MINIMAL IMMERSIONS OF 3-DIMENSIONAL SPHERE INTO SPHERES

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Introduction

Let \( S^n \) be the \( n \)-dimensional sphere with constant curvature \( c \). Let \( \Delta \) be the Laplace-Beltrami operator on \( S^n \). The spectre and eigen-functions of \( \Delta \) are well-known [2]. Let \( V^d \) be the eigen-space of \( \Delta \) corresponding to the \( d \)-th eigen-value \( \lambda_d = d(d+n-1) \). Let \( f_0, f_1, \ldots, f_{m(d)} \) be an orthonormal basis of \( V^d \) with respect to the inner product. Then

\[
\psi_{n,d}: S^n \rightarrow \mathbb{S}^m(d)(\subset \mathbb{R}^m(d)+1)
; \quad p \rightarrow 1/(m(d)+1)(f_0(p), f_1(p), \ldots, f_{m(d)}(p)),
\]

is an isometric minimal immersion, where \( k(d) \) and \( m(d) \) are as follows [6];

\[
k(d) = n/d(d+n-1),
m(d) = (2d+n-1)(d+n-2)!/d!(n-1)! - 1.
\]

It is proved that any isometric minimal immersion of \( S^n \) into \( S^n \) is equivalent to \( \psi_{2,d} \) for some \( d \), [3], [6]. But it is not true if the dimension \( n \) is greater than 3. In fact do Carmo and Wallach proved the following

**Theorem 0.1** (do Carmo and Wallach, [7]). Let \( f: S^n \rightarrow S^n \) be an isometric minimal immersion. Then

(i) there exists an integer \( d \) such that \( c = k(d) \).
(ii) There exists a positive semi-definite matrix \( A \) of size \((m(d)+1) \times (m(d)+1)\) such that \( f \) is equivalent to \( A \circ \psi_{n,d} \).
(iii) If \( n=2 \) or \( d=3 \), then \( A \) is the identity matrix.
(iv) If \( n \geq 3 \) and \( d \geq 4 \), then \( A \) is parametrized by a compact convex body \( L \) in some finite dimensional vector space, \( \dim L \geq 18 \). If \( A \) is an interior point of \( L \) then \( N = m(d) \), and if \( A \) is a boundary point of \( L \) then \( N < m(d) \).

There are some problems concerning (iv) of the above Theorem.
Problem 0.2 (Chern, [4]). Let \( S^3 \to S^1 \) be an isometric minimal immersion. Is it totally geodesic?

In [5], do Carmo posed a more general

Problem 0.3. Determine the lower bound \( 1(d) \) of the dimension \( N \) of the sphere \( S^N \) into which a given \( S^8 \) can be isometrically and minimally immersed.

Recently Problem 0.2 was negatively answered by N. Ejiri [8]. In fact he proved that there exists an isometric minimal immersion \( S^3_1 \to S^0 \).

As for the Problem 0.3, scarcely anything is known.

In this paper we confine our consideration to the case \( n=3 \). In this case \( S^3 \) has a structure of a Lie group, \( S^3 = SU(2) \). We investigate whether there exists an orbit in a representation space \( V \) of \( SU(2) \), which is a minimal submanifold in the unit sphere in \( V \). And we give an estimate for \( 1(d) \) (of the Problem 0.3 in the case \( n=3 \)). The following will be proved.

Theorem A. Let \( d \) be an integer, \( d \geq 4 \). Then there exists an isometric minimal immersion of \( S^3_{3d(d+2)} \) into \( S^1_{d(d+2)} \).

Theorem B. Let \( d \) be an even integer, \( d \geq 6 \). Then there exists an isometric minimal immersion of \( S^3_{3d(d+2)} \) into \( S^1_{d} \).

1. Complex linear representations of \( SU(2) \)

In this section we give a brief review on the complex linear representation of \( SU(2) \).

The special unitary group \( SU(2) \) is the group of matrices which acts on \( C^2 \) and leaves invariant the usual Hermitian inner product on \( C \). We can identify \( SU(2) \) with the 3-dimensional unit sphere \( S_1^3 \) by

\[
SU(2) \to S_1^3: g \rightarrow g \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad g \in SU(2).
\]

Then the induced metric on \( SU(2) \) by the above diffeomorphism is the bi-invariant metric on \( SU(2) \).

