ACTIONS OF SYMPLECTIC GROUPS ON A PRODUCT OF QUATERNION PROJECTIVE SPACES

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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0. Introduction

We shall study smooth actions of symplectic group $Sp(n)$ on a closed orientable manifold $X$ such that $X \simeq P_a(H) \times P_b(H)$, under the conditions: $a + b \leq 2n - 2$ and $n \geq 7$. Our result is stated in §2 and proved in §5. Typical examples are given in §1. Similar result on smooth actions of special unitary group $SU(n)$ on a closed orientable manifold $X$ such that $X \simeq P_a(C) \times P_b(C)$ is stated in the final section.

Throughout this paper, let $H^*(\_)$ denote the singular cohomology theory with rational coefficients, and let $P_a(H)$, $P_a(C)$ and $P_a(R)$ denote the quaternion, complex and real projective $n$-space, respectively. By $X \simeq X'$, we mean that $H^*(X) \approx H^*(X')$ as graded algebras.

1. Typical examples

1.1. We regard $S^{4k-1}$ as the unit sphere of the quaternion $k$-space $H^k$ with the right scalar multiplication. Let $Y$ be a compact $Sp(1)$ manifold. By the diagonal action, $Sp(1)$ acts freely on the product manifold $S^{4k-1} \times Y$. Here we consider the cohomology ring of the orbit manifold $(S^{4k-1} \times Y)/Sp(1)$ for the case $Y \simeq P_a(H)$.

Consider the fibration: $Y \to (S^{4k-1} \times Y)/Sp(1) \to P_{a-1}(H)$. By the Leray–Hirsch theorem, $H^*((S^{4k-1} \times Y)/Sp(1))$ is freely generated by $1, u, u^2, \ldots, u^b$ as an $H^*(P_{a-1}(H))$ module for an element $u \in H^*((S^{4k-1} \times Y)/Sp(1))$. If $u$ can be so chosen as $u^{a+1} = 0$, then we see that $(S^{4k-1} \times Y)/Sp(1) \simeq P_{a-1}(H) \times P_b(H)$.

Lemma 1.1. Denote by $F$, the fixed point set of the restricted $U(1)$ action on $Y$. If $F \simeq P_a(C)$, then $(S^{4k-1} \times Y)/Sp(1) \simeq P_{a-1}(H) \times P_b(H)$.

Proof. Consider the fibration: $Y \to (S^{4k-1} \times Y)/U(1) \to P_{2a-1}(C)$. We see that $H^*((S^{4k-1} \times Y)/U(1))$ is freely generated by $1, v, v^2, \ldots, v^b$ as an $H^*(P_{2a-1}(C))$ module for an element $v \in H^*((S^{4k-1} \times Y)/U(1))$. We shall show first that...
can be so chosen as \( v^{k+1} = 0 \). We regard \( S^m \) as the inductive limit of \( S^{4k-1} \) on which \( U(1) \) acts naturally. Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^*((S^m \times Y)/U(1)) & \xrightarrow{j^*} & H^*((S^{4k-1} \times Y)/U(1)) \\
\downarrow i^*_m & & \downarrow i^*
\end{array}
\]

\[
H^*(P_m(C) \times F) \xrightarrow{j_F^*} H^*(P_{2k-1}(C) \times F)
\]

where \( i, i_m, j, j_F \) are natural inclusions. Since \( H^{\text{odd}}(Y) = 0 \), we see that \( i^*_m \) is injective [4] and \( j^* \) is surjective. Let \( v_m \) be an element of \( H^*((S^m \times Y)/U(1)) \) such that \( j^*(v_m) = v \). Let \( x \) be the canonical generator of \( H^2(P_m(C)) \cong H^2(P_{2k-1}(C)) \). Then we can express

\[
i^*_m(v_m) = x^2 f_0 + x f_1 + 1 f_2
\]

where \( f_i \in H^{2r}(F) \) for \( r = 0, 1, 2 \). Since \( F \sim P_k(C) \), we see that there are rational numbers \( a_0, a_1, a_2 \) and a non-zero element \( y \in H^2(F) \), such that \( f_i = a_i y^r \) for \( r = 0, 1, 2 \). Then we obtain

\[
i^*_m(v_m - a_0 x^2)^{k+1} = (x f_1 + 1 x f_2)^{k+1} = 0.
\]

Since \( i^*_m \) is injective, we obtain \( (v_m - a_0 x^2)^{k+1} = 0 \). Put \( v_1 = j^*(v_m - a_0 x^2) \). Then \( v_1^{k+1} = 0 \), and hence

\[
H^*((S^{4k-1} \times Y)/U(1)) \cong Q[x, v_1]/(x^{2k}, v_1^{k+1}); \text{deg } x = 2, \text{deg } v_1 = 4.
\]

Consider next the following commutative diagram:

\[
\begin{array}{ccc}
S_p(1)/U(1) & \xrightarrow{p} & (S^{4k-1} \times Y)/U(1) \\
\downarrow & & \downarrow \\
S_p(1)/U(1) & \xrightarrow{q} & P_{2k-1}(C) \times P_k(H).
\end{array}
\]

Let \( t \in H^*(P_{2k-1}(H)) \) be the canonical generator such that \( q^*(t) = x^2 \). There exist rational numbers \( \lambda, \mu \) such that \( p^*(u) = \lambda v_1 + \mu x^2 \). Put \( u_i = u - \mu t \). Then \( p^*(u_i) = \lambda v_i \), and hence \( p^*(u_i)^{k+1} = 0 \). Since the homomorphism \( p^*: H^*((S^{4k-1} \times Y)/S_p(1)) \to H^*((S^{4k-1} \times Y)/U(1)) \) is injective, we obtain \( u_i^{k+1} = 0 \), and hence

\[
H^*((S^{4k-1} \times Y)/S_p(1)) \cong Q[t, u_i]/(t^k, u_i^{k+1}); \text{deg } t = \text{deg } u_i = 4.
\]

Thus we obtain \( (S^{4k-1} \times Y)/S_p(1) \sim P_{2k-1}(H) \times P_k(H) \). q.e.d.

1.2. We give here examples of a closed orientable \( S_p(1) \) manifold \( Y \) such that \( Y \sim P_k(H) \) and \( F \sim P_k(C) \), where \( F \) denotes the fixed point set of the restricted \( U(1) \) action on \( Y \).

Consider the \( S_p(1) \) action on \( P_k(H) = S^{4k+3}/S_p(1) \) by the left scalar multiplication. Then the fixed point set of the restricted \( U(1) \) action is naturally
diffeomorphic to $P_3(C)$, the fixed point set of the $Sp(1)$ action is naturally diffeomorphic to $P_3(R)$, and the isotropy representation at each fixed point of the $Sp(1)$ action is equivalent to $b\eta \oplus \theta^b$, where $\eta$ denotes the canonical 3-dimensional real representation of $Sp(1)$, $b\eta$ denotes the $b$-fold direct sum of $\eta$, and $\theta^b$ is the trivial representation of degree $b$.

