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INVOLUTIONS ON TORUS BUNDLES OVER S^1

Dedicated to the memory of Professor Takehiko Miyata

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Introduction

Involutions on torus bundles have been studied by several authors [10, 11, 13, 15, 17, 20, 22, 25]. In particular, involutions on $S^1\times S^1\times S^1$ and orientation-reversing involutions on orientable torus bundles have been classified by Kwun-Tollefson [13] and Kim-Sanderson [10] respectively.

The purpose of this paper is to classify all involutions on torus bundles. In fact, we will give a finite procedure for finding all involutions on a torus bundle M_A form its monodromy matrix A (see Section 2). It should be noted that involutions on a given non-orientable torus bundle are not necessarily distinguished by their quotients (Example 4.7). Here the *quotient* of an involution h on a space M means the pair $(M/h, \mathrm{Fix}(h)/h)$. As a consequence of our main theorems, we obtain the following result, which sharply improves the estimates given by Kojima [11] on the number of non-equivalent symmetries on torus bundles.

Theorem. (1) *If M_A is an orientable torus bundle, then* $1\leqq|\mathrm{Inv}(M_A)|\leqq 21$.
(2) *If M_A is a non-orientable torus bundle with $tr(A)\neq 0$, then* $1\leqq|\mathrm{Inv}(M_A)|\leqq 7$.

Here $\mathrm{Inv}(M_A)$ denotes the set of all equivalence classes of involutions on M_A, and $|S|$ denotes the cardinality of S. The following examples show that the above estimates are the best possible.

EXAMPLE. (1) If $A=\begin{bmatrix}40 & 9\\ 31 & 7\end{bmatrix}$, then $|\mathrm{Inv}(M_A)|=1$.

(2) If $A=\begin{bmatrix}89 & 20\\ 40 & 9\end{bmatrix}$, then $|\mathrm{Inv}(M_A)|=21$.

(3) If $A=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$, then $|\mathrm{Inv}(M_A)|=1$.

(4) If $A=\begin{bmatrix}21 & 4\\ 16 & 3\end{bmatrix}$, then $|\mathrm{Inv}(M_A)|=7$.

As an application, a simple sufficient condition for a torus bundle to have

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exactly one involution will be presented (Theorem VI), which generalizes a result of Tollefson [22]. Moreover, we will give a necessary and sufficient condition for a torus bundle to be a regular covering of S^3 (Theorem VII).

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1. Preliminaries

T^2 denotes the torus obtained as the quotient space $\boldsymbol{R}^2/(\theta_1, \theta_2)\sim(\theta_1+2p\pi, \theta_2+2q\pi)$ $(p, q\in\boldsymbol{Z})$, on which two specific loops $l=\boldsymbol{R}\times 0/\sim$ and $m=0\times\boldsymbol{R}/\sim$ are assigned. For each homeomorphism ϕ on T^2, ϕ_* denotes the matrix representing the automorphism on $H_1(T^2)$ induced by ϕ with respect to the base $\langle l, m\rangle$. The correspondence $\phi\mapsto\phi_*$ gives an isomorphism from the homeotopy group of T^2 to $GL(2, \boldsymbol{Z})$. For each matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $GL(2, \boldsymbol{Z})$, ϕ_A denotes the homeomorphism on T^2 defined by the equation $\phi_A(\theta_1, \theta_2)=(a\theta_1+b\theta_2, c\theta_1+d\theta_2)$. Then we have $(\phi_A)_*=A$.

There are just five non-equivalent involutions $\{r_i \mid 1\leqq i\leqq 5\}$ on T^2, which are listed as follows.

	$r_i(\theta_1, \theta_2)$	$(r_i)_*$	T^2/r_i	$\mathrm{Fix}(r_i)$
r_1	$(\theta_1, \theta_1+\pi)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	T^2	Empty
r_2	$(-\theta_1, -\theta_2)$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	S^2	Four points
r_3	$(\theta_1, -\theta_2)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Annulus	Two circles
r_4	$(\theta_1+\theta_2, -\theta_2)$	$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$	Möbius band	A circle
r_5	$(\theta_1+\pi, -\theta_2)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Klein bottle	Empty

For a matrix A in $GL(2, \boldsymbol{Z})$, M_A denotes the torus bundle whose monodromy matrix is A, and $\tilde{M}_A$ denotes the infinite cyclic cover of M_A associated to the subgroup of $\pi_1(M_A)$ generated by a fiber. For a homeomorphism ϕ on T^2, $\tilde{\phi}$ denotes the homeomorphism on $T^2\times\boldsymbol{R}$ defined by $\tilde{\phi}(x, t)=(\phi(x), t+2\pi)$. If $\phi_*=A$, then a generator of the covering transformation group of the infinite cyclic cover $\tilde{M}_A$ is identified with $\tilde{\phi}$. Then $H_1(\tilde{M}_A)$ admits a $\boldsymbol{Z}\langle t\rangle$-module structure by identifying the action of t with that of $\tilde{\phi}_*$, where $\boldsymbol{Z}\langle t\rangle$ is the integral group ring of the infinite cyclic group generated by t. As an abelian group,

$H_1(\tilde{M}_A)$ is isomorphic to $\boldsymbol{Z}^2$, and the action of t is identified with the linear action of the matrix A. Such a $\boldsymbol{Z}\langle t\rangle$-module is denoted by the symbol H_A. $\pi_1(M_A)$ is isomorphic to the semi-direct product $H_A \rtimes \langle t\rangle$ of H_A and $\langle t\rangle$ with the above operation (cf. p. 105 of [14]). The following lemma is well-known (cf. [20]).

Lemma 1.1. *Let A and B be matrices in $GL(2, \boldsymbol{Z})$. Then the following conditions are equivalent.*

(1) *M_A is homeomorphic to M_B.*

(2) *$\pi_1(M_A)$ is isomorphic to $\pi_1(M_B)$.*

(3) *The $\boldsymbol{Z}\langle t\rangle$-module H_A is isomorphic or anti-isomorphic to the $\boldsymbol{Z}\langle t\rangle$-module H_B.*

(4) *A is conjugate to B or B^{-1}.*

We now give some properties of $GL(2, \boldsymbol{Z})$. Proofs can be found in [19] (cf. [9]).

DEFINITION 1.2. (1) For a matrix A in $GL(2, \boldsymbol{Z})$ let

$$C(A) = \{B \in GL(2, \boldsymbol{Z}) \mid BAB^{-1} = A\},$$
$$C^{\varepsilon}(A) = \{B \in C(A) \mid det(B) = \varepsilon 1\} \ (\varepsilon = + \text{ or } -),$$
$$R(A) = \{B \in GL(2, \boldsymbol{Z}) \mid BAB^{-1} = A^{-1}\},$$
$$R^{\varepsilon}(A) = \{B \in R(A) \mid det(B) = \varepsilon 1\} \ (\varepsilon = + \text{ or } -),$$
$$N(A) = C(A) \cup R(A).$$

(2) A is called *exceptional*, if one of the following conditions is satisfied.

(i) $det(A) = 1$ and $|tr(A)| \leqq 2$.

(ii) $det(A) = -1$ and $tr(A) = 0$.

(3) A is called *Anosov*, if A is not exceptional.

REMARK 1.3. $N(A)$ can be identified with the group of all isomorphisms and anti-isomorphisms on H_A.

Lemma 1.4. *If A is exceptional then it is conjugate to one and only one of the following matrices.*

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \pm\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \ \pm\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} (n \geqq 0), \ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Moreover we have the followings.

$$C\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \left\{\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \pm\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\} \cong \boldsymbol{Z}_4$$

$$R\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} C\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

$$C\left(\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right) = \left\{\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \pm\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \ \pm\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\right\} \cong \boldsymbol{Z}_6$$

$$R\left(\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}C\left(\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}\right)$$
$$C\left(\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}\right)=\left\{\pm\begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}(k\in\boldsymbol{Z})\right\}\cong\boldsymbol{Z}+\boldsymbol{Z}_2 \quad (n\neq 0)$$
$$R\left(\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}C\left(\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}\right) \qquad (n\neq 0)$$
$$C\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)=R\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)=\left\{\pm\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \pm\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\}\cong\boldsymbol{Z}_2+\boldsymbol{Z}_2$$
$$C\left(\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\right)=R\left(\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\right)=\left\{\pm\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \pm\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\right\}\cong\boldsymbol{Z}_2+\boldsymbol{Z}_2$$

For an Anosov matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, consider the Möbius transformation on $\boldsymbol{R}\cup\{\infty\}$ given by $x\mapsto(ax+b)/(cx+d)$. Let $\omega(A)$ be the fixed point of this map given by

$$\omega(A)=\{(a-d)+\sqrt{(tr(A))^2-4det(A)}\}/2c\,.$$

Since $\omega(A)$ is a quadratic irrationality, its infinite continued fraction

$$\begin{aligned}\omega(A)&=[c_0, c_1, c_2, \cdots]\\ &=c_0+\cfrac{1}{c_1+\cfrac{1}{c_2+\cfrac{1}{\cdots}}},\end{aligned}$$

where $c_0\in\boldsymbol{Z}$ and $c_i\in\boldsymbol{N}\setminus\{0\}$ $(i\geqq 1)$, is ultimately periodic. Let $(a_1, \cdots, a_s)$ be a primitive period, and put

$$A_0=\begin{bmatrix}a_1 & 1\\ 1 & 0\end{bmatrix}\cdots\begin{bmatrix}a_s & 1\\ 1 & 0\end{bmatrix}.$$

We call A_0 a *primitive root* of A. Two sequences $(b_1, \cdots, b_u)$ and $(c_1, \cdots, c_v)$ are said to be *congruent*, if they are related by a cyclic permutation.