A homogeneous polynomial on \( C^2 \) is called of degree \( d \) if it satisfies

\[
P(\lambda z, \lambda w) = \lambda^d P(z, w), \quad \lambda \in C, \ z, w \in C.
\]

For each positive integer \( d \), let \( V(d) \) be the space of homogeneous polynomials of type \((d, 0)\) on \( C^2 \). Then \( SU(2) \) acts on \( V(d) \) as follows

\[
(\rho(g)(P))(z, w) = P((g^{-1} \cdot (z, w))), \quad g \in SU(2), \ z, w \in C, \ P \in V(d).
\]

Then \( (V(d), \rho) \) is a complex irreducible representation and each complex irreducible representation of \( SU(2) \) is equivalent to \( (V(d), \rho) \) for some \( d \) [12].
Define a Hermitian inner product in $V(d)$ by
\[(P, Q) = (d+1) \int_{g \in SU(2)} P((g \cdot (1.0))) Q((g \cdot (1.0))) \, dg\]
where $dg$ is the normalized Haar measure on $SU(2)$. Let $P_i$ be the polynomial in $V(d)$ defined by
\[P_i(z, w) = (\alpha C_i)^{1/2} z^{-i} w^i, \quad z, w \in \mathbb{C}.
\]
Then $P_0, P_1, \ldots, P_d$ is an orthonormal basis of $V(d)$.

Let $\mathfrak{su}(2)$ be the Lie algebra of $SU(2)$. Take the following basis of $\mathfrak{su}(2)$ and fix them once for all.

\[
X_1 = \begin{bmatrix}
(-1)^{1/2} & 0 \\
0 & -(-1)^{1/2}
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \quad X_3 = \begin{bmatrix}
0 & -(-1)^{1/2} \\
(-1)^{1/2} & 0
\end{bmatrix}.
\]

Then the bracket relations of $X_1$, $X_2$, and $X_3$ are
\[
[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.
\]

We denote also by $\rho$ the representation of $\mathfrak{su}(2)$ induced by the representation of $SU(2)$, i.e.,
\[
\rho(A)(P) = d/dt|_{t=0} \rho(\exp tA)(P), \quad A \in \mathfrak{su}(2).
\]

Then by a direct calculation we get
\[
\begin{align*}
(1.2)_1 & \quad \rho(X_1)(P_j) = (-1)^{j/2}(2j-d)P_j, \quad 0 \leq j \leq d, \\
(1.2)_2 & \quad \rho(X_2)(P_j) = -((-d-j)(j+1))^{1/2} P_{j+1} + (j(d-j+1))^{1/2} P_{j-1}, \quad 0 \leq j \leq d, \\
(1.2)_3 & \quad \rho(X_3)(P_j) = -((d-j)(j+1))^{1/2} P_{j+1} - (j(d-j+1))^{1/2} P_{j-1}, \quad 0 \leq j \leq d, 
\end{align*}
\]
where we put $P_{-1} = P_{d+1} = 0$.

2. **Real irreducible representations of $SU(2)$**

In this section we give a brief review on real irreducible representations of $SU(2)$.

Let $G$ be a compact Lie group and $(V, \rho)$ be a complex irreducible representation of $G$. Then $(V, \rho)$ is said to be self-conjugate if $V$ has a structure map $j$, i.e., a conjugate linear map on $V$ such that
\[
\begin{align*}
& j(\rho(g)v) = \rho(g)j(v), \quad g \in G, \ v \in V, \\
& j(\alpha v + \beta w) = \alpha j(v) + \beta j(w), \quad \alpha, \beta \in \mathbb{C}, \ v, w \in V, \\
& j^2 = \pm 1.
\end{align*}
\]
A self-conjugate representation \((V, \rho)\) is said to be of index 1 (resp. \(-1\)) if 
\[ j^2 = 1 \text{ (resp. } j^2 = -1). \]
For simple Lie groups self-conjugate representations and their indices are known [13]. We denote by \((V_R, \rho)\) the representation of \(G\) over \(R\) obtained by the restriction of the coefficient field from \(C\) to \(R\).

