $H^{(\pi)}$ the group of all $F^{\pi}$-stable points of $H$. Let $I_0$ be a subset of $\mathcal{S}$ such that there exists a unipotent cuspidal representation $\rho_0$ of $L^{(\pi)}_{I_0}$. To prove the lemma it suffices to prove that any unipotent representation of $G^{(\pi)}$ in the series of $\rho_0$ is $\sigma_0$-invariant. We recall a result of Lusztig [5, §5]. Let $\mathcal{W} = (N_\mathcal{W}(L_{I_0})/L_{I_0})^{\pi m}$, where $N_{\mathcal{W}}(L_{I_0})$ is the normalizer of $L_{I_0}$ in $G$. $\mathcal{W}$ has a natural structure as a Coxeter group with the canonical set of generators $\mathcal{S}$. For a subset $J$ of $\mathcal{S}$ with $I_0 \subseteq J \subseteq \mathcal{S}$, a subset $J$ of $\mathcal{S}$ is associated in a natural manner and any subset of $\mathcal{S}$ is obtained in this form. We denote by $\mathcal{W}_J$ the subgroup of $\mathcal{W}$ generated by $J(\subseteq \mathcal{S})$. Then unipotent representations (up to equivalence) of $G^{(\pi)}$ (resp. $L^{(\pi)}_J$) in the series of $\rho_0$ are parameterized by the set of irreducible representations $\mathcal{W}_J$ (resp. $(\mathcal{W}_J)^{\pi m}$) of $\mathcal{W}$ (resp. $\mathcal{W}_J$). And this parameterization is compatible with the inductions:
\[ \begin{align*}
\chi & \in R(W_T) \sim \{ \text{Z-linear combi. of unip. char. of } L_j^{(m)} \} \text{ in the series of } \rho_0 \\
\text{Ind}_W^{W_T} \chi & \in R(W) \sim \{ \text{Z-linear combi. of unip. char. of } G^{(m)} \} \text{ in the series of } \rho_0
\end{align*} \]

where \( R(W_T) \) and \( R(W) \) denote the group of all virtual characters of \( W_T \) and \( W \) respectively, and irreducible characters are mapped to the irreducible characters by the horizontal isomorphisms. Now, \((W, \tilde{S})\) is isomorphic to a classical Weyl group. Thus, if \( \text{rank}(W, \tilde{S}) \geq 2 \), then we have:

For \( \chi_1, \chi_2 \in \tilde{W} \), if \( \chi_1 | W_T = \chi_2 | W_T \) for any \( j \in \tilde{S} \), then \( \chi_1 = \chi_2 \).

Therefore to prove that any unipotent representation \( \rho \) in the series of \( \rho_0 \) is \( \sigma_0 \)-invariant, it suffices to prove the statement only when \( \rho \) is a cuspidal (i.e. \( I_0 = S \)) or subcuspidal (i.e. \(|S \setminus I_0| = 1\)) representation (see [5]). Assume that \( \rho \) is cuspidal, i.e. \( \rho = \rho_0 \). Then \( \rho \) is the unique unipotent cuspidal representation. Therefore \( \rho \) is \( \sigma_0 \)-invariant. Assume that \( \rho \) is subcuspidal. Let \( \rho' \) be another unipotent subcuspidal representation (see [5]). Since \( \dim \rho = \dim \rho' \) (cf. [4]) and there is no other unipotent subcuspidal representation, \( \rho \) and \( \rho' \) are both \( \sigma_0 \)-invariant.

Henceforth we assume that \( m \) is prime to \( 2p \) with \( p \) the characteristic of \( \mathbb{F}_q \). Then by N. Kawanaka [3], [4], the following statement is true:

For any \( \sigma_0 \)-invariant irreducible representation \( \rho^{(m)} \) of \( G^{(m)} \), there exists a \( \sigma_0 \)-invariant (or \( j \)-invariant) irreducible representation \( \rho^{(1)} \) of \( G^f \) such that

\[ \text{Tr}(xy^j, \rho^{(m)}) = c \text{Tr}(N^{(m)}(x)^j, \rho^{(1)}) \]

for any \( x \in G^f \), where \( \rho^{(m)} \) (resp. \( \rho^{(1)} \)) is an irreducible representation of \( G^f \langle \sigma \rangle \) (resp. \( G^f \)) that extends \( \rho^{(m)} \) (resp. \( \rho^{(1)} \)), and \( c \) is a root of unity. We now assume that \( m \) is sufficiently large so that the main theorem in [7] (resp. [8]) holds for the group \( G^f \) if \( G = SO_{2n+1} \) or \( Sp_{2n} \) (resp. \( G = SO_{5n}^* \)). Let \( \Phi^*_n, \Phi^*_n \) be the sets of symbol classes defined in [5, § 3]. We set

\[ \Psi^*_n = \begin{cases} 
\Phi^*_n & \text{if } G = SO_{2n+1} \text{ or } Sp_{2n} \\
\Phi^*_n (\text{resp. } \Phi^*_n) & \text{if } G = SO_{5n}^* (\text{resp. } SO_{5n}^*)
\end{cases} \]

By [5], the unipotent representations of \( G^f \) (resp. \( G^f \)) are parameterized by the symbol classes in \( \Psi^*_n \). For \( \Lambda \in \Psi^*_n \), we denote by \( \rho^{(m)}_\Lambda \) (resp. \( \rho^{(1)}_\Lambda \)) the corresponding unipotent representation of \( G^f \) (resp. \( G^f \)), and by \( \lambda^{(m)}_\Lambda \) (resp. \( \lambda^{(1)}_\Lambda \)) the normalized eigenvalue of \( (F^m)^* \) (resp. \( (F^1)^* \)) associated with the unipotent representation \( \rho^{(m)}_\Lambda \) (resp. \( \rho^{(1)}_\Lambda \)). By [1], \( \lambda^{(m)}_\Lambda = 1 \) or \(-1\) if \( G = SO_{2n+1}, Sp_{2n} \) or \( SO_{5n}^* \). By [8, 3.4], \( \lambda^{(m)}_\Lambda = \lambda^{(1)}_\Lambda = 1 \) for any \( \Lambda \in \Psi^*_n \) if \( G = SO_{5n}^* \).
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Since $m$ is odd, we may choose the extension $\tilde{\rho}^{(m)}_\Lambda \in (G^{F^m} \langle \sigma \rangle)^\wedge$ of $\rho^{(m)}_\Lambda$ by the condition

$$\mu_\rho^{(m)} = \lambda_{\rho^{(m)}_\Lambda}$$

(See Lemma 1.5). And we may choose the extension $\tilde{\rho}^{(1)}_\Lambda \in (G^{F} \langle j \rangle)^\wedge$ of $\rho^{(1)}_\Lambda$ by the condition

$$\mu_\rho^{(1)} = \lambda_{\rho^{(1)}_\Lambda};$$

Here we applied Lemma 1.5 with $m=1$. Let $(W \langle j \rangle)^*\Lambda$ be the set of irreducible representations $\chi$ (up to equivalence) of the semi-direct product $W \langle j \rangle$ such that $\chi|_W$ is irreducible. For any $\chi \in (W \langle j \rangle)^*\Lambda$, let $R^{(m)}_\chi$ be the class function of $G^{F^m}$ defined in [6, (3.17.1)], i.e.

$$R^{(m)}_\chi = |W|^{-1}\sum_{w \in W} \text{Tr}(w, \chi) R^{(m)}_w$$

where $R^{(m)}_w$ is the character of the virtual $G^{F^m}$-module $\sum_{i \geq 0} (-1)^i H^{i}_C(X^{(m)}_w)$. We are to prove

**Theorem 4.2.** Let $\tilde{\rho}^{(m)}_\Lambda$ and $\tilde{\rho}^{(1)}_\Lambda (\Lambda \in \Psi)$ be the extensions of $\rho^{(m)}_\Lambda$ and $\rho^{(1)}_\Lambda$ chosen as above. Then we have

1. $\chi_{\rho^{(m)}_\Lambda}(\tilde{\rho}^{(m)}_\Lambda) = \chi_{\rho^{(1)}_\Lambda}(\tilde{\rho}^{(1)}_\Lambda)$ for any $\chi \in G^{F^m}$,
2. $\chi_{\rho^{(m)}_\Lambda}(\tilde{\rho}^{(m)}_\Lambda) = \chi_{\rho^{(1)}_\Lambda}(\tilde{\rho}^{(1)}_\Lambda)$
3. $\langle \rho^{(m)}_\Lambda, R^{(m)}_\chi \rangle = \langle \rho^{(1)}_\Lambda, R^{(1)}_\chi \rangle$ for any $\chi \in (W \langle j \rangle)^*\Lambda$, $\Lambda \in \Psi$,
4. $f_{\rho^{(m)}_\Lambda, \chi}(X) = f_{\rho^{(1)}_\Lambda, \chi}(X)$ for any $\chi \in (W \langle j \rangle)^*\Lambda$, $\Lambda \in \Psi$.

