KLEIN BOTTLES IN GENUS TWO 3-MANIFOLDS

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Introduction

For a closed 3-manifold $M$, it is very interesting to study the relation between a Heegaard surface of $M$ and an embedded surface in $M$. For this purpose W. Haken has shown in [2] that if a closed 3-manifold $M$ is not irreducible, then there is an essential 2-sphere in $M$ which intersects a fixed Heegaard surface of $M$ in a single circle, and W. Jaco has given in [4] an alternative proof of it. M. Ochiai has shown in [8] that if a closed 3-manifold $M$ contains a 2-sided projective plane, then there is a 2-sided projective plane in $M$ which intersects a fixed Heegaard surface of $M$ in a single circle, and moreover he has shown in [9] that if a closed 3-manifold $M$ with a Heegaard splitting of genus two contains a 2-sided projective plane, then $M$ is homeomorphic to $P^2 \times S^1$. Successively T. Kobayashi has shown in [5] that if a closed 3-manifold $M$ with a Heegaard splitting of genus two contains a 2-sided non-separating incompressible torus, then there is a 2-sided non-separating incompressible torus in $M$ which intersects a fixed Heegaard surface in a single circle. In this paper we will show a similar result for a Klein bottle.

Theorem 1. Let $M$ be a closed connected orientable 3-manifold with a fixed Heegaard splitting $(V_1, V_2; F)$ of genus two. If $M$ contains a Klein bottle, then there is a Klein bottle in $M$ which intersects $F$ in a single circle.

By the way it is well known that a closed orientable 3-manifold $M$ with a Heegaard splitting of genus one contains a Klein bottle if and only if $M$ is homeomorphic to $L(4n, 2n+1)$ for some non-negative integer $n$ (c.f. [1]). Using Theorem 1 we will give a necessary and sufficient condition for a closed orientable 3-manifold with a Heegaard splitting of genus two to contain a Klein bottle. Namely we will give three families of closed orientable 3-manifolds, and we will show that a closed orientable 3-manifold $M$ with a Heegaard splitting of genus two contains a Klein bottle if and only if $M$ belongs to one of the three families (Theorem 2).

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0. Preliminaries

Throughout this paper, we will work in the piecewise linear category. \( S^n \) and \( P^n \) means the \( n \)-sphere and the real \( n \)-dimensional projective space respectively. \( [0, 1] \). \( Cl(\cdot) \), \( Int(\cdot) \) and \( \partial(\cdot) \) mean the closure, the interior and the boundary respectively. A handlebody of genus \( n \) is defined by disk sum of \( n \)-copies of \( S^1 \times D^2 \) where \( D^2 \) is a 2-disk, and we call a handlebody of genus one a solid torus. A Heegaard splitting of genus \( n \) of a closed orientable 3-manifold \( M \) is a pair \((V_1, V_2; F)\), where \( V_i \) is a handlebody of genus \( n \) \((i=1, 2)\) and \( M = V_1 \cup V_2 \) and \( V_1 \cap V_2 = \partial V_1 = \partial V_2 = F \). Then \( F \) is called a Heegaard surface of \( M \). According to J. Hempel \[3\] we call a closed orientable 3-manifold with a Heegaard splitting of genus one a lens space. A properly embedded surface \( F \) in a 3-manifold \( M \) is essential if \( F \) is incompressible in \( M \) and is not boundary parallel. \( A \# B \) and \( A \approx B \) mean the connected sum of \( A \) and \( B \) and that \( A \) is homeomorphic to \( B \) respectively. Furthermore for the definitions of standard terms in three dimensional topology and knot theory, we refer to \[3\], \[4\] and \[9\]. For the definition of a hierarchy for a 2-manifold and an isotopy of type \( A \), we refer to \[4\].

1. Proof of Theorem 1

**Lemma 1.1.** If a compact orientable 3-manifold \( M \) contains a compressible Klein bottle in \( IntM \), then \( M \cong S^2 \times S^1 \# M' \) or \( M \cong P^3 \# P^3 \# M' \) for some compact orientable 3-manifold \( M' \).

Proof. Let \( K \) be a compressible Klein bottle in \( IntM \), then there is a 2-disk \( D \) in \( IntM \) such that \( D \cap K = \partial D \) and \( \partial D \) is a 2-sided essential simple loop in \( K \). And so there is an embedding \( D \times I \subset IntM \) such that \( D \times \{1/2\} = D \) and \( (D \times I) \cap K = (\partial D \times I) \cap K = \partial D \times I \). By W. Lickorish \[7\] there are following two cases.