Let \((V, \rho)\) be a self-conjugate representation of \(G\) of index \(-1\). Then \((V_R, \rho)\) is also irreducible. But \((V_R, \rho)\) is reducible if \((V, \rho)\) is a self-conjugate representation of \(G\) of index 1. Namely \((1+j)V_R\) and \((1-j)V_R\) are mutually equivalent real irreducible representation of \(G\) and 
\[ V_R = (1+j)V_R + (1-j)V_R, \text{ (direct sum)}. \]

For these facts we refer, for instance, to [1].

Now we confine our attention to the case \(G = SU(2)\).

Let \(j\) be a conjugate-linear automorphism on \(C^2\) defined by
\[ j(z, w) = (-\bar{w}, \bar{z}), \quad z, w \in C. \]
Extend \(j\) to an automorphism on \(V(d)\) by
\[ (jP)(z, w) = P(j(z, w)), \quad z, w \in C. \]

Then \(j\) is a structure map on \(V(d)\) with \(j^2 = (-1)^d 1\). So \((V(d)_R, \rho)\) is a self-conjugate representation of index \((-1)^d\). Let \(d\) be an even integer \(d = 2d'\) and put \(Q_i = (-1)^i P_i, 0 \leq i \leq d\). Then
\[ jP_i = (-1)^i P_{d-i}, \quad jQ_i = -(-1)^i Q_{d-i}, \quad 0 \leq i \leq d. \]

Since \(P_0, P_1, \ldots, P_d, Q_0, Q_1, \ldots, Q_d\) are basis of \((V(d)_R, (1+j)P, (1+j)Q, 0 \leq i \leq d,\) are generators of \((1+j)V(d)_R\). It is easily seen that \( (1+j)P_i, (1-j)Q_i, 0 \leq i \leq d-1\) and \((1+j)P_{d'}\) [resp. \((1+j)Q_{d'}\)] are basis of \((1+j)V(d)_R\) if \(a'\) is an even [resp. odd] integer. We denote \((1+j)V(d)_R\) by \(V_0(d)\).

**Lemma 2.1.** Let \(d\) be an even integer, \(d = 2d'\). Then \(\sum_{i=0}^{d} z_i P_i\) is contained in \(V_0(d)\) if and only if
\[ z_i = (-1)^i \bar{z}_{d-i}, \quad 0 \leq i \leq d'. \]

Proof.
\[ \sum_{i=0}^{d} z_i P_i = (\text{Re } z_0 P_0 + \text{Re } z_d P_d) + (\text{Im } z_0 Q_0 + \text{Im } z_d Q_d) \]
\[ + (\text{Re } z_1 P_1 + \text{Re } z_{d-1} P_{d-1}) + (\text{Im } z_1 Q_1 + \text{Im } z_{d-1} Q_{d-1}) \]
\[ + \cdots + z_{d'} P_{d'}. \]

Remember that \(P_{-j}, (1+j)P_{-j}, Q_{-j}, (1+j)Q_{-j}, 0 \leq j \leq d' - 1\) and \(P_{d'}, \) [resp. \(Q_{d'}\)] are basis of \(V_0(d)\) if \(d'\) is an even [resp. odd] integer. So \(\sum_{i=0}^{d} z_i P_i\) is contained in \(V_0(d)\) if and only if
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\[
\text{Re } z_i = (-1)^i \text{Re } z_{d-i}, \quad \text{Im } z_i = -(-1)^i \text{Im } z_{d-i}, \quad 0 \leq i \leq d' - 1.
\]

\[
\text{Im } z_{d'} = 0 \text{ [resp. Re } z_{d'} = 0\text{] if } d' \text{ is even [resp. odd].}
\]

So we get the Lemma.

Q.E.D.

3. Orbits in a sphere

Let \( G \) be a Lie subgroup in \( SO(N+1) \). Then \( G \) acts on the unit sphere \( S^N_1 \) in \( \mathbb{R}^{N+1} \) centered at the origin in a natural manner. Take a point \( p_0 \) in \( S^N_1 \) and let \( M \) be the orbit of the action of \( G \) through \( p_0 \).

