1. Introduction. In 1931 M.H.A. Newman [N] proved the following result.

**Theorem (Newman).** If $M$ is a connected topological manifold with metric $d$, there exists a number $\varepsilon=\varepsilon(M, d)>0$, depending only upon $M$ and $d$, such that every finite group $G$ acting effectively on $M$ has at least one orbit of diameter at least $\varepsilon$.

P.A. Smith [S] in 1941 proved a version of Newman’s Theorem in terms of coverings of $M$ and Dress [D] in 1969 gave a simplified proof of Newman’s Theorem based on Newman’s original approach and using a modern version of local degree.

In another direction Cernavskii [C] in 1964 generalized Newman’s Theorem to the setting of finite-to-one open mappings on manifolds. His techniques were based upon those of Smith. Recently McAuley and Robinson [M-R] and Deane Montgomery [MO] have expanded upon Cernavskii’s results. In fact McAuley and Robinson, using the techniques of Dress, have obtained the following version of Cernavskii’s result. [M-R, Theorem 3].

**Theorem (Cernavskii-McAuley-Robinson).** If $M$ is a compact connected topological manifold with metric $d$, there exists a number $\varepsilon=\varepsilon(M, d)>0$ such that if $Y$ is a metric space and $f: M \to Y$ a continuous finite-to-one proper open surjective mapping which is not a homeomorphism, then there is at least one $y \in Y$ such that $\text{diam } f^{-1}(y) \geq \varepsilon$.

In [H-M] we gave estimates of the $\varepsilon$ in Newman’s Theorem for Riemannian manifolds $M$ in terms of convexity and curvature invariants of $M$. In this note we apply the techniques of [H-M] to obtain estimates of $\varepsilon$ for the Cernavskii-McAuley-Robinson result for the case where $M$ is a Riemannian manifold. In particular, if $S^n$ is the standard unit sphere with standard metric, we show $\varepsilon > \pi/2$, i.e. if $f: S^n \to Y$ is as above, there exists $y \in Y$ with $\text{diam } f^{-1}(y) > \pi/2$. We also obtain a cohomology version of Newman’s Theorem for compact orientable Riemannian manifolds which generalizes Theorem 3 of
We wish to thank McAuley and Robinson for sending us [M-R] prior to its publication.


We shall call an open finite-to-one proper surjective map \( f: M \to Y \), \( Y \) a metric space, which is not a homeomorphism, a pseudo-submersion, and \( f^{-1}(f(x)) \) an orbit of \( f \) at \( x \) and denoted by \( O_f(x) \).

Now let \( M \) be a connected Riemannian manifold with a metric induced from the Riemannian metric of \( M \). Assume that there exists at least one pseudo-submersion \( f: M \to Y \). Define the Newman's diameter \( d^\tau(M) \) of \( M \) by

\[
d^\tau(M) = \sup \{ \varepsilon \mid \text{for every pseudo-submersion } f: M \to Y, \exists x \in M \text{ such that } \operatorname{diam} O_f(x) \geq \varepsilon \}
\]

Define the cardinality of \( f \) by \( \text{Card } f = \max \{ \text{card } O_f(x) : x \in M \} \). Suppose there exists at least one pseudo-submersion \( f: M \to Y \) with \( \text{Card } f = p > 1 \); we define the mod \( p \) Newman's diameter \( d^p(M) \) as the supremum of the numbers \( \varepsilon > 0 \) such that for every pseudo-submersion \( g: M \to Y \) with \( \text{Card } g = p \), there exists an orbit of diameter at least \( \varepsilon \).

We call a subset \( S \) of a Riemannian manifold \( M \) convex if for every pair of points in \( S \) there exists a unique distance measuring geodesic in \( S \) joining them. For \( x \in M \), the radius of convexity of \( M \) at \( x \), which we denote by \( r_x \), is defined as the supremum of the radii of all convex embedded open balls centered at \( x \).

The following result is based upon Lemma 3 in [D] and appears as Theorem 2 in [M-R].

**Proposition 2.1** (Dress-McAuley-Robinson). Let \( U \) be an open, connected, relatively compact subset of \( \mathbb{R}^n \) and \( f: U \to Y \) a pseudo-submersion. Then

\[
D = \max \{ \min \{ ||x-y|| : y \in \partial U \} : x \in U \}
\]

where \( ||x-y|| \) is the euclidean norm.