Case 1: \( \partial D \) cuts \( K \) into an annulus. Then \((K - \partial D \times I) \cup (D \times \{0, 1\}) = S \) is a non-separating 2-sphere in \( M \), so \( M \cong S^2 \times S^1 \# M' \), because \( K \) is one-sided in \( M \).

Case 2: \( \partial D \) cuts \( K \) into two Möbius bands. Then \((K - \partial D \times I) \cup (D \times \{0, 1\}) = P_0 \cup P_1 \) is a disjoint union of two one-sided projective planes in \( M \), so \( M \cong P^3 \# P^3 \# M' \).

Proof of Theorem 1.

Let \( M \) be a closed orientable 3-manifold with a Heegaard splitting \((V_1, V_2; F)\) of genus two. If \( M \) contains a compressible Klein bottle, then by Lemma 1.1 \( M \cong S^2 \times S^1 \# L \) where \( L \) is a lens space or \( M \cong P^3 \# P^3 \). In the both cases it is clear that \( M \) contains a Klein bottle which intersects either \( V_1 \) or \( V_2 \) in a non-separating disk. Hence we may assume that \( M \) is neither homeomorphic to
Therefore any Klein bottle in $M$ is incompressible. For any Klein bottle in $M$ by thinning $V_1$ enough we may assume that the Klein bottle intersects $V_1$ in disks. Let $K$ be a Klein bottle in $M$ such that among all Klein bottles in $M$ which intersects $V_1$ in disks the number of the components of $K \cap V_1$ is minimal, and put $K_i = V_i \cap K$ ($i = 1, 2$). We may assume that $K_2$ is incompressible in $V_2$ because $K$ is incompressible in $M$. Then as in W. Jaco [4] we have a hierarchy $(K^1_1, \alpha_1), (K^2_2, \alpha_2), \ldots, (K^n_2, \alpha_n)$ for $K_1 = K_2$ which gives rise to a sequence of isotopies in $M$ where the $i$-th isotopy is an isotopy of type A at $\alpha_i$ ($i = 1, 2, \ldots, n$). In addition we may suppose that $\alpha_i \cap \alpha_j = \phi$ ($i \neq j$), so we assume that each $\alpha_i$ is a properly embedded essential arc in $K_2$.

By W. Lickorish [7], each $\alpha_i$ is one of the following five types. We say that $\alpha_i$ is of type I if $\alpha_i$ meets two distinct components of $\partial K_2$, $\alpha_i$ is of type II if $\alpha_i$ meets only one component of $\partial K_2$ and $\alpha_i$ cuts $K_2$ into a planar surface and Klein bottle with hole(s), $\alpha_i$ is of type III if $\alpha_i$ meets only one component of $\partial K_2$ and $\alpha_i$ cuts $K_2$ into an annulus (with holes), $\alpha_i$ is of type IV if $\alpha_i$ meets only one component of $\partial K_2$ and $\alpha_i$ cuts $K_2$ into two Möbius bands (with holes), $\alpha_i$ is of type V if $\alpha_i$ meets only one component of $\partial K_2$ and $\alpha_i$ cuts $K_2$ into a Möbius band (with holes). (Fig. 1.1)

In particular we say that $\alpha_i$ is a $d$-arc if $\alpha_i$ is of type I and there is a component $C$ of $\partial K_2$ such that $\alpha_i \cap C \neq \phi$ and $\alpha_j \cap C = \phi$ for all $j < i$. Put $K_i = D_1 \cup D_2 \cup \ldots \cup D_r$, where $D_i$ is a disk and $C_i = \partial D_i$, so $\partial K_2 = \partial K_1 = C_1 \cup C_2 \cup \ldots \cup C_r$.

Before the proof of Theorem 1 we show some lemmas.

**Lemma 1.2.** Any $\alpha_i$ is not a $d$-arc.

Proof. If some $\alpha_i$ is a $d$-arc, then by using the argument of the inverse
operation of an isotopy of type A defined in M. Ochiai [9] we can show that there is a Klein bottle $K'$ in $M$ such that each component of $K' \cap V_1$ is a disk and the number of the components of $K' \cap V_1$ is less than that of $K \cap V_1$. This is a contradiction.