Let \( \mathfrak{g} \) be the Lie algebra of \( G \). We denote by \( A^* \) the vector field on \( S^N_1 \) defined by

\[
A^*_{|p} = d/dt|_{t=0} \exp(tA)(p), \quad p \in S^N_1.
\]

We consider elements of \( \mathfrak{g} \) as skew symmetric \((N+1) \times (N+1)\)-matrices in a natural manner. Then we get from (3.1) the following

\[
A^*_{|p} = A(p), \quad A \in \mathfrak{g}, \quad p \in S^N_1.
\]

So the tangent space of \( M \) at \( p \) is

\[
T_p(M) = \{A(p) | A \in \mathfrak{g}\}.
\]

Let \( N_p(M) \) be the normal space at \( p \) in \( S^N_1 \). Consider the tangent space \( T_p(M) \) and the normal space \( N_p(M) \) as a subspace in \( \mathbb{R}^{N+1} \). Then \( \mathbb{R}^{N+1} \) is decomposed into the direct sum

\[
\mathbb{R}^{N+1} = \mathbb{R}p + T_p(M) + N_p(M).
\]

For a vector \( A \) in \( \mathbb{R}^{N+1} \), we denote \( A^T \) and \( A^N \) the \( T_p(M) \)-component and \( N_p(M) \)-component of \( A \) in the decomposition (3.2) respectively.

Lemma 3.1. Let \( G \) be a Lie subgroup in \( SO(N+1) \). Let \( \alpha \) be the second fundamental form of the orbit \( G \cdot p \) in \( S^N_1 \). Then

\[
\alpha(A^*, B^*)_{|p} = (A(B(p)))^N, \quad (3.3)
\]

\[
\nabla_{B^*} A^*_{|p} = (A(B(p)))^T, \quad A, B \in \mathfrak{g}. \quad (3.4)
\]

where \( \nabla \) is the Riemannian connection on \( M \).

Proof. Let \( D \) be the Riemannian connection in \( \mathbb{R}^{N+1} \). Then

\[
D_{B^*} A^*_{|p} = d/dt|_{t=0} A^*_{|\exp(tB)(p)}
\]

\[
= d/dt|_{t=0} A(\exp(tB)(p))
\]

\[
= A(B(p)).
\]
Since \( \alpha(A^*, B^*)_{\mu} = (D_{B^*}A^*_{\mu})^N \) and \( \nabla_{B^*}A^*_{\mu} = (D_{B^*}A^*_{\mu})^* \), we get the Lemma.

Q.E.D.

4. Left invariant metrics on \( SU(2) \) and \( SO(3) \)

In this section we denote by \( G \) the Lie group \( SU(2) \) or \( SO(3) \). The Lie algebras of \( SU(2) \) and \( SO(3) \) are mutually isomorphic. We denote them by \( \mathfrak{su}(2) \).

Let \( B \) be the Killing form of \( \mathfrak{su}(2) \). Then \( X_1, X_2, X_3 \) defined in § 1 are orthonormal with respect to \( -B/8 \). Let \( g_0 \) be the Riemannian metric on \( G \) which is the bi-invariant extension of \( -B/8 \).

**Lemma 4.1.** [11]. Let \( g \) be an inner product on \( \mathfrak{su}(2) \). Then there exists an element \( \sigma \) in \( G \) such that

(i) \( X_i' = \text{Ad}(\sigma)(X_i), \ i = 1, 2, 3, \) are mutually orthogonal with respect to \( g \).

(ii) \( g = \lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2, \) where \( \lambda_i \) are positive constants and \( \omega_i(\cdot) = g_0(X_i', \cdot), \ i = 1, 2, 3. \)