Lemma 1.5. (1) *A is conjugate to εA_0^n ($\varepsilon=\pm 1$, $n\in\boldsymbol{N}$), where ε and n are characterized by the identity $tr(A)=tr(\varepsilon A_0^n)$.*

(2) *Two Anosov matrices A and B are conjugate iff the following conditions are satisfied.*

(i) *$tr(A)=tr(B)$.*

(ii) *The primitive periods of $\omega(A)$ and $\omega(B)$ are congruent.*

We call the above expression εA_0^n a *standard form* of A.
Since

$$\left\{\Pi_{i=1}^{s}\begin{bmatrix}a_i & 1\\ 1 & 0\end{bmatrix}\right\}^{-1}=(-1)^s\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\left\{\Pi_{i=s}^{1}\begin{bmatrix}a_i & 1\\ 1 & 0\end{bmatrix}\right\}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^{-1},$$

we have the following.

Lemma 1.6. *If $(a_1, \cdots, a_s)$ is a primitive period of $\omega(A)$, then $(a_s, \cdots, a_1)$ is a primitive period of $\omega(A^{-1})$.*

A sequence $(a_1. \cdots, a_s)$ is said to be *invertible* if it is congruent to $(a_s, \cdots, a_1)$.

Lemma 1.7. *Let $A = \varepsilon A_0^n$ be an Anosov matrix where $A_0 = \prod_{i=1}^{s} \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix}$ is a primitive root of A. Then we have the following.*

(1) $C(A) = \{\pm A_0^i \, (i \in \boldsymbol{Z})\} \cong \boldsymbol{Z} + \boldsymbol{Z}_2$.

(2) *Suppose that $det(A) = 1$, and the primitive period $(a_1, \cdots, a_s)$ is invertible, that is, there is an integer u $(1 \leqq u \leqq s-1)$ such that $(a_s, a_{s-1}, \cdots, a_1) = (a_{u+1}, \cdots, a_s, a_1, \cdots, a_u)$. Put $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \prod_{i=u+1}^{s} \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix}$. Then $R(A) = C(A)P$.*

(3) *If the condition of (2) is not satisfied, then $R(A)$ is an empty set.*

(4) *A matrix Q in $R(A)$ has period 2 or 4, according as $det(Q) = -1$ or $+1$.*

2. Statement of results

Our fundamental tool is the following lemma, which is due to Tollefson [23].

Lemma 2.1. *Let M_A be a torus bundle whose monodomy matrix A is conjugate to neither $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ nor $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Then any involution on M_A is equivalent to a fiber-preserving involution.*

Proof. In case the first Betti number $\beta_1(M_A)$ is 1, this lemma is a special case of Theorem 2 of [23]. The proof for the case $\beta_1(M_A) > 1$ is given in Section 7 (see Lemma 7.1).

For the two exceptional torus bundles M_A with $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, the above lemma does not hold (see Section 7), and therefore different approaches are necessary for them. Thus, in this paper, they are excluded from our consideration, and all involutions on torus bundles are assumed to be fiber-preserving. So, an involution h on $M_A = T^2 \times \boldsymbol{R} / \tilde{\phi}$ $(\phi_* = A)$ is induced from a homeomorphism $\tilde{h}$ on $T^2 \times \boldsymbol{R}$, which is given by one of the following formulas.

(I) $\tilde{h}(x, t) = (\gamma_t(x), t)$, where $\phi \circ \gamma_t = \gamma_t \circ \phi$ and $(\gamma_t)^2 = id$.

(II) $\tilde{h}(x, t) = (\gamma_t(x), t + \pi)$, where $\phi \circ \gamma_t = \gamma_{t+2\pi} \circ \phi$ and $\gamma_{t+\pi} \circ \gamma_t = \phi$.

(III) $\tilde{h}(x, t) = (\gamma_t(x), -t)$, where $\phi \circ \gamma_t = \gamma_{t-2\pi} \circ \phi^{-1}$ and $\gamma_{-t} \circ \gamma_t = id$.

Here $\{\gamma_t \mid t \in \boldsymbol{R}\}$ is a continuous family of homeomorphisms on T^2. We say h is of type X, if $\tilde{h}$ is given by the formula X for each $X = 1$, II, or III. $\mathrm{Inv}_X(M_A)$ denotes the set of all equivalence classes of involutions on M_A which are of type X.

Let P be the matrix $(\gamma_t)_*$ in $GL(2, \boldsymbol{Z})$.

Case I. h is of type I. Then $P\in C(A)$ and $P^2=I$. So, by Lemmas 1.4 and 1.7, we have $P=\pm I$, in case $A\neq\pm I$. If $P=I$, each involution γ_t is equivalent to the involution r_1, and therefore, h is a free involution and M_A/h is a torus bundle. If $P=-I$, each γ_t is equivalent to r_2, and M_A/h is homeomorphic to $S^2\times S^1$ or $S^2\tilde{\times}S^1$ (the non-oiientable S^2 bundle over S^1) according to whether M_A is orientable or not. We say that h is of type $I-0$ or $I-1$, according to whether $P=I$ or $-I$. $\mathrm{Inv}_{\mathrm{I}}^j(M_A)$ denotes the set of all equivalence classes of involutions on M_A of type $I-j$ for each $j=0, 1$. If M_A is orientable and $A\neq\pm I$, then all involutions on M_A of type I are orientation-preserving.

Case II. h is of type II. Then $P\in C(A)$ and $P^2=$A. So, $\det(A)=(\det(P))^2=1$, and M_A is orientable. M_A/h is homeomorphic to the torus bundle M_P. h is orientatıon-preserving, iff $\det(P)=1$.

Case III. h is of type III. Then $P\in R(A)$ and $P^2=I$. By Lemma 1.7 (3), M_A is orientable. Assume that $A\neq\pm I$, then we have $P\in R^-(A)$, by Lemma 1.7 (4). Hence each γ_t is orientation-reversnig, and therefore h is orientation-preserving. M_A/h is obtained from $T^2\times[0,\pi]$ by identifying $(x, 0)$ with $(\gamma_0(x), 0)$ and (x,π) with $(\phi\circ\gamma_\pi(x), \pi)$.

Put $V_1=T^2\times[0, \pi/2]/(x, 0)\sim(\gamma_0(x), 0)$ and

$$V_2=T^2\times[\pi/2, \pi]/(x, \pi)\sim(\phi\circ\gamma_\pi(x), \pi)\,.$$

If γ_0 (resp. $\phi\circ\gamma_\pi$) is non-free, then it is equivalent to the involution r_3 or r_4, and V_1 (resp. V_2) is a solid torus. If γ_0 (resp. $\phi\circ\gamma_\pi$) is free, then it is equivalent to r_5, and V_1 (resp. V_2) is a twisted I-bundle over a Klein bottle. Let j be the number of V_i's $(i=1,2)$ which are homeomorphic to the solid torus. Then we say h is of type III-j. $\mathrm{Inv}_{\mathrm{III}}^j(M_A)$ denotes the set of all equivalence classes of involutions on M_A of type III-j $(j=0,1,2)$. An involution $h\in\mathrm{Inv}_{\mathrm{III}}^j(M_A)$ is free, iff $j=0$. M_A/h is homeomorphic to a lens space, a prism manifold, or a "sappihre space", according as h is of type III-2, III-1, or III-0. Here a lens space (resp. a prism manifold) is a 3-manifold which is a union of two solid tori (resp. a solid torus and a twisted I-bundle over a Klein bottle), and following [16], we call a 3-manifold M a sapphire space, if M is a union of two twisted I-bundles over a Klein bottle.

We will prove the following theorems.

Theorem 0. *If $\beta_1(M_A)=1$, then the sets $Inv_{\mathrm{I}}^j(M_A)$ $(j=0,1)$, $Inv_{\mathrm{II}}(M_A)$, and $Inv_{\mathrm{III}}^k(M_A)$ $(k=0,1,2)$ are mutually disjoint. Moreover, if $A\neq-I$, the disjoint union of them is equal to $Inv(M_A)$. If $A=-I$, it is equal to $Inv(M_A)^+$, the set of all equivalence classes of orientation-preserving involutions on M_A.*

Let ρ be the natural map $GL(2, \boldsymbol{Z})\to GL(2, \boldsymbol{Z}_2)$. Note that $GL(2, \boldsymbol{Z}_2)$ is isomorphic to the dihedral group of order 6.

Theorem I. *Suppose that $\beta_1(M_A)=1$.*

(1) *If* $||\rho(A)||=3$, *then* $|Inv_{\mathrm{I}}^0(M_A)|=0$ *and* $|Inv_{\mathrm{I}}^1(M_A)|=1$.
(2) *If* $||\rho(A)||=2$, *then* $|Inv_{\mathrm{I}}^0(M_A)|=1$ *and* $|Inv_{\mathrm{I}}^1(M_A)|=2$.
(3) *Suppose that* $||\rho(A)||=1$.
 (i) *If* $|\rho(N(A))|=3$ *or* 6, *then* $|Inv_{\mathrm{I}}^0(M_A)|=1$ *and* $|Inv_{\mathrm{I}}^1(M_A)|=2$.
 (ii) *If* $|\rho(N(A))|=2$, *then* $|Inv_{\mathrm{I}}^0(M_A)|=2$ *and* $|Inv_{\mathrm{I}}^1(M_A)|=3$.
 (iii) *If* $|\rho(N(A))|=1$, *then* $|Inv_{\mathrm{I}}^0(M_A)|=3$ *and* $|Inv_{\mathrm{I}}^1(M_A)|=4$.

In the above $||\rho(A)||$ denotes the order of $\rho(A)\in GL(2, \boldsymbol{Z}_2)$. To state Theorems II and III, we need three functions σ_i $(i=1, 2, 3)$ from $\{A\in SL(2, \boldsymbol{Z})\mid |tr(A)|\geqq 3\}$ to $\{+, -\}$, defined as follows. For an Anosov matrix A in $SL(2, \boldsymbol{Z})$, let εA_0^n be a standard form of A (cf. Lemma 1.5). Then

$$\sigma_1(A)=\operatorname{sign}(det(A_0)),$$
$$\sigma_2(A)=\operatorname{sign}(\varepsilon),$$
$$\sigma_3(A)=\begin{cases}\operatorname{sign}(-1)^n & \text{if } \sigma_1(A)=+,\\ \operatorname{sign}(-1)^{(n/2)} & \text{if } \sigma_1(A)=-.\end{cases}$$

(Note that, if $\sigma_1(A)=-$, then n is even.)