**Lemma 1.3.** Any $\alpha_i$ is not of type II.

**Proof.** If some $\alpha_i$ is of type II, then by the definition of type II there is an arc $\beta$ in $\partial K_2$ such that $\beta \cap \alpha_i = \partial \beta = \partial \alpha_i$ and $\beta \cup \alpha_i$ bounds a planar surface $P$ in $K_2$. Since each $\alpha_i$ is an essential arc in $K_2$, some $\alpha_i$ in $P$ is a $d$-arc. Hence the conclusion follows from Lemma 1.2.

**Lemma 1.4.** If some $\alpha_i$ which is of type V meets $C_i$, then $D_i$ is a non-separating disk in $V_1$.

**Proof.** By performing an isotopy of type A at $\alpha_i$, we obtain a Möbius band in $V_1$. Since $V_1$ is orientable a Möbius band in $V_1$ is one-sided, and so $D_i$ is non-separating.

**Lemma 1.5.** $\alpha_i$ is of type III, IV or V. Moreover we may suppose without loss of generality that $\alpha_i$ meets $C_1$, and $D_i$ is a non-separating disk in $V_1$.

**Proof.** By lemma 1.2 and lemma 1.3 $\alpha_i$ is of type III, IV or V. Suppose that $\alpha_i$ meets $C_1$. If $\alpha_i$ is of type V then by Lemma 1.4 $D_i$ is a non-separating disk in $V_1$. So we suppose that $\alpha_i$ is of type III or IV and $D_i$ is a separating disk in $V_1$. Let $A_i$ be an annulus in $V_1$ obtained by performing an isotopy of type A at $\alpha_i$ and $K'$ be the image of $K$ after the isotopy. Then $K' \cap V_1 = A_1 \cup D_2 \cup \ldots \cup D_r$ and there is an annulus $A'$ in $\partial V_1$ such that $K' \cap A' = A_1 \cap A' = \partial A_1 = \partial A'$. Let $K'' = (K' - A_1) \cup A'$, then $K''$ is a Klein bottle in $M$ and by pushing $A'$ into $V_2$ we obtain a Klein bottle $K$ from $K''$ such that each component of $K \cap V_1$ is a disk and the number of the components of $K' \cap V_1$ is less than that of $K \cap V_1$. This is a contradiction. Therefore $D_i$ is a non-separating disk in $V_1$.

Now by Lemma 1.2 and Lemma 1.3 $\alpha_3$ is of type III, IV or V.

Case 1: $\alpha_3$ is of type III or IV.

At first let $\alpha_2$ be of type III or IV. If $\alpha_2$ also meets $C_1$, then there are two arcs $\beta_1$, $\beta_2$ in $C_1$ such that $\partial (\beta_1 \cup \beta_2) = \partial (\alpha_1 \cup \alpha_2)$ and $(\beta_1 \cup \alpha_2) \cup (\beta_2 \cup \alpha_2)$ bounds a planar surface in $K_2$, so there is a $d$-arc $\alpha_j$ for some $j \geq 3$. Therefore, by Lemma 1.2, $\alpha_3$ meets only $C_2$. Let $K^1$ be the image of $K$ after an isotopy of type A at $\alpha_3$ and $K^2$ be the image of $K^1$ after an isotopy of type A at $\alpha_2$. Then $K^2 \cap V_1 = A_1 \cup A_2 \cup D_2 \cup \ldots \cup D_r$, where $A_i$ is an essential annulus properly embedded in $V_1 (i = 1, 2)$. By cutting $V_1$ along a disk $D$ parallel to $D_2$ missing $A_1 \cup A_2$ we obtain a solid torus $V$ containing $A_1 \cup A_2$. (Fig. 1. 2).

So we obtain an annulus $A'$ in $\partial V$ missing the image of $D$, so in $\partial V_i$, such
that $A_i \cap A' = a$ component of $\partial A_i = a$ component of $\partial A'$ ($i = 1, 2$) and $K^2 \cap A' = \partial A'$. By cutting $K$ along $A'$ and pasting $A'$ to the boundaries of the suitable component(s), we obtain a Klein bottle $K'$ such that $K' \cap V_1 = A'' \cup D_{i_1} \cup \ldots \cup D_{i_p}$ ($p \leq r - 2$) where $A''$ is an annulus and $\{D_{i_1}, \ldots, D_{i_p}\}$ is a subset of $\{D_1, \ldots, D_r\}$. In the case that $A''$ is boundary parallel, then by pushing $A''$ into $V_2$ we obtain a Klein bottle which intersects $V_1$ in $p$ disks. In the case that $A''$ is essential, then by performing an isotopy of type A we obtain a Klein bottle which intersects $V_1$ in $p+1$ disks. This is a contradiction. Therefore $\alpha_2$ must be of type V. By Lemma 1.4 and Lemma 1.5 $\alpha_2$ must meet $C_1$ and $r = 1$. This completes the proof of Case 1.