It is well-known that the exponential map locally stretches distances for manifolds of nonpositive curvature. In [H-M] the following analogous result was obtained for manifolds of bounded curvature.

**Proposition 2.2.** Suppose \( K \leq b^2, b > 0 \) (respectively \( K \leq 0 \)) on a Riemannian manifold \( M \) with distance function \( d \). Let \( B_r(z) = \{ y : d(y, z) < r \} \) be a convex embedded ball centered at \( z \) in \( M \). Suppose further that \( r < \pi b^{-1} / 2 \) (respectively \( 0 < r < \infty \) when \( K \leq 0 \)). For any \( x, y \in B_r(z) \), if \( \hat{x} = \exp_z^{-1} x \) and \( \hat{y} = \exp_z^{-1} y \), then
Newman's Theorem for Pseudo-submersions

\[ d(x, y) \geq (2/\pi) ||\dot{x} - \dot{y}|| \] (respectively \( d(x, y) \geq ||\dot{x} - \dot{y}|| \) when \( K \leq 0 \)). Here \( ||\dot{x} - \dot{y}|| \) is the euclidean norm in the tangent space \( M_x \).

Using Propositions 2.1 and 2.2 and the techniques of [H-M] we are able to prove the main result of this section.

**Theorem 2.3.** Let

\[ \varphi = \sup_{z \in K} r_z . \]

1. If \( K \leq 0 \), \( d^T(M) \geq \varphi/2 \). In particular if \( \varphi = + \infty \), there exist point inverses of arbitrarily large diameters.

2. If \( K \leq b^2 \), and \( a = \min \{\pi/2b, \varphi\} \), \( d^T(M) \geq 2a/(\pi + 2) \).

**Proof.** Fix any \( z \in M \) and let \( r_z \) the radius of convexity at \( z \). For any \( r > 0 \) satisfying

\[ r < \begin{cases} r_z & \text{if } K \leq 0 \\ \min \{r_z, \pi b^{-1}/2\} & \text{if } K \leq b^2 , \end{cases} \]

and any \( \alpha, \frac{1}{2} \leq \alpha < 1 \), suppose that

\( (H) \) \( \text{diam } O_f(x) < (1 - \alpha)r \), all \( x \in M \).

Define \( U = f^{-1}[f(B_{ar}(z))] \). Clearly \( U \) is open. We claim \( U \) is connected. Let \( V \) be a component of \( U \). Now it is known [C], [MO] that \( V \) maps onto \( f(U) = f(B_{ar}(z)) \). Hence, \( V \) intersects \( O_f(x) \). But since

\[ \text{diam } O_f(x) < (1 - \alpha)r \leq \alpha r , \]

\( O_f(z) \subset B_{ar}(z) \). Furthermore by \( (H) \),

\( B_{ar}(x) \subset U \subset B_r(z) \).

Let \( U_\wedge = \exp^{-1}_z U \). Then \( U_\wedge \) is an open and connected subset of \( R^n = M_r \).

It can be verified that

\[ U_\wedge = \exp^{-1}_z f^{-1}[f(B_{ar}(z))] . \]

Consequently we can apply Proposition 2.1 to \( f = f \circ \exp_z : \bar{U} \to Y \). Now

\[ \{ \dot{x} \in M_x \mid ||\dot{x}|| \leq \alpha r \} = \exp^{-1}_z \bar{B}_r(z) \subset U_\wedge \]

\( \subset \exp^{-1}_z B_r(z) = \{ \dot{x} \in M_x \mid ||\dot{x}|| \leq r \} \)

The left-hand inclusion implies

\[ D = \max \{ \min \{||\dot{x} - \dot{y}|| \mid \dot{y} \in \partial \bar{U}_\wedge \mid \dot{x} \in U_\wedge \} \geq \alpha r \} \] (Simply let \( \dot{x} = 0 \))

Since \( \bar{B}_r(z) \) is a convex, embedded ball with \( r < \pi b^{-1}/2 \) when \( K \leq b^2 (r < \infty \) when \( K \leq 0 \)), we may apply Proposition 2.2. So
\begin{align*}
C &= \text{Max}\{\text{diam } O_f(x) | x \in \partial \tilde{U}_r\} \\
&\leq \begin{cases} 
\text{Max}\{\text{diam } O_f(x) | x \in \partial \tilde{U}\} & \text{if } K \leq 0 \\
\pi/2 \text{Max}\{\text{diam } O_f(x) | x \in \partial \tilde{U}\} & \text{if } K \leq b^2 \\
(1-\alpha)r & \text{if } K \leq 0 \\
(1-\alpha)\pi r/2 & \text{if } K \leq b^2 
\end{cases}
\end{align*}

by (H).