Put $\sigma(A)=(\sigma_1(A), \sigma_2(A), \sigma_3(A))$.

Theorem II. *Let A be an Anosov matrix in* $SL(2, \boldsymbol{Z})$.
(1) *If* $\sigma(A)=(*, -, *)$ *or* $(+, *, -)$, *then* $|Inv_{\mathrm{II}}(M_A)|=0$.
(2) *If* $\sigma(A)=(*, +, +)$, *then* $|Inv_{\mathrm{II}}(M_A)|=2$.
(3) *If* $\sigma(A)=(-, +, -)$, *then* $|Inv_{\mathrm{II}}(M_A)|=1$ *or* 2, *according to whether the primitive period of* $\omega(A)$ *is invertible or not.*

Theorem III. *Let A be an Anosov matrix in* $SL(2, \boldsymbol{Z})$.
(1) *If $R^-(A)$ is empty, then* $|Inv_{\mathrm{III}}(M_A)|=0$.
(2) *If $R^-(A)$ is not empty, then*

$$|Inv_{\mathrm{III}}^2(M_A)|=\begin{cases}4 \text{ if } \sigma(A)=(+, +, +),\\ 1 \text{ if } \sigma(A)=(-, +, -) \text{ or } (-, -, +),\\ 2 \text{ otherwise.}\end{cases}$$

(3) *If the set $\rho(R^-(A))$ does not contain I* $(\in GL(2, \boldsymbol{Z}_2))$, *then* $|Inv_{\mathrm{III}}^0(M_A)|=|Inv_{\mathrm{III}}^1(M_A)|=0$.
(4) *Assume that $\rho(R^-(A))$ contains I. (Note that, in this case, we have* $||\rho(A_0)||=1$ *or* 2.)
 (i) $||\rho(A_0)||=2$. *Then*

$$|Inv_{\mathrm{III}}^1(M_A)|=2,$$
$$|Inv_{\mathrm{III}}^0(M_A)|=\begin{cases}2 \text{ if } \sigma(A)=(+, +, +), (-, +, +), \text{ or } (-, -, -),\\ 1 \text{ if } \sigma(A)=(+, -, +), (-, +, -), \text{ or } (-, -, +),\\ 0 \text{ if } \sigma(A)=(+, +, -), \text{ or } (+, -, -).\end{cases}$$

 (ii) $||\rho(A_0)||=1$. *Then*

$$|Inv_{\mathrm{III}}^1(M_A)|=\begin{cases}4 \text{ if } \sigma(A)=(+, *, *),\\ 2 \text{ if } \sigma(A)=(-, *, *),\end{cases}$$

$$|Inv^0_{\mathrm{III}}(M_A)| = |Inv^2_{\mathrm{III}}(M_A)|\,.$$

The following table summarizes our results for orientable torus bundles with Anosov monodromies. (For a non-orientable torus bundle M_A with an Anosov monodromy, we have $\mathrm{Inv}(M_A)=\mathrm{Inv}_{\mathrm{I}}(M_A)$.)

	Type I	Type II	Type III
Free	T^2-bundle $(0\leqq 3)$	T^2-bundle $(0\leqq 2)$	Sapphire $(0\leqq 4)$
Non-free	$S^1\times S^2$ $(1\leqq 4)$		Prism $(0\leqq 4)$ Lens $(0\leqq 4)$

Here, homeo. types and the best possible estimates of the numbers of prescribed involutions are presented. Now the Theorem in the introduction is a direct consequence of the preceeding theorems and the forthcoming Theorems IV and V. The results given in the Example can be seen from the following identities.

$$\begin{bmatrix}40 & 9\\ 31 & 7\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}3 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}4 & 1\\ 1 & 0\end{bmatrix},\quad \begin{bmatrix}89 & 20\\ 40 & 9\end{bmatrix}=\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}4 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}4 & 1\\ 1 & 0\end{bmatrix},$$
$$\begin{bmatrix}21 & 4\\ 16 & 3\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}3 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}5 & 1\\ 1 & 0\end{bmatrix}.$$

For a torus bundle M_A with an exceptional monodromy matrix A and $\beta_1(M_A)=1$, we have the following.

Theorem IV. *$Inv(M_A)$ with $A\in SL(2,\boldsymbol{Z})$ and $-2\leqq tr(A)\leqq 1$ is tabulated as follows.*

	I–0	I–1	II	III–0	III–1	III–2	
$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$	1	2	0	0	1	1	
$\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}$	0	1	2	0	0	1	
$\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}$	0	1	0	0	0	1	
$\begin{bmatrix}-1 & n\\ 0 & -1\end{bmatrix}$	1	2	0	0	2	2	n: *odd*
$\begin{bmatrix}-1 & n\\ 0 & -1\end{bmatrix}$	2	3	0	1	2	2	n: *even non-zero*
$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$	1	2	0	1	1	2	

For $A=-I$, this table presents only orientation-preserving involutions. M_{-I} has just six orientatoin-reversing involutions (see [10]).

For torus bundles with $\beta_1>1$, we have the following.

Theorem V. *Let* $A=\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$.

(1) *If n is odd, then M_A has just five involutions $\{h_i \mid 1\leqq i\leqq 5\}$, and they are characterized as follows.*

(a) $M_A/h_1\cong M_A/h_2\cong S^2\times S^1$, *$Fix(h_1)\cong$ three simple loops, $Fix(h_2)\cong$ a simple loop.*

(b) $M_A/h_3\cong L(n, 2)$, *$Fix(h_3)\cong$ three simple loops.*

(c) $M_A/h_4\cong$ *a prism manifold, $Fix(h_4)\cong$ a simple loop.*

(d) $M_A/h_5\cong M_B$ *with* $B=\begin{bmatrix}1 & 2n\\ 0 & 1\end{bmatrix}$, *$h_5$ is free.*

(2) *If n is even and non-zero, then M_A has just nine involutions $\{k_i \mid 1\leqq i\leqq 9\}$, and they are characterized as follows.*

(a) $M_A/k_i\cong S^2\times S^1\ (1\leqq i\leqq 3)$, *$Fix(k_1)\cong$ four simple loops, $Fix(k_2)\cong Fix(k_3)$ $\cong$ two simple loops.*

(b) $M_A/k_4\cong L(n/2, 1)$, *$Fix(k_4)\cong$ four simple loops.*

(c) $M_A/k_5\cong L(2n, 2n+1)$, *$Fix(k_5)\cong$ two simple loops.*

(d) $M_A/k_i\cong M_{B_i} (6\leqq i\leqq 8)$ *with* $B_6=\begin{bmatrix}1 & 2n\\ 0 & 1\end{bmatrix}$, $B_7=\begin{bmatrix}1 & n/2\\ 0 & 1\end{bmatrix}$, $B_8=\begin{bmatrix}-1 & n/2\\ 0 & -1\end{bmatrix}$, *$k_i$ $(6\leqq i\leqq 8)$ are free.*

(e) $M_A/k_9\cong$ *a sapphire space, k_9* is free.

REMARK 2.2. Free involutions on M_A with $A=\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$ were also classified by Goto [7] through a group theoretical method.

We end this section by proving Theorem 0. Let M_A be a torus bundle with $\beta_1(M_A)=1$, and let h be an involution on M_A and p: $M_A\rightarrow M_A/h$ be the projection. For a topological space Y, $B_1(Y)$ denotes the torsion-free abelian group $H_1(Y)/TorH_1(Y)$. Then, by the preceding observations, we have the followings.

(1) If h is of type I, then $p_\sharp$: $B_1(M_A)\rightarrow B_1(M_A/h)$ is an isomorphism.

(2) If h is of type II, then $p_\sharp$ is injective, and $p_\sharp(B_1(M_A))$ is a subgroup of $B_1(M_A/h)$ of index 2.

(3) If h is of type III, then $B_1(M_A/h)=0$.

Hence, $Inv_X(M_A)$ ($X=$I, II, III) are mutually disjoint. Clearly, $\mathrm{Inv}_{\mathrm{I}}^0(M_A)$ and $\mathrm{Inv}_{\mathrm{I}}^1(M_A)$ are disjoint. So, we have only to prove that $\mathrm{Inv}_{\mathrm{III}}^j(M_A)$ $(j=0, 1, 2)$ are mutually disjoint. This is done in Section 6 (see Lemma 6.2).

3. Involutions of Type I-0

Let B be a matrix in $GL(2,\boldsymbol{Z})$, and H be a $\boldsymbol{Z}\langle t\rangle$-submodule of H_B of index

2. Then there is a matrix A in $GL(2, \boldsymbol{Z})$, such that $H=H_A$, and the group $H_A \rtimes \langle t \rangle$ can be considered as a subgroup of $\pi_1(M_B)=H_B \rtimes \langle t \rangle$ of index 2. The covering space of M_B corresponding to the subgroup is homeomorphic to M_A, and the covering transformation is an involution on M_A of type I–0. We call it an involution determined by the $\boldsymbol{Z}\langle t \rangle$-module pair (H_B, H_A). Every involution on M_A of type I–0 is obtained in this way. We say that two $Z\langle t \rangle$-module pairs (H_{B_i}, H_{A_i}) $(i=1, 2)$ are *equivalent*, if there is an isomorphism or an anti-isomorphism from H_{B_1} to H_{B_2} carrying H_{A_1} to H_{A_2}. Then we have the following.