Let $D^{3b}$ denote the unit disk of the representation space $b\eta$. Let $W$ be a $(b+1)$-dimensional compact orientable smooth manifold which is rationally acyclic. Then the boundary $\partial(D^{3b} \times W)$ is a $4b$-dimensional compact orientable smooth $Sp(1)$ manifold which is a rational homology sphere, and the isotropy representation at each fixed point of the $Sp(1)$ action is equivalent to $b\eta \oplus \theta^b$. Hence we can construct an equivariant connected sum

$$Y(W) = P_3(H) \# \partial(D^{3b} \times W).$$

Denote by $F(W)$ the fixed point set of the restricted $U(1)$ action on $Y(W)$. Then $F(W)$ is naturally diffeomorphic to $P_3(C) \# d(D^{3b} \times W)$. It is easy to see that

$$Y(W) \approx P_3(H), F(W) \approx P_3(C).$$

1.3. Let $\xi$ be a quaternion $k$-plane bundle and $\xi_C$ its complexification under the restriction of the filed. Its $i$-th symplectic Pontrjagin class $e_i(\xi)$ is by definition [2, §9.6]

$$e_i(\xi) = (-1)^ic_{2i}(\xi_C),$$

where $c_{2i}(\xi_C)$ is the $2i$-th Chern class. Denote by $P(\xi)$ the total space of the associated quaternion projective space bundle. Let $\xi$ be the canonical quaternion line bundle over $P(\xi)$ and put $t = e_0(\xi)$. It is known that there is an isomorphism:

$$(1.3) \quad H^*(P(\xi)) \approx H^*(B) [t]/(\sum_{-2i-1}^\infty e_{2i-1}(t^i)), $$

where $B$ is the base space of the bundle $\xi$ (cf. [3, §3]).

Let $\xi$ be the canonical quaternion line bundle over $P_3(H)$ and $\xi^*$ its dual line bundle. Let $W$ be a $4b$-dimensional closed orientable smooth manifold and let $f: W \to P_3(H)$ be a smooth mapping such that $f^*: H^*(P_3(H)) \approx H^*(W)$. Let $c$ be a non-negative integer such that $b \leq c+1$. Then, there is a quaternion $(c+1)$-plane bundle $\xi$ over $W$ such that

$$(n+c+1)f^*\xi^* \approx \xi \oplus \theta^b_H,$$

where $\theta^b_H$ is a trivial quaternion $n$-plane bundle. Put $X = P((n+c+1)f^*\xi^*)$. Since $X$ is diffeomorphic to $\partial(D(\xi) \times D^b)/Sp(1)$, we can act $Sp(n)$ on $X$ in order that the fixed point set is diffeomorphic to $P(\xi)$. We see that by (1.3)
\[ H^*(X) \cong Q[u, v]/(u^{a+c+1}, v^{a+1}), \]
\[ H^*(P(\xi)) \cong Q[t, v]/(v^{a+1}, \sum_{i=0}^{c+1} (-1)^i t^{c+1-i} v^i), \]

where \( v = f^* e(\xi), t = e(\xi) \) and \( u+v \) is the first symplectic Pontrjagin class of the canonical line bundle over \( P((n+c+1)f^* \xi^*). \)

2. Classification theorems

We shall prove the following results in this paper.

**Theorem 2.1.** Let \( X \) be a closed orientable manifold on which \( Sp(n) \) acts smoothly and non-trivially. Suppose \( X \cong P_a(H) \times P_b(H); a \geq b \geq 1, a+b \leq 2n-2 \) and \( n \geq 7 \). Then there are four cases:

(0) \( a = n-1 \) and \( X \cong P_{n-1}(H) \times Y_0 \), where \( Y_0 \) is a closed orientable manifold such that \( Y_0 \cong P_{n-1}(H) \) and \( Sp(n) \) acts naturally on \( P_{n-1}(H) \) and trivially on \( Y_0 \).

(i) \( a = n-1 \) and \( X \cong (S^{4n-1} \times Y_1)/Sp(1) \), where \( Y_1 \) is a closed orientable \( Sp(1) \) manifold such that \( Y_1 \cong P_{n-1}(H) \), \( Sp(1) \) acts as right scalar multiplication on \( S^{4n-1} \), the unit sphere of \( H^* \), and \( Sp(n) \) acts naturally on \( S^{4n-1} \) and trivially on \( Y_1 \).

In addition, the fixed point set of the restricted \( U(1) \) action on \( Y_1 \) is \( \sim P_{3}(C) \).

(ii) \( a = b = n-1 \) and \( X \cong P_{n-1}(H) \times P_{n-1}(H) \) with the diagonal \( Sp(n) \) action.

(iii) \( a \geq n \) and \( X \cong \partial(D^{4n} \times Y_2)/Sp(1) \), where \( Y_2 \) is a compact orientable \( Sp(1) \) manifold such that \( \dim Y_2 = 4(a+b+1-n) \) and \( Y_2 \cong P_{b}(H) \), \( Sp(1) \) acts as right scalar multiplication on \( D^{4n} \), the unit disk of \( H^* \), and \( Sp(n) \) acts naturally on \( D^{4n} \) and trivially on \( Y_2 \).

In addition, the \( Sp(1) \) action on the boundary \( \partial Y_2 \) is free and the fixed point set of the restricted \( U(1) \) action on \( Y_2 \) is \( \sim P_{3}(C) \) or \( \sim P_{b}(H) \).

**Remark.** By \( X \cong X' \) we mean that \( X \) is equivariantly diffeomorphic to \( X' \) as \( Sp(n) \) manifolds. In the case (iii), the fixed point set of the \( Sp(n) \) action on \( X \) is naturally diffeomorphic to the orbit manifold \( \partial Y_2/Sp(1) \).

**Theorem 2.2.** In the case (iii) of Theorem 2.1, the cohomology ring \( H^*(\partial Y_2/Sp(1)) \) is isomorphic to one of the following:

1. \( Q[x, y]/(x^{a+1-n}, y^{a+1}), \)
2. \( Q[x, y]/(y^{a+1}, \sum_{i=0}^{b} (-1)^i (a+1) x^{a+1-n-i} y^i); b \leq a+1-n, \)

where \( \deg x = \deg y = 4 \), and \( x \) is the Euler class of the principal \( Sp(1) \) bundle \( \partial Y_2 \rightarrow \partial Y_2/Sp(1) \).