Let \( g \) be the Riemannian metric on \( G \) which is the left invariant extension of the inner product \( g \) on \( \mathfrak{su}(2) \). Extend \( X_i/\sqrt{\lambda_i}, 1 \leq i \leq 3, \) to the left invariant vector fields \( Y_i, 1 \leq i \leq 3. \) Let \( \theta_i, 1 \leq i \leq 3, \) be the dual coframe fields on \( G \) to \( Y_i, 1 \leq i \leq 3. \) Let \( \Theta_{ij} \) (resp. \( \Omega_{ij} \)) be the connection (resp. curvature) form of \( (G, g) \) with respect to the orthonormal frame fields \( Y_1, Y_2, Y_3. \) Then we get easily

\[
\begin{align*}
\theta_{12} &= -\frac{(\lambda_1 + \lambda_2 - \lambda_3)\lambda_1 \lambda_2 \lambda_3}{(\lambda_1 \lambda_2 \lambda_3)^2} \theta_3, \\
\theta_{23} &= -\frac{(\lambda_2 + \lambda_3 - \lambda_1)\lambda_1 \lambda_2 \lambda_3}{(\lambda_1 \lambda_2 \lambda_3)^2} \theta_1, \\
\theta_{31} &= -\frac{(\lambda_3 + \lambda_1 - \lambda_2)\lambda_1 \lambda_2 \lambda_3}{(\lambda_1 \lambda_2 \lambda_3)^2} \theta_2, \\
\Omega_{12} &= \frac{(\lambda_1 - \lambda_2)^2 - 3\lambda_3 + 2\lambda_3 - \lambda_1 \lambda_3 \lambda_3 - \lambda_2 \lambda_3}{(\lambda_1 \lambda_2 \lambda_3)^2} \theta_1 \Lambda \theta_2, \\
\Omega_{23} &= \frac{(\lambda_2 - \lambda_3)^2 - 3\lambda_1 + 2\lambda_1 - \lambda_1 \lambda_3 \lambda_3 - \lambda_2 \lambda_3}{(\lambda_1 \lambda_2 \lambda_3)^2} \theta_2 \Lambda \theta_3, \\
\Omega_{31} &= \frac{(\lambda_3 - \lambda_1)^2 - 3\lambda_2 + 2\lambda_2 - \lambda_1 \lambda_3 \lambda_3 - \lambda_2 \lambda_3}{(\lambda_1 \lambda_2 \lambda_3)^2} \theta_3 \Lambda \theta_1.
\end{align*}
\]

So \( (G, g) \) is a space of constant curvature \( k \) if and only if \( \lambda_1 = \lambda_2 = \lambda_3 = 1/k, \) i.e.,

\( g = (1/k)g_0. \)

Let \( (V, \rho) \) be a real representation of \( G \) and \( \langle \cdot, \cdot \rangle \) be a \( G \)-invariant inner product on \( V. \) Then an orbit \( M \) of \( G \) through a unit vector \( p \in V \) is contained in the unit sphere \( S_1 \) (in \( V \) centered at the origin).

**Lemma 4.2.** (i) The orbit \( M \) is a 3-dimensional space of constant curvature \( k \) if and only if

\[
\langle \rho(X_i)(p), \rho(X_j)(p) \rangle = \delta_{ij}/k, \quad 1 \leq i, j \leq 3.
\]

(ii) Assume that the orbit \( M \) is a 3-dimensional space of constant curvature \( k. \)
Then $M$ is a minimal submanifold in $S_1$ if and only if

$$\sum_{i=1}^{3} \rho(X_i)^2(p) = -3kp .$$

**Proof.** Define a map $f: G \to S_1$ by

$$f(\sigma) = \rho(\sigma)(p) , \quad \sigma \in S_1 .$$

Then

$$f_*(X_i) = \rho(X_i)(p) .$$

Let $g$ be the induced metric on $G$ of $f_*$. Then $g$ is a left invariant metric. So $(G, g)$ is a 3-dimensional space of constant curvature $k$ if and only if $g=(1/k)g_0$. By definition of $g$

$$g(X_i, X_j) = \langle \rho(X_i)(p), \rho(X_j)(p) \rangle$$

$$= \frac{g(X_i, X_j)}{k}$$

$$= \delta_{ij}/k , \quad 1 \leq i, j \leq 3 ,$$

if and only if $g=(1/k)g_0$.