Lemma 3.1. *Let M_A be a torus bundle with $\beta_1(M_A)=1$, and let h_1 and h_2 be involutions on M_A of type I–0, which are determined by the $\boldsymbol{Z}\langle t \rangle$—module pairs (H_{B_1}, H_A) and (H_{B_2}, H_A) respectively. Then h_1 and h_2 are equivalent, iff (H_{B_1}, H_A) and (H_{B_2}, H_A) are equivalent.*

Proof. Let p_i: $M_A \to M_A/h_i \cong M_{B_i}$ be the projection for each $i=1, 2$. Suppose that h_1 and h_2 are equivalent, then there are homeomorphisms f: $M_A \to M_A$ and g: $M_{B_1} \to M_{B_2}$ such that $p_2 \circ f = g \circ p_1$. Since $\beta_1(M_A)=1$, we have $\beta_1(M_{B_i}) =1$ $(i=1, 2)$. So M_A, M_{B_1}, and M_{B_2} have the unique infinite cyclic covers $\tilde{M}_A$, $\tilde{M}_{B_1}$, and $\tilde{M}_{B_2}$, and the maps f, g, and p_i $(i=1, 2)$ lift to maps $\tilde{f}$, $\tilde{g}$, and $\tilde{p}_i$ $(i=1, 2)$ between the infinite cyclic coverings. Note that $\tilde{p}_2 \circ \tilde{f}$ and $\tilde{g} \circ \tilde{p}_1$ coincide up to multiplication of a covering transformation. $\tilde{p}_{i*}$: $H_1(\tilde{M}_A) \to H_1(\tilde{M}_{B_i})$ is a $\boldsymbol{Z}\langle t \rangle$-homomorphism, and we have $(H_{B_i}, H_A) \cong (H_1(\tilde{M}_{B_i}), \tilde{p}_{i*}H_1(\tilde{M}_A))$ for each $i=1, 2$. Now the "only if" part follows from the fact that $\tilde{g}_*$: $H_1(\tilde{M}_{B_1}) \to H_1(\tilde{M}_{B_2})$ is an isomorphism or an anti-isomorphism (cf. Lemma 1.1). The "if" part follows from the fact that any isomorphism or anti-isomorphism from H_{B_1} to H_{B_2} is realized by a homeomorphism from M_{B_1} to M_{B_2} (cf. Lemma 1.1).

Lemma 3.2. *If H_A is a $\boldsymbol{Z}\langle t \rangle$-submodule of H_B of index 2, then $2H_B$ is a $\boldsymbol{Z}\langle t \rangle$-submodule of H_A of index 2. Moreover, the equivalence class of (H_B, H_A) is uniquely determined by that of $(H_A, 2H_B)$.*

Proof. Clear.

Each $\boldsymbol{Z}\langle t \rangle$-submodule H of H_A of index 2 is a kernel of a $\boldsymbol{Z}\langle t \rangle$-epimorphism from H_A to $\boldsymbol{Z}_2$ ($\boldsymbol{Z}_2$ can be considered as a $\boldsymbol{Z}\langle t \rangle$-module in a unique way), and such epimorphism is determined by an epimorphism from $H_A \otimes \boldsymbol{Z}_2$ to $\boldsymbol{Z}_2$. Here the tensor product is taken over $\boldsymbol{Z}\langle t \rangle$, and therefore $H_A \otimes \boldsymbol{Z}_2 \cong \mathrm{Coker}(\rho(A)-I)$. We say that two epimorphisms ξ_i: $H_A \otimes \boldsymbol{Z}_2 \to \boldsymbol{Z}_2$ are *A-equivalent*, if there is an element f of $\rho(N(A))$ such that $\xi_2 = \xi_1 \circ f$. (By Remark 1.3, f can be considered as an isomorphism on $H_A \otimes \boldsymbol{Z}_2$.) It is clear that the equivalence class of the pair (H_A, H) is uniquely determined by the A-equivalence class of the corresponding epimorphism from $H_A \otimes \boldsymbol{Z}_2$ to $\boldsymbol{Z}_2$. Thus we obtain the following.

Proposition 3.3. *Let M_A be a torus bundle with $\beta_1(M_A)=1$. Then there is a one to one correspondence between $Inv_{\mathrm{I}}^0(M_A)$ and the set of all A-equivalence classes of epimorphisms from $H_A \otimes \boldsymbol{Z}_2$ to $\boldsymbol{Z}_2$.*

The classification of $\mathrm{Inv}_{\mathrm{I}}^0(M_A)$ in Theorem I follows from the above Proposition. For example, if $\|\rho(A)\|=3$, then $H_A \otimes \boldsymbol{Z}_2=0$ and therefore $|\mathrm{Inv}_{\mathrm{I}}^0(M_A)|=0$.

4. Involutions of type I–1

Recall the involution r_2 on T^2 defined in Section 2, whose fixed point set, denoted by $\tilde{W}$, consists of four points $\tilde{w}_i$ $(1 \leqq i \leqq 4)$, and whose quotient space is homeomorphic to S^2 (see Fig. 1). Let w_i be the image of $\tilde{w}_i$ in $T^2/r_2 \cong S^2$ for each i $(1 \leqq i \leqq 4)$.

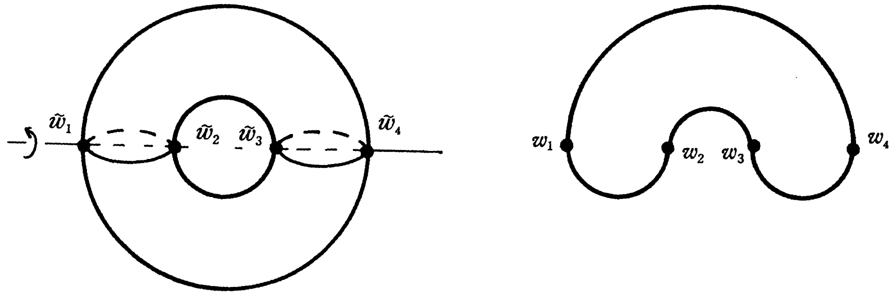

Fig. 1

Let $\tilde{\mathfrak{M}}$ be the group of all homeomorphisms on T^2 which commute with r_2 modulo homeomorphisms which are isotopic to the identity map by isotopies commuting with r_2. Let $\mathfrak{M}$ be the group of all homeomorphisms on S^2 which preserve the subset $W \equiv \{w_i \mid 1 \leqq i \leqq 4\}$ modulo homeomorphisms which are isotopic to the identity map by isotopies preserving W. Then there is a natural epimorphism ψ: $\tilde{\mathfrak{M}} \to \mathfrak{M}$ with $\mathrm{Ker}(\psi)=\{id, r_2\}$ (cf. [1]). Let η be the forgetting homomorphism from $\tilde{\mathfrak{M}}$ to the homeotopy group of T^2 (which is identified with $GL(2, \boldsymbol{Z})$).

If f is an element of $\tilde{\mathfrak{M}}$, the torus bundle $M_{\eta(f)}$ is homeomorphic to $T^2 \times [0, 1]/(x, 0) \sim (f(x), 1)$. (Here the last "$f$" denotes a homeomorphism on T^2 which commutes with r_2 and represent the class $f \in \tilde{\mathfrak{M}}$.) So the involution $r_2 \times id$ on $T^2 \times [0, 1]$ induces an involution on $M_{\eta(f)}$, which is of type I–1. We denote it by the symbol $h(f)$. (Note that it does not depend on a choice of a representative of the class $f \in \tilde{\mathfrak{M}}$.) Every involution of type I–1 is obtained in this way. The quotient of $h(f)$ is homeomorphic to $(S^2, W) \times [0, 1]/(x, 0) \sim (\psi(f)(x), 1)$, which is denoted by the symbol $L(\psi(f))$.

Proposition 4.1. *Let f_1 and f_2 be elements of $\tilde{\mathfrak{M}}$. Then the involutions $h(f_1)$ and $h(f_2)$ are equivalent, iff f_1 is conjugate to either f_2 or f_2^{-1} in $\tilde{\mathfrak{M}}$.*

Proof. Since the "if" part is clear, we prove the "only if" part. Assume that $h(f_1)$ and $h(f_2)$ are equivalent, that is, there is a homeomorphism $\tilde{G}: M_{\eta(f_1)} \to M_{\eta(f_2)}$ such that $h(f_2)=\tilde{G}\circ h(f_1)\circ\tilde{G}^{-1}$. $\tilde{G}$ induces a homeomorphism G between the quotients $L(\psi(f_1))$ and $L(\psi(f_2))$. The following lemma will be proved at the end of this section.

Lemma 4.2. *G is isotopic (as a homeomorphism between manifold pairs) to a homeomorphism G' such that $G'(S^2\times 0)=S^2\times 0$. So $\tilde{G}$ is equivariantly isotopic to a homeomorphism $\tilde{G}'$ such that $\tilde{G}'(T^2\times 0)=T^2\times 0$.*

By the above lemma, we may assume that $G(S^2\times 0)=S^2\times 0$ and $\tilde{G}(T^2\times 0)=T^2\times 0$. Thus the homeomorphism $\tilde{G}$ from $T^2\times[0, 1]/(x, 0)\sim(f_1(x), 1)$ to $T^2\times[0, 1]/(x, 0)\sim(f_2(x), 1)$ comes from a homeomorphism $\hat{G}$: $T^2\times[0, 1]\to T^2\times[0, 1]$. Let $\tilde{g}_0$ and $\tilde{g}_1$ be the homeomorphisms on T^2 defined by the equations $\hat{G}(x, 0)=(\tilde{g}_0(x), *)$ and $\hat{G}(x, 1)=(\tilde{g}_1(x), *)$. Then we have the followings.

(0) $\tilde{g}_0$ and $\tilde{g}_1$ commute with r_2, and therefore they can be considered as elements of $\tilde{\mathfrak{M}}$.

(1) If $\hat{G}(T^2\times 0)=T^2\times 0$, then $f_2=\tilde{g}_1\circ f_1\circ\tilde{g}_0^{-1}$.

(2) If $\hat{G}(T^2\times 0)=T^2\times 1$, then $f_2^{-1}=\tilde{g}_1\circ f_1\circ\tilde{g}_0^{-1}$.

Proposition 4.1 now follows from the following lemma.

Lemma 4.3. *$\tilde{g}_0$ and $\tilde{g}_1$ represent the same element of $\tilde{\mathfrak{M}}$.*

Proof. We have only to prove that $\psi(\tilde{g}_0)=\psi(\tilde{g}_1)$ in $\mathfrak{M}$. To do this, note that G is induced from a homeomorphism $\hat{G}$ on $(S^2, W)\times[0, 1]$. Let g_0 and g_1 be homeomorphisms on (S^2, W) defined by the equations $\hat{G}(x, 0)=(g_0(x), *)$ and $\hat{G}(x, 1)=(g_1(x), *)$. Then $\psi(\tilde{g}_i)=g_i$ $(i=0, 1)$ and $\hat{G}$ gives a homotopy between the maps g_0 and g_1 on (S^2, W). (Note that $\hat{G}$ is not necessarily level preserving.) By using a theorem of Baer (cf. [6]), we can see that g_0 and g_1 determine the same element of $\mathfrak{M}$.