**Corollary 4.3** The main theorems in G. Lusztig [7], [8] are true for any finite field.

**Lemma 4.4.** Let $\Lambda_1, \Lambda_2 \in \Psi$. Assume

(*) $\text{Tr}(xj\sigma, \tilde{\rho}^{(m)}_\Lambda) = c \text{Tr}(N^{(m)}(x), j, \tilde{\rho}^{(1)}_\Lambda)$

for any $x \in G^{F^m}$ with some root $c$ of 1. Then

1. $\chi_{\rho^{(m)}_{\Lambda_1}}(\tilde{\rho}^{(m)}_{\Lambda_2}) = \chi_{\rho^{(1)}_{\Lambda_2}}(\tilde{\rho}^{(1)}_{\Lambda_2})$
2. $\dim \rho^{(1)}_{\Lambda_1} = \dim \rho^{(1)}_{\Lambda_2}$
3. $\langle \rho^{(1)}_{\Lambda_2}, R^{(m)}_\chi \rangle = \langle \rho^{(1)}_{\Lambda_2}, R^{(1)}_\chi \rangle$ for any $\chi \in (W \langle j \rangle)^*\Lambda$, $\Lambda \in \Psi$,
4. $f_{\rho^{(m)}_{\Lambda_1}, \chi}(X) = f_{\rho^{(m)}_{\Lambda_2}, \chi}(X)$ for any $\chi \in (W \langle j \rangle)^*\Lambda$, $\Lambda \in \Psi$.

To prove the lemma we need some preparations. Let $H(W)$ be the generalized Hecke algebra of the Coxeter group $(W, S)$ over the polynomial ring $Q[X]$ that yields by the specialization $(X \rightarrow q)$ the $G^F$-module endomorphism algebra of the induced representation of $G^F$ from the trivial representation of
Let \( \{ a_w; w \in W \} \) be the canonical basis of \( H(W) \). \( H(W) \) is a subalgebra of an algebra \( H(W \langle j \rangle) \) defined as follows.

\[
H(W \langle j \rangle) = H(W) \oplus a_j H(W)
\]

as linear spaces, 

\[
a_j a_w a_i^{-1} = a_{w_{ij}} a_i^{-1} \quad \text{for } w \in W,
\]

\[
a_j^i = 1
\]

We put \( a_w a_j = a_{w_{ij}} \) \((w \in W)\). Let \( H^{(m)}(W \langle j \rangle) \) (resp. \( H^{(1)}(W \langle j \rangle) \)) denote the algebra obtained by specializing \( X \to q^m \) (resp. \( X \to q \)) in the defining relations of \( H(W \langle j \rangle) \). For \( w \in W \langle j \rangle \), let \( a_w^{(m)} \) (resp. \( a_w^{(1)} \)) denote the specialized element of \( a_w \) in \( H^{(m)}(W \langle j \rangle) \) (resp. \( H^{(1)}(W \langle j \rangle) \)). For \( x \in (W \langle j \rangle) \), let \( \nu_x \) be the corresponding irreducible representation of \( H(W \langle j \rangle) \otimes \Phi(X) \) and \( \nu_x^{(m)} \) (resp. \( \nu_x^{(1)} \)) its specialized representation of \( H^{(m)}(W \langle j \rangle) \) (resp. \( H^{(1)}(W \langle j \rangle) \)).