(ii) Since $(G, g)$ is a space of constant curvature, $\exp tX_i$ are geodesics in $(G, g)$. By Lemma 3.1, $(\rho(X_i))^2(p)$ is normal to $M$. Consider the vector $\sum_{i=1}^{3} (\rho(X_i))^2(p)$ in $V$, which is normal to $M$. Then its $N_{\rho}(M)$-components in the decomposition (3.2) is the mean curvature vector of $M$ in $S_1$ at $p$. Since $M$ is an orbit of a representation of $G$, $M$ is a minimal submanifold in $S_1$ if and only if the mean curvature vector of $M$ in $S_1$ at one point is 0. So $M$ is a minimal submanifold if and only if

$$(4.1) \quad \sum_{i=1}^{3} (\rho(X_i))^2(p) = cp ,$$

for some constant $c$. Assume that (4.1) holds, then

$$c = \langle \sum_{i=1}^{3} (\rho(X_i))^2(p), p \rangle$$

$$= -\sum_{i=1}^{3} \langle \rho(X_i)(p), \rho(X_i)(p) \rangle$$

$$= -3k .$$

**Q.E.D.**

5. **Proof of Theorems**

For each integer $d$, there exists a (complex) irreducible linear representation of $SU(2)$. We denote by $(V(d)_R, \rho)$ the real representation of $SU(2)$ obtained by the restriction of the coefficient field. Then $(V(d)_R, \rho)$ is irreducible if $d$ is odd. $(V(d)_R, \rho)$ is reducible if $d$ is even and we denote by $V_0(d)$ one of the irreducible component of $V(d)_R$. In this section we study whether there exists an orbit of constant curvature which is a minimal submanifold in the unit sphere in $V(d)_R$ or $V_0(d)$.
Let $\langle , \rangle$ be the real part of the $SU(2)$-invariant Hermitian inner product $(\cdot , \cdot)$ on $V(d)$ defined in (1.1). Then $\langle , \rangle$ is an $SU(2)$-invariant inner product on $V(d)$. Let $p = \sum_{i=0}^{d} z_i P_i \in S_{1}^{2d+1}$, i.e.,
\begin{equation}
\sum_{i=0}^{d} z_i \bar{z}_i = 1.
\end{equation}
Then from (1.2), we get
\begin{equation}
\rho(H)(P_j) = (2j-d)P_j , \quad 0 \leq j \leq d ,
\end{equation}
\begin{equation}
\rho(X)(P_j) = -2((d-j)(j+1))^{1/2}P_{j+1} , \quad 0 \leq j \leq d ,
\end{equation}
\begin{equation}
\rho(Y)(P_j) = -2(j(d-j+1))^{1/2}P_{j-1} , \quad 0 \leq j \leq d .
\end{equation}
where we put $P_{-1} = P_{d+1} = 0$.

**Lemma 5.1.** If an orbit $M = \rho(SU(2)) \ (p)$ is a space of constant curvature $k$, then
(i) $k = \frac{3}{d(d+2)}$,
(ii) $M$ is a minimal submanifold in $S_{1}^{2d+1}$.

By virtue of the above Lemma, we have only to verify the existence of an orbit of constant curvature in $S_{1}^{2d+1}$ to prove Theorem A.

Extend $\rho: \mathfrak{s}u(2) \rightarrow gI(d+1, \mathbb{C})$ to $\mathfrak{s}l(2, \mathbb{C}) = (\mathfrak{s}u(2))^C$ and put
\begin{align*}
H &= -(-1)^{1/2}X_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
X &= X_2 - (-1)^{1/2}X_3 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \\
Y &= -X_2 - (-1)^{1/2}X_3 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.
\end{align*}
Then from (1.2), we get
\begin{align*}
(5.3)_1 & \quad \rho(H)(P_j) = (2j-d)P_j , \quad 0 \leq j \leq d , \\
(5.3)_2 & \quad \rho(X)(P_j) = -2((d-j)(j+1))^{1/2}P_{j+1} , \quad 0 \leq j \leq d , \\
(5.3)_3 & \quad \rho(Y)(P_j) = -2(j(d-j+1))^{1/2}P_{j-1} , \quad 0 \leq j \leq d .
\end{align*}