There is an epimorphism τ from $\tilde{\mathfrak{M}}$ to S_4, the symmetric group on four letters, defined by the equation $f(\tilde{w}_i)=\tilde{w}_{\tau(f)(i)}$. Let Δ be the normal subgroup of S_4, which consists of four elements δ_i $(0\leqq i\leqq 3)$, where $\delta_0=id$, $\delta_1=(1\ 2)\ (3\ 4)$, $\delta_2=(1\ 3)\ (2\ 4)$, and $\delta_3=(1\ 4)\ (2\ 3)$.

Lemma 4.4. *The restriction of τ to the subgroup $Ker(\eta)$ gives an isomorphism from $Ker(\eta)$ to Δ.*

Proof. By Lemma 3 of [22] (cf. [11]), each element f of $\mathrm{Ker}(\eta)$ is distinguished by the image $f(\tilde{w}_1)$ $(\in\tilde{W})$, and therefore, $\mathrm{Ker}(\eta)$ consists of at most four elements, and the restriction of τ to $\mathrm{Ker}(\eta)$ is injective. On the other hand, we can easily find four elements $\tilde{\delta}_i$ $(0\leqq i\leqq 3)$ of $\mathrm{Ker}(\eta)$ such that $\tau(\tilde{\delta}_i)=\delta_i$. This completes the proof.

Corollary 4.5. *Let A be a matrix in $GL(2, \boldsymbol{Z})$, and let f_1 and f_2 be elements of $\eta^{-1}(A)$. Then f_1 is conjugate to f_2 (resp. f_2^{-1}), iff there is an element g of $\eta^{-1}(C(A))$ (resp. $\eta^{-1}(R(A))$) such that $\tau(f_2)=\tau(g)\tau(f_1)\tau(g)^{-1}$ (resp. $\tau(f_2)^{-1}=\tau(g)\tau(f_1)\tau(g)^{-1}$).*

By Lemma 4.4, τ induces an epimorphism from $\mathfrak{M}/\mathrm{Ker}(\eta)$ to S_4/Δ. S_4/Δ can be identified with $GL(2, \boldsymbol{Z}_2)$ by an isomorphism sending $(1\ 2)\Delta$ to $\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $(2\ 3)\Delta$ to $\begin{bmatrix}1&1\\0&1\end{bmatrix}$. We can see that, through this identification, the above epimorphism is identified with $\rho: GL(2, \boldsymbol{Z})\to GL(2, \boldsymbol{Z}_2)$, and we have the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc}
1 & \to & \mathrm{Ker}(\eta) & \to & \mathfrak{M} & \xrightarrow{\eta} & GL(2, \boldsymbol{Z}) & \to & 1\\
 & & \downarrow\wr\!\shortparallel & & \downarrow\tau & & \downarrow\rho & & \\
1 & \longrightarrow & \Delta & \longrightarrow & S_4 & \to & GL(2, \boldsymbol{Z}_2) & \to & 1
\end{array}$$

Let A be a matrix in $GL(2, \boldsymbol{Z})$, and let f be an element of $\eta^{-1}(A)$. Then, by Lemma 4.4, $\eta^{-1}(A)=\{f\circ\tilde{\delta}_i \mid 0\leqq i\leqq 3\}$. Proposition 4.1 reduces the classification of $\mathrm{Inv}_{\mathrm{I}}^1(M_A)$ to the classification of $\eta^{-1}(A)$ modulo a suitable equivalence relation, which can be done as follows, using Corollary 4.5.

Case 1. $\|\rho(A)\|=3$. We may assume that $\tau(f)=(1\ 2\ 3)$. Then we have $\tau(f\circ\tilde{\delta}_1)=\tau(\tilde{\delta}_3\circ f\circ\tilde{\delta}_3^{-1})$, $\tau(f\circ\tilde{\delta}_2)=\tau(\tilde{\delta}_1\circ f\circ\tilde{\delta}_1^{-1})$, and $\tau(f\circ\tilde{\delta}_3)=\tau(\tilde{\delta}_2\circ f\circ\tilde{\delta}_2^{-1})$. Since $\tilde{\delta}_i$ $(0\leqq i\leqq 3)$ are contained in $\mathrm{Ker}(\eta)\subset\eta^{-1}(C(A))$, any element of $\eta^{-1}(A)$ is conjugate to f by Corollary 4.5. Hence, $h(f)$ is the unique involution on M_A of type I–1, and we have $\mathrm{Fix}(h(f))=$two simple loops.

Case 2. $\|\rho(A)\|=2$. We may assume $\tau(f)=(1\ 2)$. Then we have $\tau(f\circ\tilde{\delta}_1)=\tau(\tilde{\delta}_2\circ f\circ\tilde{\delta}_2^{-1})=(3\ 4)$, and $\tau(f\circ\tilde{\delta}_3)=\tau(\tilde{\delta}_2\circ(f\circ\tilde{\delta}_2)\circ\tilde{\delta}_2^{-1})=(1\ 2\ 3\ 4)$. Hence, $f\circ\tilde{\delta}_1$ and $f\circ\tilde{\delta}_3$ are conjugate to f and $f\circ\tilde{\delta}_2$ respectively, and $f^{\pm1}$ is not conjugate to $f\circ\tilde{\delta}_2$. So M_A has precisely two inequivalent involutions $h(f)$ and $h(f\circ\tilde{\delta}_2)$ of type I–1, and we have $\mathrm{Fix}(h(f))=$three simple loops and $\mathrm{Fix}(h(f\circ\tilde{\delta}_2))=$one simple loop.

Case 3. $\|\rho(A)\|=1$. Note that (1) $\tau(\eta^{-1}(A))=\Delta$, and (2) the action of $GL(2, \boldsymbol{Z}_2)$ on Δ $(\cong\boldsymbol{Z}_2+\boldsymbol{Z}_2)$ by conjugation is equivalent to the standard action of $GL(2, \boldsymbol{Z}_2)$ on $\boldsymbol{Z}_2+\boldsymbol{Z}_2$. Thus we have only to count the number of orbits under the action of $\rho(N(A))$ $(\subset GL(2, \boldsymbol{Z}_2))$ on $\boldsymbol{Z}_2+\boldsymbol{Z}_2$. It is easy to see that the number is equal to 4, 3, or 2, according as $|\rho(N(A))|$ is equal to 1, 2, or a multiple of 3. Exactly one element of $\mathrm{Inv}_{\mathrm{I}}^1(M_A)$ has four simple loops as its fixed point set, and the remaining elements of $\mathrm{Inv}_{\mathrm{I}}^1(M_A)$ have two simple loops as their fixed point sets. The proof of Theorem I is now complete.

Next, we study quotients of involutions of type I–1. Since $\mathrm{Ker}(\psi)=\{id, r_2\}$, the epimorphisms η and τ induce epimorphisms η': $\mathfrak{M}\to PGL(2, \boldsymbol{Z})$ and τ': $\mathfrak{M}\to S_4$ respectively, and we have the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{Ker}(\eta') & \to & \mathfrak{M} & \xrightarrow{\eta'} & PGL(2, \boldsymbol{Z}) & \to & 1 \\ & & \downarrow \wr\!\!\parallel & & \downarrow \tau' & & \downarrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & S_4 & \to & GL(2, \boldsymbol{Z}_2) & \to & 1 \end{array}$$

Let $A\mapsto\bar{A}$ $(A\in GL(2, \boldsymbol{Z}))$ denote the natural map from $GL(2, \boldsymbol{Z})$ to $PGL(2, \boldsymbol{Z})$. For an element $\bar{A}$ of $PGL(2, \boldsymbol{Z})$, put $C(\bar{A})=\{X\in PGL(2, \boldsymbol{Z})\,|\,X\bar{A}X^{-1}=A\}$ and $R(\bar{A})=\{X\in PGL(2, \boldsymbol{Z})\,|\,X\bar{A}X^{-1}=(\bar{A})^{-1}\}$. Then, by the proof of Proposition 4.1 and the above commutative diagram, we obtain the following.

Proposition 4.6. *Let A be a matrix in $GL(2, \boldsymbol{Z})$, and f_1 and f_2 be elements of $\eta^{-1}(A)$. Then $L(\psi(f_1))$ is homeomorphic to $L(\psi(f_2))$, iff there is an element g of $\eta'^{-1}(C(\bar{A})\cup R(\bar{A}))$, such that $\tau(f_2)$ is equal to $\tau'(g)\tau(f_1)\tau'(g)^{-1}$ or $\tau'(g)\tau(f_1)^{-1}$ $\tau'(g)^{-1}$ according as $g\in\eta'^{-1}(C(\bar{A}))$ or $g\in\eta'^{-1}(R(\bar{A}))$.*

Put $\overline{C(A)}$ (resp. $\overline{R(A)}$) be the image of $C(A)$ (resp. $R(A)$) in $PGL(2, \boldsymbol{Z})$. Then, for a matrix A in $GL(2, \boldsymbol{Z})$ with $tr(A)\neq 0$, we have the following by considering $tr(\pm A^{\pm 1})$ (cf. Section 5).

(1) $C(\bar{A})=\overline{C(A)}$

(2) If $\det(A)=1$ or $\det(A)=-1$ and the primitive period of $\omega(A)$ is not invertible, then $R(\bar{A})=\overline{R(A)}$.

(3) If the condition of (2) is not satisfied, then $R(\bar{A})$ is non-empty, while $\overline{R(A)}$ is empty.

Thus, comparing Proposition 4.6 with Proposition 4.1 and Corollary 4.5, we can see that, if A satisfies the condition of (2), then each element of $\mathrm{Inv}_{\mathrm{I}}^{1}(M_A)$ is distinguished by its quotient. However, this is not always true if A does not satisfy the condition of (2). In fact, we have the following.

