ON WEAKLY TRANSITIVE TRANSLATION PLANES

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1. Introduction

Let $\pi'$ be a translation plane of order $p'$ with $p$ a prime. Let $G$ be a subgroup of the translation complement and $\Delta$ a subset of $l_\infty$ with $|\Delta| = p+1$. $\pi$ is said to be $\Delta$-transitive if the following conditions are satisfied (V. Jha [4]):

(i) $G$ leaves $\Delta$ invariant and acts transitively on $l_\infty - \Delta$.
(ii) $G$ fixes at least two points of $\Delta$.
(iii) $G$ has a normal Sylow $p$-subgroup.

On $\Delta$-transitive planes, V. Jha has proved the following theorem.

Theorem (V. Jha [4]). If $\pi'$ is $\Delta$-transitive with $|\Delta| = p+1$, then $\pi$ has order $p^2$ and $\Delta = \pi_0 \cap l_\infty$, where $\pi_0$ is a subplane of order $p$.

If $(\pi', \Delta, G)$ satisfies the conditions (i) and (ii) above, $\pi$ is said to be weakly transitive.

In his paper [4], V. Jha has conjectured that weakly transitive planes are the Hall planes of order $p^2$, the Lorimer-Rahilly plane of order 16 and the Johnson-Walker plane of order 16.

In this paper we prove the following theorems on weakly transitive planes.

Theorem 1. Let $\pi'$ be a translation plane of order $p'$ with $p$ a prime and $\Delta$ a subset of $l_\infty$ with $|\Delta| = p+1$. If a subgroup $G$ of the translation complement of $\pi$ leaves $\Delta$ invariant and acts transitively on $l_\infty - \Delta$, then one of the following holds.

(i) $O_p(G)$ is semiregular on $\Delta - \{A\}$ for some point $A \in \Delta$.
(ii) $\pi$ has order $p^2$.
(iii) $\pi$ has order $p^3$ and $G$ is transitive on $\Delta$.

The Lorimer-Rahilly plane of order 16 and the Johnson-Walker plane of order 16 are examples of the case (i). The Hall planes of order $p^2$ and the plane of order 25 constructed by M.L. Narayana Rao and K. Satyanarayana in [6] are examples of the case (ii). The desarguesian plane of order 27 is an example of the case (iii).

As an immediate corollary we have the following.
Theorem 2. Suppose \((\pi^{t_\omega}, \Delta, G)\) with \(|\Delta|=p+1\) is weakly transitive. If \(O_p(G)\neq 1\), then \(\pi\) has order \(p^2\) and \(\Delta=F(O_p(G))\cap I_\omega\).

We note that if \(\pi^{t_\omega}\) is \(\Delta\)-transitive, then it satisfies the assumption of Theorem 2.

2. Proof of Theorem 1

We prove Theorem 1 by way of contradiction. Assume that \((\pi^{t_\omega}, \Delta, G)\) is a counterexample such that \(p^r+|G|\) is minimal. Therefore \(r\geq 3\) and \(O_p(G)\neq 1\).

Throughout the paper we use the following notations.
\[ T: \text{the group of translations of } \pi \]
\[ M(=O_p(G)): \text{the maximal normal } p\text{-subgroup of } G \]
\[ F(H): \text{the fixed structure consisting of points and lines of } \pi \text{ fixed by a nonempty subset } H \text{ of } G. \]
\[ n_p: \text{the highest power of a prime } p \text{ dividing a positive integer } n \]
\[ \Gamma: \text{ } I_\omega-\Delta. \]

Other notations are taken from [1] and [2].

Lemma 1. \(F(M)\) is a subplane of \(\pi\) of order \(p\) and \(\Delta=F(M)\cap I_\omega.\)

Proof. Let \(K\) be the pointwise stabilizer of \(\Delta\) in \(G\) and assume that \(M\leq K.\) We denote by \(\bar{G}\) the restriction of \(G\) on \(\Delta.\) Clearly \(\bar{G}\triangleright \bar{M}\neq 1\) and as \(|\Delta|=p+1\), \(\bar{M}\) is a Sylow \(p\)-subgroup of \(\bar{G}.\) By the Schur-Zassenhaus' theorem (Theorem 6.2.1 of [1]), there is a subgroup \(L\) of \(\bar{G}\) such that \(K\triangleleft L\) and \(|\bar{G}:L|=p, \bar{G}=\bar{M}L.\)

Set \(N=M\cap K.\) We have \(N\neq 1,\) for otherwise \(\pi\) satisfies (i) of Theorem 1, contrary to the minimality of \(\pi.\) As \(G\triangleright K, G\triangleright N.\) It follows from the transitivity of \(G\) on \(\Gamma\) that \(N\) is \(\frac{1}{2}\)-transitive on \(\Gamma.\)