By Proposition 2.1, \(D \leq C\). Consequently
\begin{align*}
\alpha r < \begin{cases} 
(1-\alpha)r & \text{if } K \leq 0 \\
(1-\alpha)\pi r/2 & \text{if } K \leq b^2 
\end{cases}
\end{align*}
or
\begin{align*}
\alpha < \begin{cases} 
1/2 & \text{if } K \leq 0 \\
\pi/(\pi+2) & \text{if } K \leq b^2 
\end{cases}
\end{align*}

Consequently, (H) is false for
\begin{align*}
a = \begin{cases} 
1/2 & \text{if } K \leq 0 \\
\pi/(\pi+2) & \text{if } K \leq b^2 
\end{cases}
\end{align*}

So there exists an \(x \in M\) with \(\text{diam } O_f(x) \geq r/2\) if \(K \leq 0\); \(2r/(\pi+2)\) if \(K \leq b^2\).

It is possible to obtain a version of Theorem 2.3 in terms of injectivity radius. For a complete connected Riemannian manifold \(M\) define the injectivity radius \(i(M)\) by
\[i(M) = \sup \{d(x, C(x)) : x \in M\}\]

where \(C(x)\) denotes the cut locus of \(x\).

\textbf{Theorem 2.4.}
(1) If \(K \leq 0\), \(d^T(M) \geq i(M)/2\).
(2) If \(K \leq b^2\), \(M\) is compact and \(a = \text{Min}\{\pi/2b, i(M)/2\}\), \(d^T(M) \geq 2a/\pi\).

\section{3. Estimate of Newman's diameter \(d^T(S^n)\) and related topics.}

We use the notion of \textit{degree of a map} defined by Dress [D].

Let \(f: M^* \rightarrow Y\) be a pseudo-submersion. The branch set \(B_f\) of \(f\) is defined as \(B_f = \{x \in M : f \text{ is not a local homeomorphism at } x\}\). By [C] or [M-R], \(M = f^{-1}(f(B_f))\) is a dense open subset of \(M^*\).

\textbf{Lemma 3.1: Newman's Lemma} (Dress [D], McAuley-Robinson [M-R]). Let \(f: M \rightarrow Y\) be a pseudo-submersion, \(X\) a locally compact metric space, \(g: M \rightarrow X\) and \(j: Y \rightarrow X\) be a proper map such that \(g = j \circ f\). Let \(x \in X\) be such that \(g^{-1}(x) \cap f^{-1}(f(B_f)) = \phi\).
and $y \in j^{-1}(x)$. If $\text{Card} \ f^{-1}(y)=p$, then $g$ is inessential at $x$ for $Z_p$; that is, the degree of $g$ at $x$, $d(g, x)$, is zero (with $Z_p$ as coefficients).

**Theorem 3.2.** Let $M$ be a compact connected oriented topological $n$-manifold and $f: M^n \to Y$ be a pseudo-submersion with $\text{Card} \ O_f(x_0)=p>1$ for some $x_0 \in M-f^{-1}(f(B_f))$. Suppose $\varphi: M \to S^n$ is a map such that the $\deg \varphi \equiv 0 \mod p$. If we denote $\varphi(z)$ by $\tilde{z}$, then there exists $x \in M$ such that the following holds:

1. $\sum_{z \in \partial O_f(x)} \tilde{z} = cx \in \mathbb{R}^{n+1}$ for some $c \leq 0$.
2. $\sum_{z \in \partial O_f(x)} \tilde{z} = \pi$ if $\text{Card} \ O_f(x)=2$,
   $\sum_{z \in \partial O_f(x)} \tilde{z} = \sum_{z \in \partial O_f(x)} \tilde{z}$, if $\text{Card} \ O_f(x)=3$ and $O_f(x)=\{x, y, z\}$,
   $\sum_{z \in \partial O_f(x)} \tilde{z} \geq \pi - \cos^{-1}(\frac{1}{p-1}) > \pi/2$ if $\text{Card} \ O_f(x) \geq 4$

for some $z \in O_f(x)$, where $\sum_{z \in \partial O_f(x)} \tilde{z}$ denotes the angle between $oz$ and $oz$, $o \in \mathbb{R}^{n+1}$ the origin, and $S^n$ the standard unit sphere in $\mathbb{R}^{n+1}$.