**Remark.** The \( Sp(n) \) action given in §1.3 is an example of the case (iii)–(2). Lemma 1.1 assures that a converse of Theorem 2.1 (i) is true.
3. Cohomology of certain homogeneous spaces

Here we consider the cohomology of $V_{n,2}/G=Sp(n)/Sp(n-2) \times G$ for certain closed subgroups $G$ of $Sp(2)$. Let $\xi$ be the canonical quaternion line bundle over $P_{n-1}(H)$ and $\zeta$ its orthogonal complement, that is, $\xi$ is a quaternion $(n-1)$-plane bundle over $P_{n-1}(H)$ such that its total space is

$$E(\xi) = \{ (u, [v]) \in H^* \times P_{n-1}(H) : u \perp v \}.$$

It is easy to see that the total space $P(\zeta)$ of the associated quaternion projective space bundle is naturally diffeomorphic to $V_{n>2}/Sp(1) \times Sp(1)$. Since $\xi \oplus \zeta$ is a trivial bundle, we obtain $e_1(\xi) = (-1)e_1(\zeta)$. By definition, $P(\xi)$ is naturally identified with a subspace of $P_{n-1}(H) \times P_{n-1}(H)$. Let $i: P(\xi) \rightarrow P_{n-1}(H) \times P_{n-1}(H)$ be the inclusion. Put $\xi = i^*(\xi^* \times 1)$. Then by (1.3) there is an isomorphism:

$$H^*(V_{n,2}/Sp(1) \times Sp(1)) \cong Q[x, y]/(x^n, \sum_i x^i y^{n-1-i}).$$

$$\deg x = -\deg y = 4,$$

by the identification $x = i^*(1 \times e_1(\zeta))$ and $y = i^*(e_1(\xi) \times 1)$.

Let $p: V_{n,2}/Sp(1) \times Sp(1) \rightarrow V_{n,2}/Sp(2)$ be the natural projection and $\xi_2$ the standard quaternion 2-plane bundle over $V_{n,2}/Sp(2)$.

**Lemma 3.2.** The graded algebra $H^*(V_{n,2}/Sp(2))$ is generated by $e_1(\xi_2)$, $e_2(\xi_2)$. The algebra is isomorphic to the subalgebra of $Q[x, y]/(x^n, \sum_i x^i y^{n-1-i})$, consisting of symmetric polynomials.

**Proof.** Since the fibration $p$ is a 4-sphere bundle and $H^{odd}(V_{n,2}/Sp(2)) = 0$ (cf. [1, §26]), the homomorphism $p^*: H^*(V_{n,2}/Sp(2)) \rightarrow H^*(V_{n,2}/Sp(1) \times Sp(1))$ is injective. Since $p^*(e_1(\xi_2)) = i^*(\xi \times \xi)$, we obtain

$$p^*e_1(\xi_2) = i^*e_1(\xi \times \xi) = x + y,$$

$$p^*e_2(\xi_2) = i^*e_2(\xi \times \xi) = xy.$$

Then the desired result is obtained by the Leray-Hirsch theorem.

**Corollary 3.3.** $e_1(\xi_2)^{2n-4} = 0$ and $e_2(\xi_2)^{2n-3} = 0$.

**Proof.** Put $I = (x^n, \sum_i x^i y^{n-1-i})$. It is easy to see that $y^n \in I$. In the quotient ring $Q[x, y]/I$, we obtain

$$(x + y)^{2n-4} = \left( \begin{array}{c} 2n-4 \\ n-1 \end{array} \right) x^{n-1} y^{n-3} + \left( \begin{array}{c} 2n-4 \\ n-2 \end{array} \right) x^{n-2} y^{n-2} + \left( \begin{array}{c} 2n-4 \\ n-1 \end{array} \right) x^{n-3} y^{n-1}$$

$$= \left( \begin{array}{c} 2n-4 \\ n-2 \end{array} - \left( \begin{array}{c} 2n-4 \\ n-1 \end{array} \right) \right) x^{n-2} y^{n-2},$$

and hence $e_1(\xi_2)^{2n-4} = 0$. We obtain $e_2(\xi_2)^{2n-3} = 0$ similarly.

4. Preliminary results

First we state the following two lemmas which are proved by a standard
Lemma 4.1. Suppose $n \geq 7$. Let $G$ be a closed connected proper subgroup of $\text{Sp}(n)$ such that $\dim \text{Sp}(n)/G < 8n$. Then $G$ coincides with $\text{Sp}(n-i) \times K$ $(i=1,2,3)$ up to an inner automorphism of $\text{Sp}(n)$, where $K$ is a closed connected subgroup of $\text{Sp}(i)$.

Lemma 4.2. Suppose $r \geq 5$ and $k < 8r$. Then an orthogonal non-trivial representation of $\text{Sp}(r)$ of degree $k$ is equivalent to $(\nu_r)_R \oplus \theta^{k-t}$. Here $(\nu_r)_R : \text{Sp}(r) \rightarrow \mathcal{O}(4r)$ is the canonical inclusion, and $\theta^t$ is the trivial representation of degree $t$.

In the following, let $X$ be a closed connected orientable manifold with a non-trivial smooth $\text{Sp}(n)$ action, and suppose $n \geq 7$ and $\dim X < 8n$. Put

$$F(i) = \{x \in X : \text{Sp}(n-i) \subset \text{Sp}(n) \subset \text{Sp}(n-i) \times \text{Sp}(i)\}$$

$$X(i) = \text{Sp}(n)F(i) = \{gx : g \in \text{Sp}(n), x \in F(i)\}.$$ Here $\text{Sp}(n)_x$ denotes the isotropy group at $x$. Then, by Lemma 4.1, we obtain $X = X(\omega) \cup X(1) \cup X(2) \cup X(3)$.

Proposition 4.3. If $X(\omega)$ is non-empty, then $X(i)$ is empty for each $i \geq k+2$.

Proof. Let us denote by $F(\text{Sp}(n-j), X(i))$ the fixed point set of the restricted $\text{Sp}(n-j)$ action on $X(i)$. It is easy to see that the set is empty for each $j < i \leq n-i$. Suppose that $X(\omega)$ is non-empty and fix $x \in F(\omega)$. Let $\sigma$ be the slice representation at $x$. Then the restriction $\sigma | \text{Sp}(n-k)$ is trivial or equivalent to $(\nu_{n-k})_R \oplus \theta^t$ by Lemma 4.2. Anyhow, a principal isotropy group of the given action contains $\text{Sp}(n-k-1)$, and hence $F(\text{Sp}(n-k-1), X(i))$ is non-empty if so is $X(i)$. q.e.d.

Proposition 4.4. Suppose $X = X(\omega) \cup X(k+1)$. If $X(\omega)$ and $X(k+1)$ are non-empty, then the codimension of each connected component of $F(\omega)$ in $X$ is equal to $4(k+1)(n-k)$.

Proof. Fix $x \in F(\omega)$. Let $\sigma$ and $\rho$ denote the slice representation at $x$ and the isotropy representation of the orbit $\text{Sp}(n)x$, respectively. The restriction $\sigma | \text{Sp}(n-k)$ is equivalent to $(\nu_{n-k})_R \oplus \theta^t$ by Lemma 4.2 and the assumption that $X(\omega)$ is non-empty. On the other hand, $\rho | \text{Sp}(n-k)$ is equivalent to $k(\nu_{n-k})_R \oplus \theta^t$ by considering adjoint representations. Hence $(\sigma \oplus \rho) | \text{Sp}(n-k)$ is equivalent to $(k+1)(\nu_{n-k})_R \oplus \theta^{k+t}$. This shows that the codimension of $F(\omega)$ at $x$ is equal to $4(k+1)(n-k)$. q.e.d.