Let \(\Psi\) be the set of \(N\)-orbits on \(\Gamma.\) Since there is no nontrivial homology of order \(p, N\) acts faithfully on \(\Gamma.\) As \(N\neq 1\) and \(|\Gamma|_p=p, |\Psi|=|\Gamma||p=p^{r-1}-1,\) hence \(\Psi\) coincides with the set of \(M\)-orbits on \(\Gamma.\)

Since \(G=ML, L\) is transitive on \(\Psi\) by the last paragraph. Hence \(L\) is transitive on \(\Gamma\) as \(N\triangleleft L.\) From this \(\pi^{t_\omega}, \Delta, L\) satisfies (ii) or (iii) of Theorem 1 by the minimality of \((\pi^{t_\omega}, \Delta, G).\) Therefore \((\pi^{t_\omega}, \Delta, G)\) also satisfies (ii) or (iii) of Theorem 1. This is a contradiction. Thus \(M\leq K.\)

Since \(F(M)\cap \Gamma=\phi, F(M)\cap I_\omega=\Delta,\) so that \(F(M)\) is a subplane of \(\pi\) of order \(p.\)

Lemma 2. If \(p=2,\) then \(r\) is even.

Proof. Assume \(p=2.\) Let \(x\) be an involution in \(M.\) Since \(F(x)\) contains \(\Delta\) by Lemma 1, \(F(x)\) is a subplane of \(\pi.\) By a Baer's theorem (Theorem 4.3 of [2]), \(F(x)\) is of order \(\sqrt{2^r}.\) Thus \(r\) is even.
Lemma 3. Let $t$ be a prime $p$-primitive divisor of $p^r-1$ and let $x$ be a nontrivial $t$-element of $G$. If $x$ centralizes $M$, then $F(x) \cap \Delta = \phi$.

Proof. Let $A \in F(x) \cap \Delta$ and set $U=T(A)$, the set of translations of $T$ with center $A$. Clearly $|U|=p^r$. By Lemma 1, $|C_v(M)|=p$ as $U$ is regular on the set of affine points on the line $OA$. Set $R=\langle x \rangle$. Since $R$ normalizes $C_v(M)$ and $t \nmid p-1$, $C_v(R)$ contains $C_v(M)$.

If $C_v(R) \neq C_v(M)$, $R$ acts trivially on $U/C_v(R)$ as $|U/C_v(R)|<p^r-1$ and $t$ is a $p$-primitive divisor of $p^r-1$. Hence $[R, U]=1$ by Theorem 5.3.2 of [1]. Therefore $x$ is a homology with axis $OA$ and so $t|(p^r-1, p-1)=p-1$, a contradiction. Thus $C_v(R)=C_v(M)$.

By Theorem 5.2.3 of [1], $U=C_v(R) \times [U, R]$. Since $M$ centralizes $R$ and normalizes $U$, it also normalizes $[U, R]$. Hence $1 \neq C_{v, R}(M) \leq C_v(M) = C_v(R)$, a contradiction. Thus $F(x) \cap \Delta = \phi$.

Lemma 4. If $r=3$, then $p \equiv -1 \pmod{4}$.

Proof. By a Baer's theorem and Lemma 1, $p \equiv 2$ and $|M|=p$ as $r=3$. Assume $p \equiv 1 \pmod{4}$ and let $t$ be an odd prime dividing $p+1$. Clearly $t$ is a prime $p$-primitive divisor of $p^r-1=2^r-1$. Since $|M|=p$ and $t \nmid p-1$, a Sylow $t$-subgroup $R$ of $G$ centralizes $M$. Applying Lemma 3, $t \neq 1$ as regular on $\Delta$. As $p+1|\mid G\mid$ and $t$ is arbitrary, the length of each $G$-orbit on $\Delta$ is divisible by $(p+1)/2$. Since $\pi$ is a counterexample of Theorem 1, $G$ has two orbits of length $(p+1)/2$ on $\Delta$.