Case 2: $\alpha_1$ is of type V.

At first let $\alpha_2$ be of type III or IV. If $\alpha_2$ also meets $C_1$ and $\alpha_2$ is of type III, then $\alpha_1 \cup \alpha_2$ cuts $C_l(K - D_1)$ into a disk, and so $r = 1$ by Lemma 1.2. If $\alpha_2$ also meets $C_1$ and $\alpha_2$ is of type IV, then by Lemma 1.2 $\alpha_2$ is an inessential arc in $K^2_2$ where $K^2_2$ is a surface obtained by cutting $K^2_2 = K \cap V_2$ along $\alpha_1$. This is a
contradiction. Therefore $\alpha_2$ meets only $C_2$ and is of type IV. Let $A_1$ be a Möbius band obtained by an isotopy of type A at $\alpha_1$, and $A_2$ be an annulus obtained by an isotopy of type A at $\alpha_2$. If there is a properly embedded 2-disk $D$ in $V_1$ such that $D$ cuts $V_1$ into two solid tori $T_1$ and $T_2$ and $A_i$ is properly embedded in $T_i$ ($i=1, 2$). (Fig 1.3)

Then by the argument of Lemma 1.5 we obtain a Klein bottle $K'$ such that each component of $K' \cap V_1$ is a disk and the number of the components of $K' \cap V_1$ is less than that of $K \cap V_1$. This is a contradiction. Hence there is a non-separating 2-disk $D$ properly embedded in $V_1$ with $D \cap A_i = \emptyset$ ($i=1, 2$). (Fig. 1.4)

Let $T$ be a solid torus obtained by cutting $V_1$ along $D$. Since $\partial A_1$ and $\partial A_2$ are mutually parallel simple loops in $\partial T$, there is an annulus $A'$ in $\partial_i T$ missing the image of $D$, so in $\partial_i V_1$, such that $A_1 \cap A' = \partial A_1 = a$ component of $\partial A'$ and $A_2 \cap A' = a$ component of $\partial A_2 = a$ component of $\partial A'$. By cutting $K$ along $\partial A'$ and pasting $A'$ to the boundaries of the suitable components we obtain a Klein bottle $K'$ such that $K' \cap V_1 = S \cup D_{1} \cup \ldots \cup D_p$, $p \leq r-2$ where $S$ is a Möbius band and $\{D_1, \ldots, D_p\}$ is a subset of $\{D_3, \ldots, D_r\}$. Then by performing an isotopy of type A we obtain a Klein bottle which intersects $V_1$ in $p+1$ disks. This is a contradiction.

Secondly let $\alpha_2$ be of type V. If $\alpha_2$ also meets $C_1$ then we have the following two cases.

Case (a): Each component of $C_1 \cup \partial \alpha_1$ contains one point of $\partial \alpha_2$.

Case (b): $\partial \alpha_2$ is contained in a component of $C_1 \cup \partial \alpha_1$.

If Case (a) holds, then by Lemma 1.2 $\alpha_2$ is an inessential arc in $K_2$ where $K_2$ is a surface obtained by cutting $K_{1} = K \cap V_1$ along $\alpha_1$. This is a contradiction.

If Case (b) holds, then $\alpha_1 \cup \alpha_2$ cuts $Cl(K-D_1)$ into a disk, so $r=1$ by Lemma 1.2.

If $\alpha_2$ meets only $C_2$, then $\alpha_3$ meets $C_1$, $C_2$, or $C_3$. If $\alpha_3$ meets only $C_3$, then $\alpha_3$ must be of type IV. By a similar argument of the first case of Case 2, we get
Fig. 1.4

a contradiction. If $\alpha_3$ meets either only $C_1$ or only $C_2$, then $\alpha_3$ is an inessential arc in $K^2$. Hence $\alpha_3$ is of type I and meets both $C_1$ and $C_2$. Let $K'$ be the image of $K$ after a sequence of isotopies of type A at $\alpha_1$, at $\alpha_2$ and at $\alpha_3$. Then $K' \cap V_2$ is a single disk. This completes the proof.