Proof of Lemma 4.4. By Corollary 1.4 and Lemma 3.1 we have

\[
\sum_{\lambda \in \Psi_*} f_{\rho_\Lambda, \psi}(q) \lambda_{\rho_\Lambda} \text{ Tr} \left( (Xj)^{-1}, \rho_\Lambda \right)
\]

for any \( w \in W \) and \( x \in G^{m} \). The relation (1) and the relation (*) in the lemma together with the orthogonality relations (cf. [1]) imply

\[
f_{\rho_\Lambda, \psi}(q) \lambda_{\rho_\Lambda} = f_{\rho_\Lambda, \psi}(q) \lambda_{\rho_\Lambda}^{(1)}
\]

for any \( w \in W \). By [1, 2.4.7] and by [8, 3.5], we have

\[
f_{\rho_\Lambda, \psi}(X) = \delta^{-1} \sum_{x \in (W \langle j \rangle)_{\lambda}} \text{Tr} (a_w, \nu_x) \langle R_x^{(m)}, \rho_\Lambda \rangle
\]

for \( a=1, m \) and \( \Lambda \in \Psi_* \). By (2) and (3),

\[
\{ \delta^{-1} \sum_{x \in (W \langle j \rangle)_{\lambda}} \text{Tr} (a_w^{(1)}, \nu_x^{(1)}) \langle R_x^{(1)}, \rho_\Lambda \rangle \lambda_{\rho_\Lambda} \}
\]

\[
= \{ \delta^{-1} \sum_{x \in (W \langle j \rangle)_{\lambda}} \text{Tr} (a_w^{(1)}, \nu_x^{(1)}) \langle R_x^{(1)}, \rho_\Lambda \rangle \lambda_{\rho_\Lambda}^{(1)} \}
\]

Let \( \{ a_w'; w \in W \} \) be the dual basis of \( \{ a_w; w \in W \} \). We put \( a_w^* = a_{w'}^{-1} a_w \) for \( w \in W \). Then for \( \chi, \chi' \in (W \langle j \rangle)^* \),

\[
\sum_{w \in W} \text{Tr} (a_w^* \chi, \nu_x^{(1)}) \text{Tr} (a_w^{(1)}, \nu_x^{(1)}) \neq 0
\]

if and only if \( \chi \mid W = \chi' \mid W \), where \( a_w^{*(1)} \) is the specialized element of \( a_w^* \). Thus by (4),

\[
\langle R_x^{(m)}, \rho_{\Lambda}^{(m)} \rangle \lambda_{\rho_\Lambda} = \langle R_x^{(1)}, \rho_{\Lambda_2}^{(1)} \rangle \lambda_{\rho_\Lambda}^{(1)}
\]

for any \( \chi \in (W \langle j \rangle)^* \). By [6, 3.12],

\[
\]
\[
\dim \rho_{\Lambda_1}^{(m)}(x) = \delta^{-1} \sum_{\chi \in \Psi_{\Lambda_1}^{(m)}} <R_{\chi}^{(m)}, \rho_{\Lambda_1}^{(m)}> \dim R_{\chi}^{(m)}
\]

By [4], \(\dim \rho_{\Lambda_1}^{(m)}\) and \(\dim R_{\chi}^{(m)}\) are expressed as polynomials in \(q^m\). By Lusztig [7] and [8], \(<R_{\chi}^{(m)}, \rho_{\Lambda_1}^{(m)}>\) is independent of \(m\), since we have assumed that \(m\) is a sufficiently large odd integer. Thus the relation (6) holds with each term regarded as polynomials in \(q^m\). Hence by replacing \(q^m\) with \(q\) in (6) we have

\[
\dim \rho_{\Lambda_1}^{(1)} = \delta^{-1} \sum_{\chi \in \Psi_{\Lambda_1}^{(1)}} <R_{\chi}^{(1)}, \rho_{\Lambda_1}^{(1)}> \dim R_{\chi}^{(1)}
\]

By (5) and (7),

\[
\dim \rho_{\Lambda_1}^{(1)} = c \lambda^{-1}_{\Lambda_1} \lambda_{\Lambda_2} \delta^{-1} \sum_{\chi \in \Psi_{\Lambda_2}^{(1)}} <R_{\chi}^{(1)}, \rho_{\Lambda_2}^{(1)}> \dim R_{\chi}^{(1)}
\]

Since \(c\) is of absolute value 1, \(c \lambda^{-1}_{\Lambda_1} \lambda_{\Lambda_2}\) is also of absolute value 1. Considering that \(\dim \rho_{\Lambda_1}^{(1)}\) and \(\dim \rho_{\Lambda_2}^{(1)}\) are positive integers, we see that (i), (ii) of the lemma are true. (iii) is obtained by (5) and (i). (iv) is obtained by (3), (4) and (iii).