**Lemma 5.2.** An orbit $M = \rho(SU(2)) \ (p)$ is a space of constant curvature $3/d(d+2)$ if and only if
\begin{align*}
(5.4)_1 & \quad (\rho(H)(p), \rho(X)(p)) + (\rho(H)(p), \rho(Y)(p)) = 0 , \\
(5.4)_2 & \quad (\rho(X)(p), \rho(Y)(p)) = 0 , \\
(5.4)_3 & \quad (\rho(H)(p), \rho(H)(p)) = d(d+2)/3 .
\end{align*}

Proof. By definition of $H, X$ and $Y$
\begin{align*}
X_1 &= (-1)^{1/2}H , \quad X_2 = X - Y , \quad X_3 = (-1)^{1/2}(X + Y) .
\end{align*}
A simple computation shows
\[ \langle \rho(X_i)(p), \rho(X_j)(p) \rangle = \langle (-1)^{d_i} \rho(H)(p), \rho(X)(p) - \rho(Y)(p) \rangle = -\text{Im} \left( \rho(H)(p), \rho(X)(p) \right) + \text{Im} \left( \rho(H)(p), \rho(Y)(p) \right). \]
Similarly
\[ \langle \rho(X_i)(p), \rho(X_j)(p) \rangle = \text{Re} \left( \rho(H)(p), \rho(X)(p) \right) + \text{Re} \left( \rho(H)(p), \rho(Y)(p) \right), \]
\[ \langle \rho(X_i)(p), \rho(X_j)(p) \rangle = 2 \text{Im} \left( \rho(X)(p), \rho(Y)(p) \right), \]
\[ \langle \rho(X_i)(p), \rho(X_j)(p) \rangle = \langle \rho(H)(p), \rho(H)(p) \rangle, \]
\[ \langle \rho(X_i)(p), \rho(X_j)(p) \rangle = \langle \rho(X)(p), \rho(X)(p) \rangle + \langle \rho(Y)(p), \rho(Y)(p) \rangle - 2 \text{Re} \left( \rho(X)(p), \rho(Y)(p) \right), \]
\[ \langle \rho(X_i)(p), \rho(X_j)(p) \rangle = \langle \rho(X)(p), \rho(X)(p) \rangle + \langle \rho(Y)(p), \rho(Y)(p) \rangle + 2 \text{Re} \left( \rho(X)(p), \rho(Y)(p) \right). \]
An orbit \( M = \rho(SU(2))(p) \) is a space of constant curvature \( \frac{3}{d(d+2)} \) if and only if
\[ \langle \rho(X_i)(p), \rho(X_j)(p) \rangle = \frac{d(d+2)}{3} \delta_{ij}, \quad 1 \leq i, j \leq 3, \]
by Lemma 4.2. Taking (5.2) into account, the Lemma is an immediate consequence. Q.E.D.

Proof of Theorems. Let \( p = \sum_{j=0}^{d} z_j P_j \) be a point in \( S^d_{2d+1} \), i.e.,
\[
\sum_{j=0}^{d} z_j = 1.
\]
From (5.3), (5.3) and (5.3), we get
\[
\rho(H)(p) = \sum_{j=0}^{d} (2j-d)z_j P_j, \\
\rho(X)(p) = -2 \sum_{j=0}^{d} ((d-j)(j+1))^{1/2} z_j P_{j+1}, \\
\rho(Y)(p) = -2 \sum_{j=0}^{d} (j(d-j+1))^{1/2} z_j P_{j+1}.
\]
Then
\[
\rho((H)(p), \rho(X)(p)) + \rho(H)(p), \rho(Y)(p)) = -2 \sum_{j=0}^{d} (2j-d)(j(d-j+1))^{1/2} z_j \bar{z}_{j-1} - 2 \sum_{j=0}^{d} (2j-d)(j+1)(d-j))^{1/2} z_j \bar{z}_{j+1}, \\
\rho(X)(p), \rho(Y)(p)) = 4 \sum_{j=0}^{d} (j(d-j+1)(d-j))^{1/2} z_j \bar{z}_{j+1}, \\
\rho(H)(p), \rho(H)(p)) = \sum_{j=0}^{d} (d^2 - 4dj + 4j^2) z_j \bar{z}_j.
So (5.4)\(_1\) and (5.4)\(_2\) is equivalent to the following
\[(5.5)_1\Sigma_{j=1}^{d} (2j-d)(j(j+1))^{1/2}z_j z_{j-1} + \Sigma_{j=1}^{d-1} (2j-d)(j+1)(d-j))^{1/2}z_{j+1} = 0,\]
\[(5.5)_2\Sigma_{j=1}^{d-1} (j(j+1)(d-j+1)(d-j))^{1/2}z_{j+1} = 0.\]