**Proof.** (1) Suppose on the contrary, then $\sum_{z \in \partial O_f(x)} \tilde{z} \neq 0$ for all $x \in M$. Define a map $g: M^n \to S^n$ by

$$g(x) = \sum_{z \in \partial O_f(x)} \tilde{z} / \sum_{z \in \partial O_f(x)} \tilde{z}.$$

Then for any $z \in O_f(x)$, $g(z)=g(x)$. Hence $g$ induces a map $j: Y \to S^n$ such that $g=j \circ f$. It follows from Lemma 3.1 that $g$ is inessential at $g(x)$ for $Z_p$.

On the other hand, by hypothesis there is a well defined homotopy $H: M \times [0, 1] \to S^n$ between $\varphi$ and $g$ defined by

$$H(x, t) = \frac{t \varphi(x) + (1-t)g(x)}{|t \varphi(x) + (1-t)g(x)|}.$$

Hence, $\deg \varphi = \deg g = d(g, g(x)) = 0 \mod p$. This is a contradiction.

(2) For any $y, z \in O_f(x)$, set $\theta_{yz}=\sum_{z \in \partial O_f(x)} \tilde{z}$. Let $\langle, \rangle$ be the standard inner product in $\mathbb{R}^{n+1}$. From (1) there exists an element $x \in M$ such that

$$\langle x, \tilde{z} \rangle + \sum_{z \in \partial O_f(x)} \langle x, \tilde{z} \rangle = c\langle x, x \rangle$$

for some $c \leq 0$; that is,

$$\sum_{x \in \partial O_f(x)} \cos \theta_{xz} = c \leq -1$$

If $\text{Card} f=2$, it is easy to see from $(**)$ that $c=0$, and $\theta_{xz}=\pi$.

If $\text{Card} f=3$, then $\cos \theta_{xy} + \cos \theta_{xz}=c-1$. From (1) we have

$$|(1-c)x + \tilde{z}|^2 = | - \tilde{y} |^2.$$

Hence $\cos \theta_{xy} = \cos \theta_{xz} = (c-1)/2$. That is, $\theta_{xy} = \theta_{xz} \geq 2\pi/3$. If $\text{Card} f=p \geq 4$, there exists at least one $z \in O_f(x)$ such that $\cos \theta_{xz} \leq -1/(p-1)$; that is, $\theta_{xz} \geq \pi$.
Theorem 3.2 implies the following:

Corollary 3.3. (1) $d^2_p(S^n) = \pi$, i.e., for any pseudo-submersion $f: S^n \to Y$ with Card $f = 2$, there exists $x \in S^n$ such that $f^{-1}(f(x)) = \{x, -x\}$.
(2) $d^3_3(S^n) = 2\pi/3$.
(3) $(p-1)\pi/p \geq d^2_p(S^n) \geq \pi - \cos^{-1}(1/(p-1)) > \pi/2$ if $p \geq 4$.
(4) $2\pi/3 \geq d^3(S^n) > \pi/2$.

Proof. In [K], the equivariant diameter $D(M)$ and modulo $p$ equivariant diameter $D_p(M)$ have been defined. They are precisely defined by the pseudo-submersions $\pi: M \to MG$ which are orbit maps of isometric actions of compact Lie groups $G$ or $G = \mathbb{Z}_p$ on $M$ respectively. Hence $D(M) \geq d^2(M)$ and $D_p(M) \geq d^2_p(M)$ for some $p$. But $D(S^n) = 2\pi/3$ and $D_p(S^n) = (p-1)\pi/p$ if $p \geq 3$ by [K]. Hence, the result follows from Theorem 3.2 by applying it to the identity map $S^n \to S^n$.