Let $S$ be a Sylow 2-subgroup of $G$ and let $X \in F(S) \cap \Delta$. Set $\pi_0=F(M)$, $S_0=S_{\langle \pi_0, \tau_p \rangle}$ and $K=G_\Delta$, the pointwise stabilizer of $\Delta$ in $G$. Since $M$ is a nontrivial normal subgroup of $G$, $\pi_0$ is $G$-invariant and isomorphic to $PG(2, p)$. The restriction of $\text{Aut}(PG(2, p))$ on the line at infinity is isomorphic to $PGL(2, p)$ in its usual 2-transitive permutation representation. Hence $G/K$ is isomorphic to a subgroup of $PGL(2, p)$. As $|G/K|$ is divisible by $(p+1)/2$, $G/K$ is isomorphic to a subgroup of the dihedral group of order $2(p+1)$ by a Dickson's theorem (Theorem 14.1 of [5]). Since $G/K$ is not transitive on $\Delta$, $|G/K|=(p+1)/2$ or $p+1$. Therefore $|S: S \cap K|=1$ or 2. Hence $S \cap K$ is semiregular on $F(M) \cap (OX-\{O, X\})$ and so $|S \cap K|\mid (p-1)^2$. From this, $|S| \leq 2(p-1)$. But, as $S \cap K \neq 1$, $S_0 \neq 1$ and so $|S/S_0| \geq |S/2|=2(p-1)$. This implies $|S| \geq 4(p-1)^2$, a contradiction.

Lemma 5. Let $S$ be a 2-group acting faithfully on an elementary abelian $p$-group $W$ of order $p^r$ with $p^r \equiv -1 \pmod{4}$. If an element $x \in S$ inverts $W$, then $S=\langle x \rangle \times S_1$ for a subgroup $S_1$ of $S$.

Proof. We may assume that $S \leq GL(r, p)$ and $x=-I$, where $I$ is the unit matrix of degree $r$. Since $r$ is odd, $\det(x)=(-1)^r=-1$ and so $x \in SL(r, p)$.
Since $2|p-1$ and $4Xp-1$, $\langle x \rangle \times SL(r, p)$ is a normal subgroup of $GL(r, p)$ of odd index. Thus $S = \langle x \rangle \times S_1$, where $S_1 = S \cap SL(r, p)$.

**Lemma 6.** Let $S$ be a Sylow 2-subgroup of $G$. If $r=3$, then the length of every $S$-orbit on $\Delta$ is divisible by $|\Delta|_2$.

**Proof.** By Lemma 4, $p \equiv -1 \pmod{4}$. Since $G$ is transitive on $\Gamma$, $|\Gamma| = p(p^2-1)|G|$ and so $2(p+1)|S/S_0$, where $S$ is a Sylow 2-subgroup of $G$ and $S_0 = S_{(0,1)}$. Hence $|S_X| \geq 2 \times |S_0|$ for some point $X \in \Delta$. Here $S_X$ denotes the stabilizer of $X$ in $S$. Let $Y \in F(S_X) \cap (\Delta - \{X\})$.

First we show that $S_0 \neq 1$. Assume that $S_0 = 1$ and let $u$ be an involution in $Z(S_X)$. By Baer's theorem, any involution in $S$ is a homology. Hence either $u$ is a $(X, OY)$-homology or $u$ is a $(Y, OX)$-homology. In either case $C_S(u) \leq S_X$. As $u \in Z(S_X)$, $C_S(u) = S_X$. In particular $|S_X| \geq 4$.

We note that either $S_{(x,oy)} = 1$ or $S_{(y,oy)} = 1$, for otherwise $S_0 \neq 1$ by Lemma 4.22 of [2]. Let $A \in \{X, Y\}$ such that $S_{(a,0)} = 1$, where $\{B\} = \{X, Y\} - \{A\}$. Then $S_X$ acts faithfully on $T(A)$. In particular every involution in $S_X$ fixes no affine point on $OY - \{O\}$. Therefore every involution in $S_X$ inverts $T(A)$. From this $S_X$ has exactly one involution. But, by Lemma 5, $S_X$ contains a subgroup isomorphic to $Z_2 \times Z_2$, a contradiction. Thus $S_0 \neq 1$.

Let $z$ be an involution in $S_0$. Since $O$ is the only affine fixed point of $z$, $z$ inverts $T$. As $(p-1)_2 = 2$, $\langle z \rangle$ is a unique Sylow 2-subgroup of $G_{(0,1)}$.

Set $V = S_X$. If $V_{(x,oy)} = 1$, then $V$ acts faithfully on $T(Y)$ and moreover $z$ inverts $T(Y)$. By Lemma 5, $V$ contains a subgroup $U$ such that $z \in U$ and $U$ is isomorphic to $Z_2 \times Z_2$. By Lemma 4.22 of [2], we obtain a contradiction. Hence $V_{(x,oy)} \neq 1$.