EXAMPLE 4.7. Let $A=\begin{bmatrix}5&4\\4&3\end{bmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}3&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}$. Then M_A has precisely four involutions of type I–1, and two of them have the same quotient.

Finally we give a proof of Lemma 4.2. To do this, we need the following.

Sublemma 4.8. *Let $\tilde{B}$ be a 3-ball in $(S^2\times\boldsymbol{R}, \tilde{K}_1\cup\tilde{K}_2)$ with $\tilde{K}_i=z_i\times\boldsymbol{R}$ $(z_i\in S^2, i=1, 2)$, such that $\tilde{B}\cap\tilde{K}_1$ is a connected arc and $\tilde{B}\cap\tilde{K}_2$ is empty. Then $(\tilde{B}, \tilde{B}\cap\tilde{K}_1)$ is a trivial ball pair.*

Proof. Note that $\pi_1(S^2\times\boldsymbol{R}-(\tilde{K}_1\cup\tilde{K}_2))$ is the infinite cyclic group generated by a meridian m of $\tilde{K}_1$. By van-Kampen theorem, we can see that $\pi_1(S^2\times\boldsymbol{R}-(\tilde{K}_1\cup\tilde{K}_2))$ is an amalgamation product of $\pi_1(S^2\times\boldsymbol{R}-(\tilde{B}\cup\tilde{K}_1\cup\tilde{K}_2))$

and $\pi_1(\tilde{B}_1-\tilde{K}_1)$ with amalgamated subgroup $\langle m\rangle$. So $\pi_1(\tilde{B}-\tilde{K}_1)$ is a subgroup of $\pi_1(S^2\times \boldsymbol{R}-(\tilde{K}_1\cup\tilde{K}_2))\cong \boldsymbol{Z}$, and therefore it is isomorphic to $\boldsymbol{Z}$. Thus $(\tilde{B}, \tilde{B}\cap\tilde{K}_1)$ is a trivial ball pair.

Let (M, L) denote the manifold pair $L(\psi(f_2))$ and let S and S' be the 2-spheres $S^2\times 0$ and $G(S^2\times 0)$ respectively. We may assume that S and S' intersect transversely. If $S\cap S'\neq\emptyset$, then there is an "innermost" disk D in S (i.e. $D\cap(S\cap S')=\partial D$), such that $|D\cap L|\leq 2$. Let D' be the closure of a component of $S'-\partial D$ such that $D\cup D'$ bounds a 3-ball B in M.

Step 1. We show that $(B, B\cap L)$ is a trivial tangle. The infinite cyclic cover $(\tilde{M}, \tilde{L})$ of (M, L) can be identified with $(S^2, W)\times\boldsymbol{R}$, where a lift $\tilde{S}$ of S is identified with $S^2\times 0\subset S^2\times\boldsymbol{R}$. Let $\tilde{D}$ be the lift of D contained in $\tilde{S}$, and let $\tilde{B}$ be the lift of B containing $\tilde{D}$. We will show that $(\tilde{B}, \tilde{B}\cap\tilde{L})$ is a trivial tangle. Since $\tilde{S}$ and $\tilde{S}'$ (a lift of S') meet each $w_i\times\boldsymbol{R}$ at exactly one point, $\tilde{B}\cap(w_i\times\boldsymbol{R})$ is a connected arc or an empty set. If $\tilde{B}\cap\tilde{L}$ is empty, the assertion is trivial. If $\tilde{B}\cap\tilde{L}$ is a connected arc, the assertion is a direct consequence of Sublemma 4.8. Suppose that $\tilde{B}\cap\tilde{L}$ has two components. We may assume that $\tilde{B}\cap(w_i\times\boldsymbol{R})$ is a connected arc $w_i\times[0, t_i]$ for $i=1, 2$, and is empty for $i=3, 4$. Let J be an arc in $\mathrm{int}(\tilde{D})$ $(\subset\tilde{S}=S^2\times 0)$ joining $w_1\times 0$ and $w_2\times 0$. Put $\tilde{D}'=\partial\tilde{B}-\mathrm{int}(\tilde{D})$. Then we may assume that $J\times\boldsymbol{R}$ intersects $\partial\tilde{B}$ transversely, and $\tilde{D}'\cap(J\times\boldsymbol{R})$ consists of simple loops and an arc J' joining $w_1\times t_1$ and $w_2\times t_2$. If $\tilde{D}'\cap(J\times\boldsymbol{R})$ contains simple loops, then there is an "innermost" disk C in $J\times\boldsymbol{R}$ (i.e. $C\cap\tilde{D}'=\partial C$). Let C' be a disk in $\tilde{D}'$ bounded by ∂C. Then the 2-sphere $C\cup C'$ is inessential in $S^2\times\boldsymbol{R}$, since $(C\cup C')\cap(z_j\times\boldsymbol{R})=\emptyset$ for $j=3, 4$. On the other hand, $(C\cup C')\cap(z_i\times\boldsymbol{R})=C'\cap(z_i\times\boldsymbol{R})$ consists of at most one point for $i=1, 2$. So $C\cup C'$ does not intersect $z_i\times\boldsymbol{R}$ $(1\leqq i\leqq 4)$, and it bounds a 3-ball which is disjoint from $\tilde{L}$. Using this 3-ball, we can eliminate the intersection ∂C. Hence we may assume that $\tilde{D}'\cap(J\times\boldsymbol{R})=J'$, and therefore $(\tilde{B}, \partial\tilde{B})\cap(J\times\boldsymbol{R})$ is homeomorphic to $J\times(I, \partial I)$. By Sublemma 4.8, the core $*\times I$ of the band $J\times(I, \partial I)$ is a trivial arc in $\tilde{B}$. Hence $(\tilde{B}, \tilde{B}\cap\tilde{L})$ is a trivial tangle.

Step 2. By Step 1, we can eliminate the intersection ∂D through an ambient isotopy of (M, L). So we can deform S' so that it does not intersect S.

Step 3. Let X be a region in M bounded by $S\cup S'$. By a similar argument as Step 1, we can find a band $J\times I$ in X, such that $J\times I$ contains $X\cap L$ and $(J\times I)\cap\partial X=J\times\partial I$. Then, by the "Light Bulb Theorem" (see p. 257 of [18]), the core of the band is unknotted in $X\cong S^2\times I$. Hence $(X, X\cap L)\cong(S^2, W)\times I$, and we can deform S' so that it coincides with S. This completes the proof of Lemma 4.2.

5. Involutions of type II

Let M_A be a torus bundle with an Anosov monodromy. We may assume that $A=\varepsilon A_0^n$, where A_0 is a primitive root of A. To classify $\mathrm{Inv}_{\mathrm{II}}(M_A)$, we have only to classify matrices $\{P\}$ such that $P^2=A$. Note that, if $P^2=A$, then $P\in C(A)$. Hence, by Lemma 1.7, such matrices exist, iff $\varepsilon=+1$ and n is even. If this condition is not satisfied, $\mathrm{Inv}_{\mathrm{II}}(M_A)$ is empty. Suppose that the above condition is satisfied. Then the equation $X^2=A$ has precisely two solutions, $\pm P$ with $P=A_0^{(n/2)}$, and $\mathrm{Inv}_{\mathrm{II}}(M_A)$ consists of one or two elements according to whether M_P is homeomorphic to M_{-P} or not. So, Theorem II follows from the following facts and Lemma 1.1.

(1) Since $tr(P)\neq tr(-P)$, P and $-P$ are not conjugate.

(2) If $\det(P)=+1$, then $tr(P^{-1})=tr(P)\neq tr(-P)$. So P^{-1} and $-P$ are not conjugate.

(3) If $\det(P)=-1$, then $tr(P^{-1})=-tr(P)=tr(-P)$. So P^{-1} is conjugate to $-P$, iff the primitive period of $\omega(A)$ is invertible.

6. Involutions of type III

Let $M_A=T^2\times \boldsymbol{R}/\tilde{\phi}$ ($\phi=\phi_A$) be an orientable torus bundle with $A\neq\pm I$, and let h be an involution on M_A of type III. Recall that h is induced from a homeomorphism $\tilde{h}$ on $T^2\times\boldsymbol{R}$ given by the equation $\tilde{h}(x, t)=(\gamma_t(x), -t)$, where $\{\gamma_t\,|\,t\in\boldsymbol{R}\}$ is a continuous family of orientation-reversing homeomorphisms on T^2. h has two invariant fibers $T^2\times 0$ and $T^2\times\pi$, and M_A/h is obtained from $T^2\times[0, \pi]$ by identifying $(x, 0)$ with $(\gamma_0(x), 0)$ and (x, π) with $(\phi\circ\gamma_\pi(x), \pi)$. It can be seen that the equivalence class of h is determined by the strong equivalence classes of the involutions γ_0 and $\phi\circ\gamma_\pi$ on T^2. (Two involutions γ and γ' on a space X is said to be strongly equivalent, if there is a homeomorphism f on X, which is isotopic to the identity map, such that $\gamma'=f\circ\gamma\circ f^{-1}$.) The strong equivalence class of an orientation-reversing involution γ on T^2 is determined by the matrix γ_* and whether γ is free or not. Let F be a subset of $\{0, \pi\}$ defined as follows.

(1) $0\in F$, iff γ_0 is non-free.

(2) $\pi\in F$, iff $\phi\circ\gamma_\pi$ is non-free.

Then, by the above arguments, the equivalence class of h is determined by the pair (P, F), where $P=(\gamma_0)_*\in GL(2, \boldsymbol{Z})$. (Note that $(\phi\circ\gamma_\pi)_*=AP$.)

Lemma 6.1. *The pair (P, F) satisfies the following conditions.*

(1) $P\in R^-(A)$.

(2) If $0\notin F$ (resp. $\pi\notin F$), then $\rho(P)=I$ (resp. $\rho(AP)=I$).

Proof. (1) is proved in Section 2. (2) follows from the fact that an orientation reversing free involution on T^2 has a matrix which is conjugate to $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ (see Section 1).