Remarks. (i) The statement (1) extends the following well known result: For any non-trivial involution $g$ of $S^n$, there exists $x \in S^n$ such that $gx = -x$.
(ii) By using the arguments of Milnor in [MI] we can also show the following: Let $f: M^* \to Y$ and $\tilde{f}: \tilde{M} \to \tilde{Y}$ be pseudo-submersions with Card $f = \text{Card} \tilde{f} = 2$, $B_f = B_\tilde{f} = \phi$, where $M$ is a compact connected oriented $n$-manifold and $\tilde{M}$ a mod 2 homology $n$-sphere. Suppose there exists a map $\varphi: M \to \tilde{M}$ of odd degree. Then there exists $x$ in $M$ such that $\varphi O_f(x) = O_\tilde{f}(\varphi x)$.

Theorem 3.4. Let $M$ be a compact connected $n$-dimensional submanifold of $R^{n+1}$, $n \geq 2$, and let $y \in R^{n+1} - M$ be in a bounded component. Suppose $f: M \to Y$ is a pseudo-submersion. Then there exists $x \in M$ such that
(1) If Card $f = 2$, $\{O_f(x), y\}$ lies on a line in $R^{n+1}$.
(2) If Card $f = 3$, and $O_f(x) = \{x, u, v\}$, then
\[ \angle xyu = \angle uvy = \angle vyx = 2\pi/3. \]
In particular $\{O_f(x), y\}$ lies in a 2-plane in $R^{n+1}$.
(3) If Card $f = p \geq 4$, then $\angle uvy \geq \pi - \cos^{-1}(1/(p-1)) > \pi/2$ for some $u, v \in O_f(x)$, and $\{O_f(x), y\} \subset R^{n-1} \cap M$, for some $(p-1)$-plane $R^{n-1}$ of $R^{n+1}$ (if $n \geq p-2$) passing through the origin.

Proof. Apply Theorem 3.2 to the map $\varphi: M \to S^n$ defined by $\varphi(x) = (y-x)/||y-x||$ because deg $\varphi = \pm 1$. The equality in (2) follows from Corollary 3.3 (2).

4. Cohomology version of Newman's theorem for pseudo-submersions

Let $f: M \to Y$ be a pseudo-submersion. A subset $A$ of $M$ is called satur-
ated if \( A = O_f(A) \), where \( O_f(A) = \bigcup \{ O_f(x) : x \in A \} \), or equivalently \( A = f^{-1}(f(A)) \). Let \( x \in M - f^{-1}(f(B_f)) \). Then there exists an open neighborhood \( V \) of \( x \) which is homeomorphic to \( R^n \) and \( f|_V : V \to f(V) \) is a homeomorphism. Hence by excision we have

\[
H_n(Y, Y - f(x); Z_p) = H_n(f(V), f(V) - f(x); Z_p) = Z_p,
\]
where \( p = \text{Card } O_f(x) \).

We shall say a pseudo-submersion \( f : M \to Y \) satisfies the (LOA) (local orientable condition for \( A \)) if \( A \) is a closed saturated subset of \( M \), \( B = f(A) \) is closed in \( Y \) and such that the inclusion \( i_B : (Y, B) \to (Y, Y - x) \) induces an isomorphism

\[
i_B^* : H_n(Y, B; Z_p) \to H_n(Y, Y - f(x); Z_p)
\]
for some \( x \in M - f^{-1}(f(B_f)) \), \( \text{Card } O_f(x) = p \).

The following result extends the cohomology version of Newman’s Theorem for group actions [B], [S] due to Smith.

**Theorem 4.1.** Let \( A \) be a closed subspace of a compact oriented \( n \)-manifold \( M \) such that \( H_n(M, A; Z_p) \approx Z_p \). Let \( \mathcal{U} \) be any open covering of \( M \) such that

\[
H^n(K(\mathcal{U}), K(\mathcal{U}|A); Z_p) \to H^n(M, A; Z_p)
\]
is surjective, where \( K(\mathcal{U}) \) denotes the nerve of the covering \( \mathcal{U} \). Then there does not exist a pseudo-submersion \( f : M \to Y \) satisfying (LOA) and such that each orbit of \( f \) is contained in some open set in \( \mathcal{U} \).