Corollary 4.5. Suppose $X = X(\omega) \cup X(\omega)$. Then either $X(\omega)$ or $X(\omega)$ is empty.

Remark. $\dim \text{Sp}(n)/\text{Sp}(n-k) \times \text{Sp}(k) = 4k(n-k)$ and $\chi(\text{Sp}(n)/\text{Sp}(n-k)$
× Sp(k) = \binom{n}{k}, \text{ where } \chi(\ ) \text{ denotes the Euler characteristic, and } \binom{n}{k} \text{ denotes the binomial coefficient.}

5. Proof of the classification theorems

Throughout this section, suppose that X is a closed orientable manifold with a non-trivial smooth Sp(n) action such that

\(*) \quad H^*(X) = \mathbb{Q}[u, v]/(u^{a+1}, v^{b+1}); \deg u = \deg v = 4.\)

Moreover, suppose that \(n \geq 7, 1 \leq b \leq a \) and \(a + b \leq 2n - 2\). By arguments and notations in the preceding section, we see that \(X = X_0 \cup X_{k+1}\) for \(k = 0, 1, 2\).

5.1. We shall show first that \(X \neq X_0 \cup X_1\). Suppose \(X = X_0 \cup X_1\). Then \(X = X_0\) or \(X = X_1\) by Corollary 4.5. Looking at the Euler characteristic of X, we see that \(X \neq X_0\).

Suppose \(X = X_1\). Then \(X = (V_{n,2} \times F_0)/(Sp(2))\). Here we consider the following commutative diagram of natural projections:

\[
\begin{array}{ccc}
(V_{n,2} \times F_0)/T & \xrightarrow{p_1} & V_{n,2}/T \\
\downarrow & & \downarrow \\
X = (V_{n,2} \times F_0)/(Sp(2)) & \xrightarrow{p} & V_{n,2}/Sp(2),
\end{array}
\]

where \(T\) is a maximal torus of \(Sp(2)\). Since \(\chi(F_0) \neq 0\), we see that the restricted \(T\) action on \(F_0\) has a fixed point, and hence the projection \(p_1\) has a cross-section. Therefore \(p_1^*: H^*(V_{n,2}/T) \rightarrow H^*((V_{n,2} \times F_0)/T)\) is injective. On the other hand, \(q^*: H^*(V_{n,2}/Sp(2)) \rightarrow H^*(V_{n,2}/T)\) is injective, because \(H^{odd}(V_{n,2}/Sp(2)/T) = 0\) (cf. [1, §26]). Consequently, we see that \(p^*: H^*(V_{n,2}/Sp(2)) \rightarrow H^*(X)\) is injective. In particular, we obtain \(a + b \geq 2n - 4\). If \(a + b = 2n - 4\), then \(X = V_{n,2}/Sp(2)\). Because rank \(H^i(X) = 2\) and rank \(H^i(V_{n,2}/Sp(2)) = 1\), we get a contradiction.

Suppose \(a + b \geq 2n - 3\), and put \(p^*e_{i}(\xi_2) = \alpha u + \beta v; \alpha, \beta \in \mathbb{Q}\). Since \(e_i(\xi_2)2n-3 = 0\) by Corollary 3.3, we obtain

\[0 = p^*e_i(\xi_2)^{a+b} = \left(\frac{a+b}{a}\right)(\alpha u)^a(\beta v)^b,\]

and hence \(\alpha \beta = 0\). On the other hand, \(e_i(\xi_2)2n-4 \neq 0\) by Corollary 3.3, and hence \(p^*e_i(\xi_2)2n-4 \neq 0\). Thus we obtain \(a = 2n - 4\). Looking at the Euler characteristic of \(F_0\), we get a contradiction.

5.2. We consider now the case \(X = X_0 \cup X_2\). Suppose that both \(X_0\) and \(X_2\) are non-empty. We see that \(\text{codim } F_0 = 8n - 8\) by Proposition 4.4. Since \(\dim X \leq 2n - 8\), we obtain \(\dim F_0 = 0\) and \(a + b = 2n - 2\).
Fix $x \in F_{(0)}$. Since $X_{(0)}$ is non-empty, we see that the slice representation $\sigma$ at $x$ is equivalent to $\nu_{n-1} \otimes \nu_{n-1}^*$ or $\nu_{n-1}^* \otimes \nu_{n-1}$ by Lemma 4.2, where $\pi$ is a natural projection of $Sp(n-1) \times Sp(1)$ onto $Sp(n-1)$. Then the principal isotropy group is of the form $Sp(n-2) \times K$, where $K = \Delta Sp(1)$ (resp. $1 \times Sp(1)$) for $\sigma = \nu_{n-1} \otimes \nu_{n-1}^*$ (resp. $\sigma = \nu_{n-1}^* \otimes \nu_{n-1}$). Here $\Delta Sp(1)$ (resp. $1 \times Sp(1)$) is a closed subgroup of $Sp(2)$ consisting of the matrices of the form $\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$).

Anyhow, we see that the $Sp(n)$ action on $X$ has a codimension one orbit, and hence $X$ is a union of closed invariant tubular neighborhoods of just two non-principal orbits (cf. [6]). We already see that one of the non-principal orbits is $P_{n-1}(H)$. Looking at the Euler characteristic of $X$, we see that $a = b = n - 1$ and another non-principal orbit is $V_{n-1}/Sp(1) \times Sp(1)$.

Suppose $K = 1 \times Sp(1)$. Then the normalizer of the principal isotropy group is connected, and hence such an $Sp(n)$ manifold is unique up to equivariant diffeomorphism (cf. [6, §5.3]). On the other hand, the product manifold $P_{n-1}(H) \times P_{n-1}(H)$ with the diagonal $Sp(n)$ action is such one. Therefore $X$ is equivariantly diffeomorphic to $P_{n-1}(H) \times P_{n-1}(H)$ with the diagonal $Sp(n)$ action.

Suppose next $K = \Delta Sp(1)$. Then the normalizer of the principal isotropy group has just two connected components, and its generator corresponds to the antipodal involution of the slice representation at a point of $V_{n-1}/Sp(1) \times Sp(1)$. Hence such an $Sp(n)$ manifold is unique up to equivariant diffeomorphism (cf. [6, §5.3]). Here we construct such one. Let $\xi$ be the canonical quaternion line bundle over $P_{n-1}(H)$ and $\xi$ its orthogonal complement (see §3). Then $Sp(n)$ acts naturally on the total space $E(\xi)$ as the bundle mappings. Denote by $\theta_{H}^i$ a trivial quaternion line bundle. We see that the $Sp(n)$ action on the total space $P(\xi \oplus \theta_{H}^i)$ of the associated quaternion projective space bundle is the desired one. On the other hand, we see that by (1.3)

$$H^k(P(\xi \oplus \theta_{H}^i)) \simeq Q[x, y]/(x^k, \sum_i x^i y^{k-i}); \deg x = \deg y = 4.$$ 

Hence the cohomology ring of $P(\xi \oplus \theta_{H}^i)$ is not isomorphic to that of $P_{n-1}(H) \times P_{n-1}(H)$.