Taking (5.1) into account, (5.4)\(_3\) is equivalent to
\[(5.5)_3\Sigma_{j=0}^{d} (6j^2-6dj+d^2-d) z_j z_j = 0.\]

Now we prove the system of equations (5.5)\(_1\), (5.5)\(_2\) and (5.5)\(_3\) has a solution under the condition (5.1).

When \(d=4\) we put
\[z_i = \begin{cases} 1/2 & , \text{ if } i = 0, 4, \\ (-2)^{i/2}/2, \text{ if } i = 2, \\ 0 \text{, if } i = 1, 3. \end{cases}\]

When \(d\) is an even integer \(d=2d'\) and \(d\geq 6\), we put
\[z_i = \begin{cases} ((d'+1)/6d')^{1/2} & , \text{ if } i = 0, d, \\ (-1)^{d/2}(2d'-1)/3d')^{1/2}, \text{ if } j = d', \\ 0 \text{, if otherwise.} \end{cases}\]

When \(d\) is an odd integer \(d=2d'+1\), \(d'\geq 2\), we put
\[z_i = \begin{cases} ((d'+2)/(3d'+3))^{1/2} & , \text{ if } i = 0, \\ ((2d'+1)/(3d'+3))^{1/2}, \text{ if } i = d'+1, \\ 0 \text{, if otherwise.} \end{cases}\]

Then it is easily verified that \((z_0, z_1, \ldots, z_d)\) is a solution of the equation. So Theorem A is proved.

When \(d\) is an even integer, \(d\geq 6\), \(\Sigma_{i=0}^{d} z_i P_i\) is contained in \(V_0(d)\) by Lemma 2.1. So the orbit passing this point must be contained in the unit sphere in \(V_0(d)\). So we get Theorem B. Q E.D.

In Theorem B the case \(d=4\) is excluded. But this is a natural consequence of the following

**Theorem 5.7** (J.D. Moore, [10]). Let \(M\) be a connected \(n\)-dimensional Riemannian manifold of constant curvature \(k\) isometrically and minimally immersed in a simply connected \((2n-1)\)-dimensional Riemannian manifold \(N\) of constant curvature \(K\). Then either \(M\) is totally geodesic or it is flat.

Recently Li [9] proved the following
Theorem. If \( \Phi: S^m \to S^1 \) is an isometric minimal immersion, then \( \Phi(S^m) \) is either an embedded sphere or an embedded projective space.

But this is not true if the codimension is not maximal. Let \( M \) be the orbit passing \((2^{1/2}P_0-(-5)^{1/2}P_3+2^{1/2}P_6)/3 \) in \( V_3(6) \). As we proved, \( M \) is a space of constant curvature \( 1/16 \) and is a minimal submanifold in \( S^6 \). But the orbit is neither an embedded sphere nor an embedded projective space in \( S^6 \). Namely we have the following Proposition 5.8. Let \( \pi \) be the covering map
\[
\pi: SU(2) \to M; \ g \to \rho(g)((2^{1/2}P_0-(-5)^{1/2}P_3+2^{1/2}P_6)/3).
\]
Then \( \pi \) is at least 6-fold.

Proof. Put \( g = \left[ \begin{array}{cc} \alpha & \alpha^{-1} \\ \end{array} \right] \), \( \alpha = e^{-i \sqrt{k} \pi/3} \) (\( 0 \leq k \leq 5 \)). Then
\[
\begin{align*}
\rho(g)((2^{1/2}P_0-(-5)^{1/2}P_3+2^{1/2}P_6)/3) \\
= (2^{1/2}P_0-(-5)^{1/2}P_3+2^{1/2}P_6)/3 \\
= (2^{1/2}P_0-(-5)^{1/2}P_3+2^{1/2}P_6)/3
\end{align*}
\]
So the covering \( \pi \) is at least 6-fold. Q.E.D.

References


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