2. Statement and proof of Theorem 2

Let $K$ be a Klein bottle and $KI$ be the (orientable) twisted $I$-bundle over $K$. Then $KI$ admits two Seifert fibrations $\mathcal{F}_1, \mathcal{F}_2$ where the orbit manifold of $\mathcal{F}_1$ is a disk with two exceptional points of each index 2, and the orbit manifold of $\mathcal{F}_2$ is a Möbius band without exceptional points. (see Ch. VI of W. Jaco [4]). Let $\alpha$ be a fiber of $\mathcal{F}_1$ in $\partial KI$ and $\beta$ be a fiber of $\mathcal{F}_2$ in $\partial KI$. In the following we give three families of closed orientable 3-manifolds containing a Klein bottle.

$C(1)$: Let $M(k)$ be a two bridge knot exterior in $S^3$ where $k$ is a two bridge knot (possibly trivial) (c.f. Ch.4 of D. Rolfsen [10]). Let $\mu_1, \mu_2$ be two disjoint meridians of $k$ in $\partial M(k)$ and $\bar{\mu}_1, \bar{\mu}_2$ be two disjoint simple loops in $Int M(k)$ obtained by pushing $\mu_1$ and $\mu_2$ into $Int M(k)$. Let $M_1$ be a 3-manifold obtained
from $M(k)$ by performing arbitrary Dehn surgeries on $M(k)$ along $\partial_1$ and $\partial_2$. Then $C(1)$ is the family which consists of all 3-manifolds obtained from $M_1$ and $K^I$ by identifying $\partial K^I$ with $\partial M_1$ by a homeomorphism which takes $\beta$ to $\mu_1$.

$C(2)$: Let $M(k)$, $\mu$, and $\partial_1$ be a two bridge knot exterior, a meridian of $k$ in $\partial M(k)$ and a simple loop in $\text{Int} M(k)$ as in $C(1)$ respectively. Let $M_2$ be a 3-manifold obtained from $M(k)$ by performing an arbitrary Dehn surgery on $M(k)$ along $\partial_1$. Then $C(2)$ is the family which consists of all 3-manifolds obtained from $M_2$ and $K^I$ by identifying $\partial K^I$ with $\partial M_2$ by a homeomorphism which takes $\alpha$ to $\mu_1$.

$C(3)$: Let $L=V_1 \cup V_2$ be a lens space where $V_i$ is a solid torus ($i=1, 2$) and $V_1 \cap V_2 = \partial V_1 = \partial V_2$. Let $L(k)$ be a one bridge knot exterior in $L$ (i.e. $k$ is a simple loop in $L$ and for $i=1, 2$ ($V_i, V_i \cap k$) is homeomorphic to ($A \times I$, $\{p\} \times I$) as pairs where $A$ is an annulus and $p$ is a point in $\text{Int} A$). Let $\mu$ be a meridian of $k$ in $\partial L(k)$. Then $C(3)$ is the family which consists of all 3-manifolds obtained from $L(k)$ and $K^I$ by identifying $\partial K^I$ with $\partial L(k)$ by a homeomorphism which takes $\alpha$ to $\mu$.

**Theorem 2.** Let $M$ be a closed connected orientable 3-manifold with a Heegaard splitting of genus two. Then $M$ contains a Klein bottle if and only if $M$ belongs to one of $C(1)$, $C(2)$ or $C(3)$.

For the proof of Theorem 2 we prepare the following two Lemmas.

**Lemma 2.1** (Lemma 3.2 of T. Kobayashi [6]). Let $V$ be a handlebody of genus two and $A$ be a non-separating essential annulus properly embedded in $V$. Then $A$ cuts $V$ into a handlebody $V'$ of genus two and there is a complete system of meridian disks $\{D_1, D_2\}$ of $V'$ such that $D_1 \cap A$ is an essential arc of $A$. (Fig. 2.1)

![Fig. 2.1](image)

**Lemma 2.2.** Let $S$ be a Möbius band properly embedded in a handlebody $V$ of genus $n$. Then there is a 2-disk $D$ properly embedded in $V$ which cuts $V$ into $V_1$ and $V_2$ where $V_1$ is a solid torus and $V_2$ is a handlebody of genus $n-1$ and $S$ is
properly embedded in $V_1$.