5.3. We consider next the case $X = X_{(0)} \cup X_{(1)}$ for $c < n$. We shall show first that $X_{(0)}$ is empty.

Suppose that $X_{(0)}$ is non-empty. Let $U$ be an invariant closed tubular neighborhood of $X_{(0)}$ in $X$, and put $E = X - \text{int} U$. Put $W = E \cap F_{(0)}$. Then $W$ is a compact connected orientable manifold with non-empty boundary $\partial W$, and $Sp(1)$ acts naturally on $W$. Since there is a natural diffeomorphism $E = (S^{4n-1} \times W)/Sp(1)$, we obtain

$$\dim W = 4(a + b + 1 - n) = 4k, \quad k \leq b \leq a < n.$$ 

Let $i: E \to X$ be the inclusion. Then $i^*: H^i(X) \to H^i(E)$ is an isomorphism.
for each \( t \leq 4n - 2 \), because the codimension of each connected component of \( X(\omega) \) is \( 4n \) by Lemma 4.2. By the Gysin sequence of the principal \( \text{Sp}(1) \) bundle \( S^{4n-1} \times W \rightarrow E \) and the cohomology ring of \( X \), we obtain rank \( H^4(W) - \text{rank} \ H^4(W) = 1 \). On the other hand, we see that \( H^k(W) \approx H_k(W) = 0 \) and rank \( H^4(W) \geq 0 \); this is a contradiction. Thus we see that \( X(\omega) \) is empty.

Consequently, we obtain \( X = X(1) = (S^{4n-1} \times F(\omega))/\text{Sp}(1) \). Put \( Y = F(\omega) \). We see that

\[
\dim Y = 4(a + b + 1 - n) = 4k, \quad k \leq b \leq a < n < a + b.
\]

We shall show next that \( a = n - 1 \) and \( Y \sim P_b(H) \).

By the Gysin sequence of the principal \( \text{Sp}(1) \) bundle \( p: S^{4n-1} \times Y \rightarrow X \), we obtain \( H^{4i+2}(S^{4n-1} \times Y) = 0 \) and an exact sequence:

\[
0 \rightarrow H^{4i-1}(S^{4n-1} \times Y) \rightarrow H^{4i-4}(X) \begin{array}{c}
\mu
\end{array} H^{4i}(X) \rightarrow H^{4i}(S^{4n-1} \times Y) \rightarrow 0
\]

for any \( i \), where \( \mu \) is the multiplication by \( e_1(p) \), the first symplectic Pontrjagin class of the quaternion line bundle associated with the \( \text{Sp}(1) \) bundle \( p \). We can represent \( p^*u = 1 \times u_1, \ p^*v = 1 \times v_1 \) for \( u_1, v_1 \in H^4(Y) \). Then we see that \( H^{4i+2}(Y) = 0 \) and \( H^*(Y) \) is generated by at most two elements \( u_1, v_1 \). We can represent \( e_1(p) = \alpha u + \beta v; \ \alpha, \beta \in \mathbb{Q} \). By definition, the \( \text{Sp}(1) \) bundle \( p \) is a pull-back of a bundle over \( P_{n-1}(H) \), and hence \( e_1(p)^s = 0 \). Since \( n \leq a + b \), we see that \( \alpha \beta = 0 \). Suppose \( e_1(p) = 0 \). Then \( p^* \) is injective, and hence \( 1 \times u_1^i v_1^j \neq 0 \) Thus we get a contradiction. Therefore we see that \( e_1(p) = \alpha u + \beta v; \ \alpha \neq 0 \) or \( e_1(p) = \beta v (\beta \neq 0) \), and hence \( u_1 = 0 \) or \( v_1 = 0 \), respectively. Looking at the Euler characteristic of \( X \) we see that \( a = n - 1 \) and \( Y \sim P_b(H) \).

When \( b < a - 1 \), we see that \( e_1(p) = \alpha u + (\alpha \neq 0) \) and \( H^*(Y) \approx \mathbb{Q}[v_1](v_1^{b+1}) \). When \( b = a - 1 \), interchanging \( u \) and \( v \) if necessary we can assume that \( e_1(p) = \alpha u (\alpha \neq 0) \) and \( H^*(Y) \approx \mathbb{Q}[v_1](v_1^b) \). It remains to consider the \( \text{Sp}(1) \) action on \( Y = F(\omega) \). We shall show that either \( F \sim P_b(C) \) or the \( \text{Sp}(1) \) action on \( Y \) is trivial, where \( F \) denotes the fixed point set of the restricted \( U(1) \) action on \( Y \).

Put \( w = \pi^*(v) \), where \( \pi \) is a natural projection of \((S^{4n-1} \times Y)/U(1)\) onto \( X = (S^{4n-1} \times Y)/\text{Sp}(1) \). Consider the fibration: \( Y \rightarrow (S^{4n-1} \times Y)/U(1) \rightarrow P_{2n-1}(C) \). We see that \( w_{b+1}^s = 0 \) and \( H^*((S^{4n-1} \times Y)/U(1)) \) is freely generated by \( 1, w, w_2, \ldots, w_b \) as an \( H^*(P_{2n-1}(C)) \) module. Consider next the following commutative diagram:

\[
\begin{array}{ccc}
H'(\mathbb{S}^\infty \times Y)/U(1) & \xrightarrow{j^*} & H'((S^{4n-1} \times Y)/U(1)) \\
\downarrow i^* & & \downarrow i^* \\
H'(P_{2n-1}(C) \times F) & \xrightarrow{j_F^*} & H'(P_{2n-1}(C) \times F)
\end{array}
\]

where \( i, i_{\infty}, j, j_F \) are natural inclusions. Since \( H^{4d}(Y) = 0 \), we see that \([4] i_{\infty}^* \) is injective for each \( r \) and surjective for each \( r > 4b \) and \( j^* \) is surjective. Let
\( w_\omega \) be an element of \( H'^{(S^m \times Y)/U(1)} \) such that \( j^* (w_\omega) = w \). Let \( x \) be the canonical generator of \( H'(P_{n}(C)) \cong H'(P_{2n-1}(C)) \). Then we can express

\[
i_*(w_\omega) = x^2 \times f_0 + x \times f_1 + 1 \times f_2
\]

where \( f_i \in H'^{(i)}(F) \) for \( i=0, 1, 2 \). It is known that \( [4] \) \( F_0 \sim P_d(C) \) or \( F_0 \sim P_d(H) \) \( (0 \leq d \leq b) \) for each connected component \( F_0 \) of \( F \). We shall show that \( F \) is connected.

Consider first the case \( b < n-1 \). We see that \( i_*(w_\omega) = x \times f_1 + 1 \times f_2 \), that is, \( f_0 = 0 \) by the relation \( (x^2 \times f_0 + x \times f_1 + 1 \times f_2)^{k+1} = 0 \) in \( H'^{(k+1)}(P_{2n-1}(C) \times F) \). Consequently, we can show that if \( F \) is not connected then \( i_*(w_\omega) = 0 \) and hence \( w_b = 0 \); this is a contradiction.