We say that a pair (P, F) ($P\in GL(2, \boldsymbol{Z})$, $F\subset\{0, \pi\}$) is *A-admissible,* if it satisfies the condition of Lemma 6.1. We can see that, if (P, F) is A-admissible, it is derived from an involution on M_A of type III–i, denoted by $h(P, F)$, where $i=|F|$. We study when two A-admissible pairs determine equivalent involutions. In the rest of this section, we assume $\beta_1(M_A)=1$.

Lemma 6.2. *Let (P, F) and (P', F') be A-admissible pairs such that h (P, F) is equivalent to $h(P', F')$. Then $|F|=|F'|$.*

Proof. Since $\beta_1(M_A)=1$, M_A has the unique infinite cyclic cover $\tilde{M}_A$ $(=T^2 \times \boldsymbol{R})$. A generator of the covering transformation group is identified with $\tilde{\phi}$, where $\phi=\phi_A$ (see Section 1). The involution $h\equiv h(P, F)$ is induced from an involution $\tilde{h}$ on $\tilde{M}_A$ given by $\tilde{h}(x, t)=(\gamma_t(x), -t)$. Then the set of all lifts of h to M_A is equal to $\{\tilde{\phi}^n\circ\tilde{h}\,|\,n\in\boldsymbol{Z}\}$. Note that

$$\begin{aligned}\mathrm{Fix}(\tilde{\phi}^n\circ\tilde{h}) &= \{(x, t)\,|\,(x, t)=(\phi^n\circ\gamma_t(x), -t+2n\pi)\}\\ &\cong \mathrm{Fix}(\phi^n\circ\gamma_{n\pi})\\ &\cong \begin{cases}\mathrm{Fix}(\phi^k\circ\gamma_0\circ\phi^{-k})\cong\mathrm{Fix}(\gamma_0) & \text{if } n=2k\,,\\ \mathrm{Fix}(\phi^k\circ(\phi\circ\gamma_\pi)\circ\phi^{-k})\cong\mathrm{Fix}(\phi\circ\gamma_\pi) & \text{if } n=2k+1\,.\end{cases}\end{aligned}$$

So $|F|=0$, 1, or 2, according as (0) all lifts of h are free, (1) half of the lifts of h are free, or (2) all lifts of h are non-free. The same result holds for the involution $h'\equiv h(P', F')$, and a homeomorphism on M_A, which gives the equivalence of h and h', lifts to a homeomorphism on $\tilde{M}_A$, which gives equivalence of the lifts of h and h'. Hence we can conclude $|F|=|F'|$.

REMARK 6.3. (1) If we assume uniqueness of equivariant fiberings, this lemma is an immediate consequence of the definition of the set F.

(2) For torus bundles with $\beta_1\neq 1$, this lemma does not holds (see Section 7).

Lemma 6.4. *Let (P, F) be an A-admissible pair.*

(1) *For any matrix B in $N(A)$, the pair (BPB^{-1}, F) is A-admissible, and $h(BPB^{-1}, F)$ is equivalent to $h(P, F)$.*

(2) *If $|F|=0$ or 2, then (AP, F) is A-admissible, and $h(AP, F)$ is equivalent to $h(P, F)$.*

(3) *If $|F|=1$, then (AP, F^c) is A-admissible, and $h(AP, F^c)$ is equivalent to $h(P, F)$. Here $F^c=\{0, \pi\}-F$.*

Proof. (1) Let $\tilde{f}$ be the homeomorphism on $\tilde{M}_A$ given by $\tilde{f}(x, t)=(\phi_B(x), t)$ or $(\phi_B(x), -t)$ according to whether $B\in C(A)$ or $B\in R(A)$. Then $\tilde{f}$ induces a homeomorphism f on M_A, such that $h(BPB^{-1}, F)=f\circ h(P, F)\circ f^{-1}$.

(2) and (3) Let $\tilde{g}$ be a homeomorphism on $\tilde{M}_A$ given by $\tilde{g}(x, t)=(x, t-\pi)$. Then it induces a homeomorphism g on M_A, such that $g\circ h(P, F)\circ g^{-1}=h(AP$ $F)$ or $h(AP, F^c)$ according to whether $|F|=0$, 2, or $|F|=1$.

Proposition 6.5. *Let (P, F) and (P', F') be A-admissible pairs. Then $h(P, F)$ and $h(P', F')$ are equivalent, iff the following conditions are satisfied.*

(1) $|F|=|F'|$.

(2) *There is a matrix B of $N(A)$ and an integer u, such that*

(i) $P'=A^u BPB^{-1}$,

(ii) *if $|F|=1$, then u is even or odd according to whether $F'=F$ or F^c.*

Proof. The "if" part follows from Lemma 6.4. We prove the "only if" part. The condition (1) follows from Lemma 6.2. Put $h=h(P, F)$ and $h'=h(P', F')$, and let $\tilde{h}$ and $\tilde{h}'$ be the "standard" lifts of h and h' to $\tilde{M}_A$ $(=T^2 \times R)$ respectively. Then we have $\tilde{h}_*=P$ and $\tilde{h}'_*=P'$. Assume that there is a homeomorphism f on M_A such that $f\circ h\circ f^{-1}=h'$. Let $\tilde{f}$ be a lift of f to $\tilde{M}_A$. Then $\tilde{h}'\circ\tilde{f}\circ\tilde{h}^{-1}\circ\tilde{f}^{-1}$ is a lift of the identity map, and therefore, it is equal to a covering transformation $\tilde{\phi}^u$. Put $B=\tilde{f}_*$, then $B\in N(A)$ (cf. Lemma 1.1) and $P'BP^{-1}B^{-1}=A^u$. In case $|F|=1$, we may assume that $F=F'=\{0\}$, by virtue of Lemma 6.4. Then Fix$(\tilde{h})$ and Fix$(\tilde{h}')$ are not empty, and their projections to M_A are equal to Fix(h) and Fix(h') respectively. Since $f(\text{Fix}(h))$ $=\text{Fix}(h')$, we can choose $\tilde{f}$, so that $\tilde{f}(\text{Fix}(\tilde{h}))=\text{Fix}(\tilde{h}')$. Then $\tilde{h}'\circ\tilde{f}\circ\tilde{h}^{-1}\circ\tilde{f}^{-1}$ is the identity map, and we have $P'BP^{-1}B^{-1}=I$. This completes the proof.

If A-admissible pairs (P, F) and (P', F') satisfy the conditions of Proposition 6.5, we say that they are *A-equivalent* and denote $(P, F)\sim(P', F')$.

Let A be an Anosov matrix, and assume that $A=\varepsilon A_0^n$, where A_0 is a primitive root of A. Suppose that $R^-(A)$ is not empty, and let P_0 be an element of $R^-(A)$. Then, by Lemma 1.7, $R^-(A)=\{\pm A_0^i P_0 | i\in \boldsymbol{Z}\}$ or $\{\pm A_0^{2i} P_0 | i\in \boldsymbol{Z}\}$ according to whether $\det(A_0)=+1$ or -1 (that is, $\sigma_1(A)=+$ or $-$). Since $N(A)=C(A)\cup C(A)P_0$ and $C(A)=\{\pm A_0^i | i\in \boldsymbol{Z}\}$, the equivalence relation $\sim$ is generated by the following relations.

$$(1)\quad (P, F)\sim(A_0PA_0^{-1}, F)=\begin{cases}(A_0^2P, F) & \text{if } \sigma_1(A)=+\,,\\ (-A_0^2P, F) & \text{if } \sigma_1(A)=-\,.\end{cases}$$

$$(2)\quad (P, F)\sim\begin{cases}(AP, F) & =(\varepsilon A_0^nP, F) \quad \text{if } |F|=0 \text{ or } 2\,,\\ (AP, F^c) & =(\varepsilon A_0^nP, F^c) \quad \text{if } |F|=1\,.\end{cases}$$

By the relation (1), any A-admissible pair is A-equivalent to a pair (P, F) with $P=P_0$, $-P_0$, A_0P_0, or $-A_0P_0$. (If $\sigma_1(A)=-$, we have $P=\pm P_0$.) In case $|F|$ $=0$ or 2, the relation (2) can be replaced by the following relation, by virtue of the relation (1).

$$(P, F)\sim\begin{cases}(P, F) & \text{if } \sigma(A)=(+,+,+), (-,+,+), \text{ or } (-,-,-)\,,\\ (-P, F) & \text{if } \sigma(A)=(+,-,+), (-,+,-), \text{ or } (-,-,+)\,,\\ (A_0P, F) & \text{if } \sigma(A)=(+,+,-)\,,\\ (-A_0P, F) & \text{if } \sigma(A)=(+,-,-)\,.\end{cases}$$

By using the above facts, we can easily classify $\mathrm{Inv}^2_{\mathrm{III}}(M_A)$. $\mathrm{Inv}^i_{\mathrm{III}}(M_A)$ $(i=0, 1)$ are empty, if $R^-(A)\cap \mathrm{Ker}(\rho)$ is empty (cf. Lemma 6.1). So, assume that $R^-(A)\cap \mathrm{Ker}(\rho)$ is not empty (and therefore, $||\rho(A_0)||=1$ or 2). We may assume that $P_0\in \mathrm{Ker}(\rho)$. Then

$$R^-(A)\cap \mathrm{Ker}(\rho)=\begin{cases}\{\pm A_0^i P_0 \,|\, i\in \boldsymbol{Z}\} & \text{if } \sigma_1(A)=+ \text{ and } ||\rho(A_0)||=1\,,\\ \{\pm A_0^{2i} P_0 \,|\, i\in \boldsymbol{Z}\} & \text{otherwise.}\end{cases}$$

Noting this fact, the classifications of $\mathrm{Inv}^i_{\mathrm{III}}(M_A)$ $(i=0, 1)$ can be done similarly.

7. Torus bundles with exceptional monodromies

In this section, we study involutions on a torus bundle M_A with an exceptional monodromy A. If $\beta_1(M_A)=1$ and $A\neq -I$, then the preceding arguments are also applicable to $\mathrm{Inv}(M_A)$, and by using Lemma 1.4 in stead of Lemma 1.7, we can classify $\mathrm{Inv}(M_A)$. For $A=-I$, the preceding arguments are applicable to orientation-preserving involutions. Thus we can obtain Theorem IV.