Proof. Since Möbius band can not be properly embedded in a 3-ball, by using a complete system of meridian disks in $V$, we can find a non-separating disk $D_1$ properly embedded in $V$ such that $D_1 \cap S \neq \emptyset$ and there is a component $\alpha$ of $D_1 \cap S$ which is an essential arc in $S$ and is innermost in $D_1$. Therefore there is a 2-disk $D_2$ in $D_1$ such that $\partial D_1 \cap D_2 = \beta$ is an arc and $\alpha \cap \beta = \partial \alpha = \partial \beta$ and $\alpha \cup \beta = \partial D_2$. Then there is a proper embedding $D_2 \times I \subset V$ such that $D_2 \times \{1/2\} = D_2$ and $(D_2 \times I) \cap S = \alpha \times I$. Let $D_3 = (S - (\alpha \times I)) \cup (D_2 \times \{0\}) \cup (D \times \{1\})$. Since $S$ is one-sided in $V$, $D_3$ is a non-separating disk properly embedded in $V$. (Fig. 2.2)

![Fig. 2.2](image)

Let $S_1 = D_3 \cup (\beta \times I)$, then $S_1$ is a Möbius band and $S$ is obtained by pushing $S_1$ slightly into $\text{Int} V$. Let $N$ be a regular neighborhood of $S_1$ in $V$, then $N$ is a solid torus and $S$ may be supposed to be properly embedded in $N$. Therefore $D = \text{Cl}(\partial N - \partial V)$ is the 2-disk satisfying the conditions of this Lemma.

Proof of Theorem 2.

Let $(V_1, V_2; F)$ be a Heegaard splitting of genus two of $M$. If $M$ contains a compressible Klein bottle, then by Lemma 1.1 $M \approx S^2 \times S^1 \# L$ where $L$ is a lens space or $M \approx P^3 \# P^3$. If $M \approx S^2 \times S^1 \# L$, then $M$ belongs to $C(3)$ because $S^2 \times S^1$ is obtained from $K \times I$ and a solid torus by identifying their boundaries by some homeomorphism. If $M \approx P^3 \# P^3$, then $M$ belongs to $C(2)$ by the same reason as above. If $M$ contains an incompressible Klein bottle, then by Theorem 1 we can suppose without loss of generality that there exists a Klein bottle $K$ in $M$ which intersects $V_1$ in a non-separating disk. For $i=1, 2$ put $K_i = K \cap V_i$ then $K_1$ is a non-separating disk in $V_1$ and $K_2$ is a Klein bottle with one hole in $V_2$. Let $\alpha$ be an essential arc in $K_2$ which gives rise to an isotopy of type A at $\alpha$ and $\tilde{K}$ be the image of $K$ after an isotopy of type A at $\alpha$ and put $K_i = \tilde{K} \cap V_i$ ($i=1, 2$). Then we have the following three cases.
Case (1): $\alpha$ is of type III. For $i=1, 2 \, K_i$ is a non-separating essential annulus in $V_i$. So by using a similar argument of §4 of T. Kobayashi [5] and noting Lemma 2.1, we can show that $M$ belongs to $C(1)$.

Case (2): $\alpha$ is of type IV. $K_1$ is a non-separating essential annulus in $V_1$ and $K_2$ is a disjoint union of two Mobius bands in $V_2$. So by using a similar argument of §4 of T. Kobayashi [5] and noting Lemma 2.1 and Lemma 2.2, we can show that $M$ belongs to $C(2)$.

Case (3): $\alpha$ is of type V. For $i=1, 2 \, K_i$ is a Mobius band in $V_i$. So by using a similar argument of §4 of T. Kobayashi [5] and noting Lemma 2.2, we can show that $M$ belongs to $C(3)$.

Conversely if $M$ belongs to one of $C(1)$, $C(2)$ or $C(3)$, then by tracing back the above procedure it is easy to see that $M$ has a Heegaard splitting of genus two and contains a Klein bottle. This completes the proof.

Remarks.

(1) In the case that $M$ is irreducible and has a non-trivial torus decomposition and has a Heegaard splitting of genus two, then $M$ is completely characterized by T. Kobayashi [6].

(2) In the case that $M$ is connected sum of two lens spaces $L_1$ and $L_2$ and contains a Klein bottle, then it is easily checked that either $L_1$ or $L_2$ is homeomorphic to $L(4n, 2n+1)$ for some non-negative integer $n$ or both $L_1$ and $L_2$ are homeomorphic to $P^3$.

References


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