Consider next the case \( b = n-1 \). Since \( j^*(w_\omega) = w^\omega = 0 \), we see that \( w_\omega = \gamma \times x^n \) for some \( \gamma \in Q \), and hence \( i_*(w_\omega) = x^n \times \gamma \). Suppose \( \gamma = 0 \). Then \( f_0 = 0 \), and hence we can show that \( F \) is connected by the same argument as above. Suppose next \( \gamma \neq 0 \). We shall show that \( i_*(w_\omega) = x^2 \times f_0 \), that is \( f_1 = 0 \) and \( f_2 = 0 \). For any connected component \( F_0 \) of \( F \), we have an equation

\[
(x^2 \times f_0 | F_0 + x \times f_1 \times f_2 | F_0)^{\gamma} = x^n \times \gamma
\]

in \( H'^{(n)}(P_{\infty}(C) \times F_0) \). Then we see that \( (f_0 | F_0)^{\gamma} = \gamma \neq 0 \) and \( f_1 | F_0 = 0 \) for \( i=1, 2 \). Thus we obtain \( i_*(w_\omega) = x^2 \times f_0 \) and \( f_0 = \gamma \). Let \( F_1 \) (resp. \( F_2 \)) be the union of connected components \( F_\sigma \) of \( F \) on which \( f_0 | F_\sigma \) is positive (resp. negative). Since \( f_0 = \gamma \), we can regard \( f_0 | F_1 \) and \( f_0 | F_2 \) as constant rational numbers. Then each element of \( H'((P_{\infty}(C) \times F_\sigma) \) for \( r \geq 4n \) is expressed as a polynomial of \( x \times 1 \) with rational coefficients for \( s=1, 2 \) because \( H'^{(S^m \times Y)/U(1)} \) is generated by an element \( w_\omega \) as a graded \( H'^{(P_{n}(C))} \) algebra and \( i_\ast \) is surjective for \( r \geq 4n \). Then we see that \( F_\sigma \) (\( s=1, 2 \)) consists of just one point, and hence \( F \) consists of at most two points. This is a contradiction to the fact: \( \chi(F) = \chi(Y) = n \geq 7 \).

Anyhow we see that \( F \) is connected, and hence \( F \sim P_d(C) \) or \( F \sim P_d(H) \).

The \( Sp(1) \) action on \( Y \) is trivial for the latter case.

**5.4.** Finally, we consider the case \( X = X_{(\omega)} \cup X_{(\iota)} \) for \( a \geq n \). We shall show first that \( X_{(\omega)} \) is non-empty.

Suppose that \( X_{(\omega)} \) is empty. Then \( X = X_{(\iota)} = (S^{4n-1} \times F_{(\iota)})/Sp(1) \). By the Gysin sequence of the principal \( Sp(1) \) bundle \( S^{4n-1} \times F_{(\iota)} \to X \), we see that \( F_{(\iota)} \sim P_d(H) \). Looking at the Euler characteristic of the fibration: \( F_{(\iota)} \to X \to P_{n-1}(H) \) we obtain \( a = n - 1 \); this is a contradiction.

Consequently, we see that (cf. [8]) there is an equivariant decomposition \( X = \partial(D^{4n} \times Y)/Sp(1) \), where \( Y \) is a compact connected orientable manifold with a smooth \( Sp(1) \) action, and \( Y \) has a non-empty boundary \( \partial Y \) on which the \( Sp(1) \) action is free. We see that

\[
\dim Y = 4(a+b+1-n)
\]
and the fixed point set of the $Sp(n)$ action on $X$ is naturally diffeomorphic to the orbit manifold $\partial Y/Sp(1)$. Moreover, we see that there is a natural decomposition $X=X_1 \cup X_2$, where

$$X_1 = (S^{a-1} \times Y)/Sp(1) \text{ and } X_2 = (D^{a} \times \partial Y)/Sp(1).$$

Put $X_0 = X_1 \cap X_2 = (S^{a-1} \times \partial Y)/Sp(1)$.

Let $\pi: \partial(D^a \times Y) \to X$ be the projection of the principal $Sp(1)$ bundle. Denote by $\pi_i$ the projection of the restricted principal $Sp(1)$ bundle over $X_i$. Let $j_i: X_i \to X$ and $i_i: X_0 \to X_i$ be inclusions. Put $u_i = j_i^*(u)$ and $v_i = j_i^*(v)$. We can express

$$e(\pi) = \alpha u + \beta v; \quad \alpha, \beta \in \mathbb{Q},$$

where $e(\pi)$ is the Euler class of the principal $Sp(1)$ bundle $\pi$. Then we obtain

$$e(\pi_i) = j_i^*e(\pi) = \alpha u_i + \beta v_i.$$

Since $H^r(X, X_1) \cong H^r(X_2, X_0) \cong H^{r+a}(\partial Y/Sp(1))$ for each $r$, we obtain an isomorphism $j_i^*: H^r(X) \cong H^r(X_1)$ for each $r \leq 4n-2$. Because $Y$ is a compact connected manifold with non-empty boundary and $\dim Y \leq 4n-4$, we see that $\pi_i^*(u_i^{-1}) = 0$ and hence $u_i^{-1} = x' e(\pi_i)$ for some $x' \in H^{a-1}(X_i)$. Then $u_i^{-1} = xe(\pi)$ for some $x \in H^{a-1}(X)$ by the isomorphism $j_i^*$. In particular we see that $\alpha \neq 0$ in the expression: $e(\pi) = \alpha u + \beta v$. Looking at the isomorphism $j_i^*$ and the Gysin sequence of the principal $Sp(1)$ bundle $\pi$, we see that $\pi_i^*(v_i) = 0$ and the algebra $H^*(S^{a-1} \times Y)$ is generated by $\pi_i^* v_i$. Hence we obtain $Y \cong P(H)$. In addition, we see that $X_1 \cong P_{a-1}(H) \times P_1(H)$ by the fibration: $Y \to X_1 \to P_{a-1}(H)$.

Since $b \leq n-2$, by the same argument as in the second half of §5.3, we see that $F \cong P_1(C)$ or $F \cong P_1(H)$, where $F$ denotes the fixed point set of the restricted $U(1)$ action on $Y$.

Here we complete the proof of Theorem 2.1.

**REMARK.** The case $\alpha \beta \neq 0$ in the expression $e(\pi) = \alpha u + \beta v$ occurs only when $b \leq a+1-n$, because

$$(e(\pi_1) - \beta v_1)^{s+1} = (\alpha u_1)^{s+1} = 0$$

in $H^*(X_i) = \mathbb{Q}[e(\pi_1), v_1])/e(\pi_1)^s, v_1^{s+1})$.

**5.5.** In the following, we consider the cohomology of $\partial Y/Sp(1)$. Regarding $au$ and $\beta v$ as new $u$ and $v$ if necessary, we can assume that $e(\pi) = u$ if $\beta = 0$ and $e(\pi) = u + v$ if $\beta \neq 0$.