**Lemma 4.5.** Let \(n_0\) be a non-negative integer. We assume that there exists a symbol class \(\Lambda_0 \in \Psi_{n_0}\) of defect \(d\) corresponding to the unipotent cuspidal representation. Let \(\Lambda_1 \neq \Lambda_2 \in \Psi_{n_0 + 1}\) be the symbol classes of defect \(d\) corresponding to the subcuspidal representations.

(i) Assume \(\text{Tr}(xj \sigma, \rho_{\Lambda_0}^{(m)}) = \text{Tr}(N^{(m)}(xj), \rho_{\Lambda_1}^{(1)})\) for any \(x \in G_{\Psi_{n_0}}^{E}\). Then

\[
\text{Tr}(xj \sigma, \rho_{\Lambda}^{(m)}) = \text{Tr}(N^{(m)}(xj), \rho_{\Lambda}^{(1)})
\]

for any \(x \in G_{\Psi_{n_0 + 1}}^{E}\) with \((\Lambda, \Lambda')\) one of the following conditions (A) and (B):

(A) \((\Lambda, \Lambda') = (\Lambda_1, \Lambda_1), (\Lambda_2, \Lambda_2)\)

(B) \((\Lambda, \Lambda') = (\Lambda_1, \Lambda_2), (\Lambda_2, \Lambda_1)\)

(ii) Let \(n \geq n_0 + 1\) and assume that the statement (i) with the condition (A) is true. Then

\[
\text{Tr}(xj \sigma, \rho_{\Lambda}^{(m)}) = \text{Tr}(N^{(m)}(xj), \rho_{\Lambda}^{(1)})
\]

for any \(x \in G_{\Psi_{n}}^{E}\) and any \(\Lambda \in \Psi_{n}\) of defect \(d\).

Proof. By Lemma 2.1, we can apply the arguments employed in [1, 2.2.3]. (See Lemma 4.1)

Proof of Theorem 4.2. By Lemma 4.4, to prove the theorem it suffices to prove (i) of the theorem for any \(\Lambda \in \Psi_{n}\). And Lemma 4.5 shows that it suffices to prove (i) of the theorem only when \(\rho_{\Lambda}^{(m)}\) is cuspidal or subcuspidal.
Let \( n_0, \Lambda_0, \Lambda_1, \Lambda_2 \) be as in Lemma 4.5.

Assume \( n = n_0 \). \( \rho_{\Lambda_0}^{(m)} \) (resp. \( \rho_{\Lambda_0}^{(1)} \)) is the unique unipotent cuspidal representation of \( G_f^m \) (resp. \( G_f \)) and there is no unipotent subcuspidal representation of \( G_f^m \) (resp. \( G_f \)). By the induction, the statements of the theorem are true if \( \Lambda = \Lambda_0 \).

In particular, the lifting of a non-cuspidal unipotent representation is a non-cuspidal unipotent representation, whereas the relation (1) in the proof of Lemma 4.4 shows that the lifting of \( \tilde{\rho}_{\Lambda_0}^{(1)} \) is unipotent (or its restriction to \( G_f^m \) is unipotent if \( G = SO_{2n} \)), and therefore must be \( \tilde{\rho}_{\Lambda_0}^{(m)} | G_f^m \). Thus

\[
\text{Tr}(xj^\sigma, \tilde{\rho}_{\Lambda_0}^{(m)}) = c \text{ Tr}(N(x)j, \tilde{\rho}_{\Lambda_0}^{(1)})
\]

for any \( x \in G_f^m \) with a constant \( c \). Assume \( G = SO_{2n} \). Then \( \lambda_{\rho_{\Lambda_0}^{(m)}} = \lambda_{\rho_{\Lambda_0}^{(1)}} = 1 \).