Let $u$ be an involution in $V_{(x,oy)}$. Then, as $u \in Z(V)$, we have $C_S(u) = V$. Assume $|V| > 4$. $\mathcal{P} = V/\langle u \rangle$ normalizes $T(Y)$ and $z$ inverts $T(Y)$. Hence $\mathcal{P} = \langle z \rangle \times L$ for a subgroup $L$ of $V$ with $u \in L$ by Lemma 5. Since $L_{(0,1)} = 1$ and $u \in L$, $L$ acts faithfully on $T(X)$ and $u$ inverts $T(X)$. Hence $L = \langle u \rangle \times Z$ for a subgroup $Z$ of $L$ by Lemma 5. As $|L| \geq 4$, $Z$ contains an involution. Therefore $Z_{(0,1)} \neq 1$ or $Z_{(y,ox)} \neq 1$, a contradiction. Thus $|V| = 4$.

As $V \leq S_Y$ and $F(V) \cap L = \{X, Y\}$, we have $V = S_Y$. Since $V$ is isomorphic to $Z_2 \times Z_2$ and $C_S(u) = V$, $S$ is dihedral or semidihedral by a lemma of [7]. Therefore any involution in $S$ is $S$-conjugate to an involution in $V$. Hence, if $S_0 \neq 1$ for some $Q \in \Delta$, then $Q = X^s$ or $Y^s$ for some $s \in S$. Thus $|S_0| = |V| = 4$. Therefore $|Q^s| \geq 2|\Gamma|_2/4 = (p+1)_2$ for all $Q \in \Delta$.

**Lemma 7.** $r \neq 3$.

**Proof.** Assume that $r = 3$. Let $t$ be an odd prime dividing $p+1$. Then
is a prime \( p \)-primitive divisor of \( p^2 - 1 \). Let \( R \) be a Sylow \( t \)-subgroup of \( G \). Since \( G \) is transitive on \( \Gamma \), \( p(p^2 - 1) = |\Gamma| \cdot |G| \) and so \( R \neq 1 \). By Lemma 1, \( |M| = p \) as \( r = 3 \). Hence \( R \) centralizes \( M \). Applying Lemma 3, \( R \) acts semi-regularly on \( \Delta \). Since \( t \) is arbitrary, using Lemma 6 we have that \( G \) acts transitively on \( \Delta \). As \( \pi \) is a counterexample, this is a contradiction. Thus we have the lemma.

**Lemma 8.** There exists a prime \( p \)-primitive divisor \( t \) of \( p^{r-1} - 1 \) such that \( t \mid |G| \) and \( t \notdivides |C_G(M)| \).

**Proof.** \( |G| \) is divisible by \( p^{r-1} - 1 \) as \( |\Gamma| \cdot |G| \). By Lemmas 1 and 7, \( r - 1 \geq 3 \) and by Lemma 2, \((p, r-1) \neq (2, 6)\). It follows from a Zsigmondy’s theorem (Theorem 6.2 of [5]) that there exists a prime \( p \)-primitive divisor \( t \) of \( p^{r-1} - 1 \).

Assume \( t \mid |C_G(M)| \) and let \( R \) be a Sylow \( t \)-subgroup of \( C_G(M) \). By Lemma 3, \( R \) is semiregular on \( \Delta \). Hence \( t \mid p+1 \) and so \( t \mid p^2 - 1 \). Since \( t \) is a \( p \)-primitive divisor of \( p^{r-1} - 1 \), we have \( r - 1 = 2 \), contrary to Lemma 7.

**Lemma 9.** Each \( M \)-orbit on \( \Gamma \) is of length \( p \).

**Proof.** Since \( p \mid |\Gamma| \), \( \rho \notdivides |\Gamma| \) and \( M \) is \( \frac{1}{2} \)-transitive on \( \Gamma \), using Lemma 1 each \( M \)-orbit on \( \Gamma \) has length \( p \).

**Proof of Theorem 1.**

Let \( t \) be a prime as in Lemma 8 and let \( R \) be a Sylow \( t \)-subgroup of \( G \). By Lemma 8, \( R \neq 1 \) and acts faithfully on \( M \). Since \( t \) is a \( p \)-primitive divisor of \( p^{r-1} - 1 \), we have \( |M| \geq p^{r+1} \). Hence, by Proposition 6.12 of [3], \( p^r = 16 \). From this, \( p = 2 \), \( r = 7 \) and \( |M| \geq 8 \).

Let \( A \in \Gamma \) and set \( N = M_A \). By Lemma 1, \( F(N) \supseteq \Delta \cup \{A\} \). Therefore \( F(N) \) is a subplane of order 4. Let \( B \in l_w - F(N) \cap l_w \). Clearly \( F(N_B) = \pi \) and so \( N_B = 1 \). By Lemma 9, \( |M : N| = 2 \) and \( |N : N_B| = 2 \). Hence \( |M| = 4 \), a contradiction. Thus we have Theorem 1.

**References**


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