Since the algebra $H^*(X_1)$ is generated by $e(\pi_1)$ and $v_1$, we obtain an short exact sequence:
0 \to H^*(X, X_1) \xrightarrow{j^*_1} H^*(X, X_1) \xrightarrow{H^*(X)} 0 .

Moreover, we see that the kernel of $j^*_1$ is an ideal generated by $e(\pi)$, that is, \( \ker j^*_1 = H^*(X) e(\pi) \). Let $\tau \in H^*(X, X_1)$ be an element such that $k^*_1(\tau) = e(\pi)$. Then $H^*(X, X_1)$ is generated by $\tau$ as an $H^*(X)$ module, that is, $H^*(X, X_1) = H^*(X) \tau$.

Let $j^* : H^*(X, X_1) \approx H^*(X_2, X_0)$ be an excision isomorphism. Denote by $t \in H^*(X_2, X_0)$ the Thom class of the quaternion $n$-plane bundle over $\partial Y/Sp(1)$. Then $j^*(\tau) = \lambda t$ for non-zero $\lambda \in \mathbb{Q}$. Since $j^*(w\tau) = j^*_2(w) j^*(\tau) = \lambda j^*_2(w) t$ for each $w \in H^*(X)$, we see that $j^*_2 : H^*(X) \to H^*(X_2)$ is surjective. In addition, $j^*_2(w) = 0$ if and only if $e(\pi) w = 0$ for $w \in H^*(X)$. Then we can show that \( \{ j^*_2(u^p v^q); 0 \leq p \leq a-n, 0 \leq q \leq b \} \) are linearly independent in the graded module $H^*(X_2) \approx H^*(X)/\ker j^*_2$. On the other hand, we obtain

\[
\text{rank } H^*(X_2) = \text{rank } H^*(X) - \text{rank } H^*(X_1) = (a+1-n) (b+1) .
\]

Therefore the set \( \{ u^p v^q; 0 \leq p \leq a-n, 0 \leq q \leq b \} \) is an additive base of the graded module $H^*(X_2)$.

Suppose first $e(\pi) = u$, i.e. $\beta = 0$. Then $j^*_2(u^{a-n+1}) = 0$, and hence $H^*(X_2) \approx Q[u_a, v_2]/(u_2^{a-n+1}, v_2^{b+1})$. Therefore $\partial Y/Sp(1) \sim P_{a-n}(H) \times P_b(H)$.

Suppose next that $b \leq a+1-n$ and $e(\pi) = u+v$, i.e. $\beta \neq 0$. We see that

\[
e(\pi) \sum_{i=0}^{b} (-1)^i \binom{a+1}{i} (u+v)^{a+1-n-i} v^i = ((u+v)-v)^{b+1} = 0 ,
\]

hence we obtain

\[
H^*(\partial Y/Sp(1)) \approx H^*(X_2) \approx Q[x, y]/(y^{b+1}, \sum_{i=0}^{b} (-1)^i \binom{a+1}{i} x^{a+1-n-i} y^i) ,
\]

where $x = u_v + v_2$ and $y = v_2$.

Here we complete the proof of Theorem 2.2.

6. **Construction**

We regard $D^*u$ as the unit disk of the quaternion $n$-space $H^n$ with the right scalar multiplication and the left $Sp(n)$ action. Let $Y$ be a compact orientable smooth $Sp(1)$ manifold such that the $Sp(1)$ action is free on the non-empty boundary $\partial Y$. By the diagonal action, $Sp(1)$ acts freely on the boundary $\partial(D^*u \times Y)$. Here we consider the cohomology ring of the orbit manifold $X = \partial(D^*u \times Y)/Sp(1)$ on which $Sp(n)$ acts naturally.

Suppose that $\dim Y = 4d+4$, $Y \sim P_d(H)$, $1 \leq b \leq d \leq n-2$, and $F \sim P_d(C)$ or $F \sim P_{n-2}(H)$, where $F$ denotes the fixed point set of the restricted $U(1)$ action on $Y$. Moreover suppose that $\iota^* : H^*(Y) \approx H^*(\partial Y)$, where $\iota$ is an inclusion. Put $c = d - b$. In addition, we suppose that the graded algebra $H^*(\partial Y/Sp(1))$
is isomorphic to one of the following:

1. \( Q[x, y]/(x^{c+1}, y^{b+1}) \),
2. \( Q[x, y]/(y^{b+1}, \sum_{i=0}^{b} (-1)^{i} (n+c+1) x^{c+1-i} y^{i}) \); \( b \leq c+1 \),

where \( \deg x = \deg y = 4 \), and \( x \) is the Euler class of the principal \( Sp(1) \) bundle \( \partial Y \to \partial Y/Sp(1) \).

Put \( X = (S^{4n-1} \times Y)/Sp(1), X_0 = (D^{4n} \times \partial Y)/Sp(1) \) and \( X_0 = X_1 \cap X_2 = (S^{4n-1} \times \partial Y)/Sp(1) \). Then \( X = X_1 \cup X_2 \). Let \( \pi: \partial(D^{4n} \times Y) \to X \) be the projection of the principal \( Sp(1) \) bundle. Let us denote by \( \pi_i \), the projection of the restricted principal \( Sp(1) \) bundle over \( X_i \). Let \( j_i: X_i \to X \) and \( i_i: X_0 \to X_i \) be the inclusions. Let \( p: X_2 \to \partial Y/Sp(1) \) be the natural projection of \( 4n \)-disk bundle, and put \( p_0 = p|_{X_0} \).

Since \( d \leq n-2 \), we see that \( H^*(X_0) \) is freely generated by \( 1, \sigma \) as an \( H^*(\partial Y/Sp(1)) \) module for an element \( \sigma \in H^{4n-4}(X_0) \) and \( i^*: H^*(X_2) \to H^*(X_0) \) is injective. Put \( x_0 = p^*(x), y_0 = p^*(y) \), \( x_2 = p^*(x) \) and \( y_2 = p^*(y) \). Then \( x_0 = c(\pi_0) \) and \( x_2 = c(\pi_2) \), the Euler classes of the principal \( Sp(1) \) bundles.

By the fibration: \( Y \to X_1 \to P_{n-1}(H) \) and the assumption that \( F \sim P_{\delta}(C) \) or \( F \sim P_{\delta}(H) \) and \( Y \sim P_{\delta}(H) \), we see that by Lemma 1.1,

\[
H^*(X_1) = Q[x, y]/(x^{c+1}, y^{b+1}); \quad \deg x = \deg y = 4,
\]

where \( x_1 = c(\pi_1) \), the Euler class of the principal \( Sp(1) \) bundle.