To study $\mathrm{Inv}(M_A)$ with $\beta_1(M_A)>1$, we need the following lemma, by which the proof of Lemma 2.1 is completed.

Lemma 7.1. *Let h be an involution on M_A with $A=\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$. Then h is equivalent to a fiber-preserving involutoin.*

Proot. For $n=0$, this lemma is proved in [13]. So assume that $n\neq 0$. Put $G=\pi_1(M_A)$, then $G=\langle f, x, y \,|\, xfx^{-1}=f,\ yfy^{-1}=f,\ [x, y]=f^n\rangle$ and the center $Z(G)$ of G is the infinite cyclic group generated by f. The involution h induces an isomorphism $h_\sharp$ on $G/Z(G)\cong \boldsymbol{Z}+\boldsymbol{Z}$. Since $h_\sharp^2=id$, $h_\sharp$ is conjugate to an isomorphism represented by the matrix $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$, $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, or $\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$. So, there is a subgroup H of $G/Z(G)$, such that (1) $H\cong (G/Z(G))/H \cong \boldsymbol{Z}$, and (2) $h_\sharp(H)=H$. Let $\tilde{H}$ be the inverse image of H in G. Then $\tilde{H}$ is a normal subgroup of G, such that (1) $\tilde{H}\cong \boldsymbol{Z}+\boldsymbol{Z}$, $G/\tilde{H}\cong \boldsymbol{Z}$, and (2) $h_*(\tilde{H})=\tilde{H}$. By the fibration theorem of Stallings [21], there is a fibering $q\colon M_A\to S^1$ with fiber a torus T such that $\pi_1(T)=\tilde{H}$. By Lemma 1.1, this fibering is equivalent to the original fibering. Using the fact that $h_*(\pi_1(T))=\pi_1(T)$, we can prove that there is a fibering $q'\colon M_A\to S^1$ isotopic to q, such that h is fiber-preserving with respect to q', by a similar argument to that of [23] (cf. Section 3 of [20]).

REMARK 7.2. For a torus bundle M_A with $A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ or $\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$, there s an involution on M_A which is not equivalent to an involution that preserves he torus-fibering structure. However, we can see that, for any involution h

on M_A, there is a fibering $q: M_A \rightarrow S^1$ with fiber a torus or a Klein bottle, with respect to which h is fiber-preserving.

By virtue of Lemma 7.1, we can list all involutions on M_A with $A=\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ $(n \neq 0)$, by the methods given in the preceding sections. The listed involutions can be easily distinguished, and finally, we obtain the following table, from which Theorem V follows.

	I–0	I–1	II	III–0	III–1	III–2
n: odd	h_5	$h_1\ h_2$			$h_2\ h_4$	$h_1\ h_3$
n: even non-zero	$k_6\ k_7$	$k_1\ k_2\ k_3$	$k_7\ k_8$	$k_8\ k_9$	$k_3\ k_5$	$k_1\ k_2\ k_4\ k_5$

8. Torus bundles admitting unique involutions

Tollefson [22] showed that a family of non-orientable torus bundles $\{M(n)\}$ has the property that each $M(n)$ admits a unique involution. The following theorem generalizes this result.

Theorem VI. *If a torus bundle M_A satisfies the following conditions, then M_A admits exactly one involution.*

(1) $\|\rho(A)\|=3$.

(2) *A is Anosov, primitive, and if $\det(A)=+1$, then the primitive period of $\omega(A)$ is not invertible.*

Proof. This is a direct consequence of Theorems I, II, and III.

EXAMPLE 8.1 (Tollefson [22]). For a positive odd integer n, let $M(n)$ be the non-orientable torus bundle with the monodromy matrix $A_n=\begin{bmatrix} n-1 & 1 \\ n & 1 \end{bmatrix}$. Then $\|\rho(A_n)\|=3$, $\omega(A_n)=[0, 1, n, n, \cdots]$, and therefore, A_n is conjugate to $\begin{bmatrix} n & 1 \\ 1 & 0 \end{bmatrix}$. So, A_n satisfies the condition of Theorem VI, and $M(n)$ admits exactly one involution.

The condition given in Theorem VI imposes restrictions not only on involutions but also on periodic maps of arbitrary periods. In fact, if h is a periodic map of period $n\,(>2)$ on a torus bundle M_A which satisfies the condition of Theorem VI, then M_A/h is again a torus bundle, and the projection $p: M_A \rightarrow M_A/h$ gives an n-fold unbranched cyclic covering, such that $q' \circ p=q$, where q (resp. q') is the torus fibering $M_A \rightarrow S^1$ (resp. $M_A/h \rightarrow S^1$). This can be proved by using Theorem 5.2 of Edmond-Livingston [5]. Note that, such periodic map h exists, iff $\boldsymbol{Z}_n$ is a quotient of Coker $(A-I)$ (cf. Section 3). For the matri

A_n in Example 8.1, we have $\mathrm{Coker}(A_n - I) \cong \boldsymbol{Z}_n$. So, any periodic map on $M(n)$ has period 2 or a divisor of n (cf. Conner-Raymond [3]). In particular $M(1)$ admits exactly one periodic map (cf. [3, 22]). Conversely $M(1)$ is the only torus bundle which admits a unique periodic map. This follows from the following facts. (1) If $|\mathrm{Coker}(A-I)|=1$, then A is conjugate to A_1, $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, or $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. (2) The torus bundle M_A with $A=\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$) has precisely two (resp. three) involutions.

9. Torus bundles which are regular coverings of S^3

In [20], we determined torus bundles which are 2-fold branched coverings of S^3 (cf. [8, 15, 17]). In this section, we give a necessary and sufficient condition for a torus bundle to be a regular covering of S^3.

Theorem VII. *An orientable torus bundle M_A is a regular covering of S^3, iff $R^-(A)$ is a non-empty set. Moreover, if this condition is satisfied, M_A becomes a $\boldsymbol{Z}_2+\boldsymbol{Z}_2$ covering of S^3.*

REMARK 9.1. For a matrix A in $SL(2, \boldsymbol{Z})$, $R^-(A)$ is not empty, iff A is exceptional or A is Anosov and one of the following conditions holds.

(1) $\sigma_1(A)=-$, and the primitive period of $\omega(A)$ is invertible.

(2) $\sigma_1(A)=+$, and the primitive period of $\omega(A)$ is "negatively invertible", that is, there is an *odd* integer u such that $(a_s, \cdots, a_1)=(a_{u+1}, a_{u+2}, \cdots, a_s, a_1, a_2, \cdots, a_u)$, where $(a_1, \cdots, a_s)$ is the primitive period of $\omega(A)$.

Proof. Suppose that $R^-(A)$ is non-empty. If $A \neq \pm I$, let P be any matrix in $R^-(A)$. If $A=\pm I$, let $P=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, by Lemmas 1.4 and 1.7, we have $P^2=I$, and therefore, we obtain an involution h_1 on $M_A=T^2\times \boldsymbol{R}/\tilde{\phi}_A$ induced from an involution $\tilde{h}_1$ on $T^2\times\boldsymbol{R}$, defined hy $\tilde{h}_1(x, t)=(\phi_P(x), -t)$. Let h_2 be the involution on M_A induced from an involution $\tilde{h}_2$ on $T^2\times\boldsymbol{R}$ given by $\tilde{h}_2(x, t)=(r_2(x), t)$. Then h_1 and h_2 are commutative, and they generate a $\boldsymbol{Z}_2+\boldsymbol{Z}_2$ action on M_A. Let h_1' be the involution on M_A/h_2 induced by h_1. Then $M_A/(\boldsymbol{Z}_2+\boldsymbol{Z}_2)\cong(M_A/h_2)/h_1'\cong S^2\times S^1/h_1'\cong S^3$. So, M_A is a $\boldsymbol{Z}_2+\boldsymbol{Z}_2$ covering of S^3.

Conversely, suppose that M_A is a regular covering of S^3, that is, there is a finite group G acting on M_A such that $M_A/G\cong S^3$. If A is exceptional, $R^-(A)$ is non-empty by Lemma 1.4. So assume that A is Anosov, and therefore, $H_1(M_A; \boldsymbol{Q})\cong\boldsymbol{Q}$. By p. 120 of [2], $H_1(M_A; \boldsymbol{Q})^G$, the part of $H_1(M_A; \boldsymbol{Q})$ fixed by the operation of G, is isomorphic to $H_1(M_A/G; \boldsymbol{Q})\cong 0$. So, there is an element g of G which acts on $H_1(M_A; \boldsymbol{Q})\cong\boldsymbol{Q}$ as the multiplication by -1. Let $\tilde{g}$ be a lift of g to the unique infinite cyclic cover $\tilde{M}_A$ of M_A. Then we have $\tilde{g}\circ\tilde{\phi}_A\circ\tilde{g}^{-1}=\tilde{\phi}_A^{-1}$, and therefore, $PAP^{-1}=A^{-1}$, where $P=\tilde{g}_*\in GL(2, \boldsymbol{Z})$. Since

g is orientation-preserving, we have $\det(P)=-1$, completing the proof.

REMARK 9.2. The figures of the branch lines can be found in Dunber [4]. In it, oriented closed geometric orbifolds, which are not hyperbolic and whose underlying topological spaces are S^3, are classified.

10. Concluding Remark

Tollefson [24] proved that, if M is a closed, orientable, irreducible 3-manifold with $\beta_1(M)\geqq 1$ and the center of $\pi_1(M)$ is trivial, then there is a one to one correspondence between the strong equivalence classes of involutions on M and the 2-torsions of $\mathrm{Out}(\pi_1(M))$, the outer-automorphism group of $\pi_1(M)$. For a torus bundle M_A with an Anosov monodromy, $\mathrm{Out}(\pi_1(M_A))$ is a finite group (see Kojima [12]), and it can be calculated explicitly by using methods of [3] and [12] and the results in Section 1 of this paper. So the estimate of $|\mathrm{Inv}(M_A)|$ given in this paper may also be obtained through a group theoretical approach.

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