Thus \( c = 1 \) by Lemma 4.4, (i). Assume \( G \neq SO_{2n} \), (hence \( j = \text{id.} \)). We are to prove \( c = 1 \). By [1, 2.4.6], for any \( \chi \in W^\wedge \),

\[
\begin{align*}
\text{(1)} & \quad \dim \rho_{\chi}^{(m)} = \sum_{\Lambda \in W^\wedge} \langle R_{\chi}^{(m)}, \rho_{\Lambda}^{(m)} \rangle \chi_{\rho_{\Lambda}^{(m)}} \dim \rho_{\Lambda}^{(m)}, \\
\text{(2)} & \quad \dim \rho_{\chi}^{(1)} = \sum_{\Lambda \in W^\wedge} \langle R_{\chi}^{(1)}, \rho_{\Lambda}^{(1)} \rangle \chi_{\rho_{\Lambda}^{(1)}} \dim \rho_{\Lambda}^{(1)},
\end{align*}
\]

where \( \rho_{\chi}^{(m)} \) (resp. \( \rho_{\chi}^{(1)} \)) denotes the unipotent representation of \( G_f^m \) (resp. \( G_f \)) in the principal series corresponding with \( \chi \) (cf. [1]). Since \( \langle R_{\chi}^{(m)}, \rho_{\Lambda}^{(m)} \rangle \) is independent of the odd integer \( m \) (\( m \) sufficiently large), the relation (1) holds with each term regarded as a polynomial in \( q^m \). Thus by replacing \( q^m \) with \( q \) in (1),

\[
\text{(3)} \quad \dim \rho_{\chi}^{(1)} = \sum_{\Lambda \in W^\wedge} \langle R_{\chi}^{(1)}, \rho_{\Lambda}^{(1)} \rangle \chi_{\rho_{\Lambda}^{(1)}} \dim \rho_{\Lambda}^{(1)}
\]

If \( \Lambda \neq \Lambda_0 \), we have already \( \langle R_{\chi}^{(m)}, \rho_{\Lambda}^{(m)} \rangle = \langle R_{\chi}^{(1)}, \rho_{\Lambda}^{(1)} \rangle \) and \( \lambda_{\rho_{\Lambda}^{(m)}} = \lambda_{\rho_{\Lambda}^{(1)}} \). Thus, by comparing the relation (2) and the relation (3), we obtain

\[
\langle R_{\chi}^{(m)}, \rho_{\Lambda_0}^{(m)} \rangle \lambda_{\rho_{\Lambda_0}^{(m)}} = \langle R_{\chi}^{(1)}, \rho_{\Lambda_0}^{(1)} \rangle \lambda_{\rho_{\Lambda_0}^{(1)}}
\]

for any \( \chi \in W^\wedge \). Thus by (iii) of Lemma 4.4, we have \( \lambda_{\rho_{\Lambda_0}^{(m)}} = \lambda_{\rho_{\Lambda_0}^{(1)}} \). (Note that there exists \( \chi \in W^\wedge \) such that \( \langle R_{\chi}^{(1)}, \rho_{\Lambda_0}^{(1)} \rangle \neq 0 \).) Hence by (i) of Lemma 4.4, we have \( c = 1 \). Therefore we have proved the theorem for \( \Lambda = \Lambda_0 \).

Assume \( n = n_0 + 1 \). \( \rho_{\Lambda_1}^{(m)} \) (resp. \( \rho_{\Lambda_2}^{(1)} \)) (\( i = 1, 2 \)) are subcuspidal representations of \( G_f^m \) (resp. \( G_f \)) and the other unipotent representations of \( G_f^m \) (resp. \( G_f \)) are neither cuspidal nor subcuspidal. Let \( i = 1 \) or 2. By Lemma 4.5, there exists \( i' = 1 \) or 2 such that

\[
\text{Tr}(xj^\sigma, \tilde{\rho}_{\Lambda_i}^{(m)}) = \text{Tr}(N(x)j, \tilde{\rho}_{\Lambda_i}^{(1)})
\]

for any \( x \in G_f^m \). Then by Lemma 4.4, \( \dim \rho_{\Lambda_i}^{(1)} = \dim \rho_{\Lambda_i}^{(1)} \). Since \( \dim \rho_{\Lambda_i}^{(1)} = \dim \rho_{\Lambda_{i'}}^{(3)} \), we must have \( i = i' \). This proves the theorem for \( \Lambda = \Lambda_1, \Lambda_2 \).
References


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