Consider the Mayer–Vietoris sequence of a triad \( (X; X_1, X_2) \):

\[
i^* \to H^{r-1}(X_0) \xrightarrow{\Delta^*} H^r(X) \to H^r(X_1) \oplus H^r(X_2) \to H^r(X_0) \xrightarrow{i^*} \]

where \( j^*(a) = (j_1^*(a), j_2^*(a)) \) and \( i^*(b_1, b_2) = i_1^*(b_1) - i_2^*(b_2) \). We see that \( H^r(X) = 0 \) for each \( r \neq 0 \) (mod 4) and there is the following short exact sequence for each \( k \):

\[
(*) \quad 0 \to H^{4k-1}(X_0) \xrightarrow{\Delta^*} H^{4k}(X) \xrightarrow{j_1^*} H^{4k}(X_1) \to 0.
\]

Notice that \( \dim X = 4(n+d) \) and

\[
(**) \quad j_1^*: H^{4k}(X) \simeq H^{4k}(X_1) \quad \text{for} \quad k < n.
\]

Let \( u, v \) be elements of \( H^r(X) \) such that \( j_1^*(u) = x_1, j_1^*(v) = y_1 \). We see that \( u = c(\pi) \), the Euler class of the principal \( Sp(1) \) bundle. Moreover, we see that \( v^{b+1} = 0 \) by \( (**) \) and the assumption \( b \leq n-2 \). Since \( j_1^*(u^{b+1}) \neq 0 \), there is an element \( z \in H^{4k+4}(X) \) such that \( u^{b+1} v^{b+1} = 0 \), by the Poincaré duality. Then we see that \( u^{b+1} v^{b+1} = 0 \), by \( (**) \) and the fact \( v^{b+1} = 0 \). In particular, we obtain \( u^* = 0 \). Looking at the exact sequence \( (*) \), we can assume that \( u^* = \Delta^*(\sigma) \).

We can express \( i_1^*(y_1) = \lambda x_0 + \mu y_0 \); \( \lambda, \mu \in Q \). Since \( \pi_1^*(y_1) \neq 0 \), we see that
\[ \mu \neq 0 \] by the assumption \( \varepsilon^*: H^*(Y) \cong H^*(\partial Y) \). Then

\[ \Delta^*(\sigma x_y y^\mu) = \mu^{-\mu}(v - \lambda u)^\mu \]

because \( \Delta^*(\sigma j^*(w)) = \Delta^*(\sigma)w \) for each \( w \in H^*(X) \). Looking at the exact sequence \((*)\), we see that the graded algebra \( H^*(X) \) is generated by two elements \( u, v \) and rank \( H^*(X) = (n + c + 1)(b + 1) \).

In the expression \( i_t^*(y_i) = \lambda x_\mu + \mu y_0 \), if \( \lambda = 0 \) then we see that \( u^{n+c+1} = 0 \) in the case (1) and \( (u - \mu v)^{n+c+1} = 0 \) in the case (2), and hence \( X \sim P_{n+c}(H) \times P_b(H) \).

Since \( i_t^* : H^*(X) \rightarrow H^*(X_0) \) is injective, we see that \( j^*(v) = \lambda x_\mu + \mu y_2 \), and hence \( (\lambda x_\mu + \mu y_2)^{n+c+1} = 0 \). Then we obtain \( \lambda = 0 \) in the case (1), because \( H^*(X_0) \cong Q[x_\mu, y_2]/(x_\mu^{n+c+1}, y_2^{n+c+1}) \).

Next we consider the case (2). We obtain a relation

\[ (\gamma x_\mu + y_2)^{n+c+1} \in I = (y_2^{n+c+1}, \sum_{i=0}^{n+c} (-1)^i \binom{n+c+1}{i} x_\mu^{i+1} y_2^i), \]

where \( \gamma = \lambda \mu^{-1} \). We see that \( \gamma = 0 \) for the case \( b < c \) or \( b = c \geq 2 \). Suppose \( b = c + 1 \). Looking at the relation \( (\gamma x_\mu + y_2)^{n+c+1} \in I \), we obtain \( \gamma = 0 \) or

\[ (-c-2) (\gamma)^{n+c+1} \binom{n+c+1}{k-1} = 0 \]

for each \( k = 2, 3, \ldots, c + 1 \). Suppose \( \gamma \neq 0 \) and \( c \geq 2 \). Then we get a contradiction from \( (A_2) \) and \( (A_3) \). Hence we obtain \( \gamma = 0 \) for \( c \geq 2 \). Suppose \( \gamma \neq 0 \) and \( c = 1 \). We see that the quadratic equation \( (A_2) \) has a rational solution \( \gamma \) if and only if \( 3n(n+2) \) is a square number.

Summing up the above arguments, we obtain a partial converse of Theorem 2.1 (iii).

REMARK. For a positive integer \( n \), \( 3n(n+2) \) is a square number if and only if \( n+1 \) is one of the following:

\[ \sum_{i \geq 0} \binom{k}{2i} 2^{k-2i} 3^i; k = 1, 2, 3, \ldots \]

7. Concluding remark

By parallel arguments, we obtain the following result which is a generalization of a theorem [7].

**Theorem 7.1.** Let \( X \) be a closed orientable manifold on which \( SU(n) \) acts smoothly and non-trivially. Suppose \( X \sim P_{a}(C) \times P_{b}(C) \); \( a \geq b \geq 1 \), \( a+b \leq 2n-2 \) and \( n \geq 7 \). Then there are three cases:
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(0) $a = n - 1$ and $X \cong \mathbb{P}_n - 1(C) \times Y_0$, where $Y_0$ is a closed orientable manifold such that $Y_0 \sim \mathbb{P}_n(C)$, and $SU(n)$ acts naturally on $\mathbb{P}_n - 1(C)$ and trivially on $Y_0$.

(i) $a = b = n - 1$ and $X \cong \mathbb{P}_n - 1(C) \times \mathbb{P}_n - 1(C)$ with the diagonal $SU(n)$ action,

(ii) $a \geq n$ and $X \cong \partial(D^{2a} \times Y_1)/U(1)$, where $Y_1$ is a compact orientable $U(1)$ manifold such that $\dim Y_1 = 2(a + b + 1 - n)$ and $Y_1 \sim \mathbb{P}_n(C)$, $U(1)$ acts as right scalar multiplication on $D^{2a}$, the unit disk of $\mathbb{C}^a$, and $SU(n)$ acts naturally on $D^{2a}$ and trivially on $Y_1$. In addition, the $U(1)$ action on the boundary $\partial Y_1$ is free and the fixed point set of the $U(1)$ action on $Y_1$ is $\sim \mathbb{P}_n(C)$.

Theorem 7.2. In the case (ii) of Theorem 7.1, the cohomology ring $H^*(\partial Y_1/U(1))$ is isomorphic to one of the following:

1. $Q[x, y]/(x^{a+1-n}, y^{b+1})$,

2. $Q[x, y]/(y^{b+1}, \sum_{i=0}^{b} (-1)^i \binom{a+1}{i} x^{a+1-n-i}y^i); \ b \leq a + 1 - n$,

where $\deg x = \deg y = 2$, and $x$ is the Euler class of the principal $U(1)$ bundle $\partial Y_1 \rightarrow \partial Y_1/U(1)$.

References


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