**Proof.** Suppose the conclusion is false. Then there exists a pseudo-submersion \( f : M \to Y \) satisfying (LOA) and each orbit \( O_f(x) \) is contained in a saturated open set \( V_x \) which is contained in some member of \( \mathcal{U} \). Let \( \mathcal{U} = \{ f(V_x) : x \in V \} \). Then \( f^{-1}(\mathcal{U}) \) is a refinement of \( \mathcal{U} \). By [B, p. 154], \( f^* : H^n(Y, B; Z_p) \to H^n(M, A; Z_p) \) is an epimorphism. But the Kronecker product induces a canonical epimorphism [G, p. 132]

\[
\alpha : H^n(M, A; Z_p) \to H_*(M, A; Z_p)^* = \text{Hom}(H_*(M, A; Z_p); Z_p);
\]
hence we have an isomorphism \( f_\#: H_*(M, A; Z_p) \to H_*(Y, B; Z_p) \).

Let \( K = O_f(x) \), and \( O_f \subseteq H_*(M, M - K; Z_p) \) be the fundamental class which is the element such that for any \( z \in K \), the inclusion \( i_z : (M, M - K) \to (M, M - z) \) satisfies \( i_z^*(O_f) = 1_z \), the identity element of \( H_*(M, M - z; Z_p) \approx Z_p \) (cf. [D]). We have the following commutative diagram

\[
\begin{array}{ccc}
Z_p & \xrightarrow{i_*} & H_n(M, A; Z_p) \\
\downarrow & & \downarrow \\
H_*(M, M - K; Z_p) & \xrightarrow{k_*} & H_*(M, M - z; Z_p) = Z_p \\
\end{array}
\]

\[
H_*(M, M - K; Z_p) \xrightarrow{i_*} H_*(M, M - z; Z_p) \approx Z_p.
\]
where all homomorphisms are induced by inclusions. Since \( k_{z} \) is an isomorphism for all \( z \) in \( K \), there exists an element \( a \) in \( H_{n}(M, A; Z_{p}) \) such that \( i^{*}(a) = 0 \). Now we consider the following commutative diagram

\[
\begin{array}{ccc}
H_{n}(M, A; Z_{p}) & \overset{f_{*}}{\longrightarrow} & H_{n}(Y, B; Z_{p}) \\
i_{*} \downarrow & \approx & \approx i_{*}^{*} \\
H_{n}(M, M-K; Z_{p}) & \overset{f_{*}}{\longrightarrow} & H_{n}(Y, Y-f(x); Z_{p})
\end{array}
\]

By definition, \( d(f, f(x)) = f_{*}(O_{k}) \) (cf. [D]). It follows that

\[
d(f, f(x)) = f_{*}i_{*}(a) = i_{*}f_{*}(a) \neq 0.\]

On the other hand, we can apply Lemma 3.1 to the map \( f \), with \( f = j \circ f \), to obtain \( d(f, f(x)) = 0 \), where \( j \) is the identity map. This is an obvious contradiction and the proof of the theorem is complete.

**Corollary 4.2.** Let \( M \) be a compact connected oriented \( n \)-manifold, and \( \mathcal{U} \) an open covering of \( M \) such that

\[
(*) \quad H^{q}(|\sigma|; Z_{p}) = 0 \quad \text{for any } \sigma \in K(\mathcal{U}) \text{ and any } q \geq 1.
\]

Then there does not exist a pseudo-submersion \( f: M \to Y \) such that

1. \( i_{*}: H_{n}(Y; Z_{p}) \cong H_{n}(Y, Y-x; Z_{p}) \), where \( i_{*}: Y \to (Y, Y-x) \) is inclusion, \( x \in M-f^{-1}(f(B_{x})) \), \( \text{Card } O_{f}(x) = p \), and
2. Each orbit of \( f \) is contained in some member of \( \mathcal{U} \).

**Proof.** The hypothesis (*) implies that

\[
H^{q}(K(\mathcal{U}); Z_{p}) \cong H^{q}(M; Z_{p})
\]

for all \( q \geq 0 \) by Leray's Theorem [G-R, p. 189].

As an example, if \( M \) is a compact connected oriented Riemannian manifold, and \( \mathcal{U} \) consists of all open convex proper subsets of \( M \), then the condition (*) of Corollary 4.2 is satisfied.

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**References**


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