0. Introduction

Let $G$ be a compact, simply connected, simple Lie group of rank $l$. $G$ has $l$ irreducible representations $\rho_1, \ldots, \rho_l$, whose highest weights are the fundamental weights $\omega_1, \ldots, \omega_l$, respectively (see [19]). Then the representation ring $R(G)$ of $G$ is a polynomial algebra $\mathbb{Z}[\rho_1, \ldots, \rho_l]$. By the theorem of Hodgkin [16], the $\mathbb{Z}/2$-graded $K$-theory $K^*(G)$ of $G$ is an exterior algebra $A(z_1, \ldots, z_l)$, where $\beta: R(G) \to K^*(G)$ is the map introduced in [16]. Therefore the Chern character $ch: K^*(G) \to H^*(G; \mathbb{Q})$ is injective [5]. We may write

$$H^*(G; \mathbb{Q}) = \Lambda_0(x_{2m_1-1}, x_{2m_2-1}, \ldots, x_{2m_l-1})$$

where $2 = m_1 \leq m_2 \leq \cdots \leq m_l$ and $\deg x_{2m_j-1} = 2m_j - 1$. If each $x_{2m_j-1}$ is chosen to be integral and not divisible by any other integral classes, we can assign to a representation $\lambda: G \to U(n)$ the rational numbers $a(\lambda, 1), \ldots, a(\lambda, l)$ by the equation

$$ch\beta(\lambda) = \sum_{j=1}^l a(\lambda, j)x_{2m_j-1}. \tag{1}$$

In view of [21] and [23], the $a(\lambda, j)$ are closely related to the Dynkin coefficients of $\lambda$ [14]. On the other hand, as is noted by Atiyah [4, Proposition 1], the determinant of the $l \times l$ matrix $(a(\rho_i, j))$ is equal to 1. We remark that for any system of generators $\{\lambda_1, \ldots, \lambda_l\}$ of the ring $R(G)$, the determinant of $(a(\lambda_i, j))$ is also 1.

In this paper, with a suitable system of generators of $R(G)$, we shall describe the resulting matrix explicitly for the groups $G$ with $l \leq 4$ without using the above informations. Indeed, we deal with the following cases:

- $l = 2$, $G = SU(3), \text{Sp}(2), G_2$.
- $l = 3$, $G = SU(4), \text{Spin}(7), \text{Sp}(3)$.
- $l = 4$, $G = SU(5), \text{Spin}(9), \text{Sp}(4), \text{Spin}(8), F_4$. 
Results are stated in Theorems 2 (SU(/+1)), 3 (Sp(/)), 4 (Spin(7)), 5 (Spin(8)), 6 (Spin(9)), 7 (G2) and 8 (F4).

The careful reader should notice that "up to sign" is implicitly added to some of the statements of this paper.

For later use we fix some notations. Let \( T \) be a maximal torus of \( G \). The inclusion \( i: T \rightarrow G \) induces a map of classifying spaces \( \rho = Bi: BT \rightarrow BG \). The action of the normalizer \( N_G(T) \) on \( T \) induces that of the Weyl group \( \Phi(G) = N_G(T)/T \) on \( BT \) and hence on \( H^*(BT; \mathbb{Z}) = \mathbb{Z}[\omega_1, \ldots, \omega_l] \) (see [9]). Let \( H^*(BT; \mathbb{Z})^{\Phi(G)} \) denote the module of \( \Phi(G) \)-invariants. For a based space \( X \), let \( \Omega X \) be its loop space, and let \( \sigma^*: H^i(X; \mathbb{Z}) \rightarrow H^{i-1}(\Omega X; \mathbb{Z}) \) be the cohomology suspension. For the rational cohomology, by [8] and [10] we have

\[
\text{Imp}^* = H^*(BT; \mathbb{Q})^{\Phi(G)} = \mathbb{Q}[f_{2m_1}, \ldots, f_{2m_j}]
\]

where all the generators, whose degrees are indicated by a subscript, are chosen to be integral and not divisible by any other integral classes.

The paper is organized as follows. The key point of our work is to characterize the generator \( x_{2m_j-1} \). For this purpose we present two methods in Section 1: in the first method we characterize the generator \( y_{2m_j} \) and relate it to \( x_{2m_j-1} \); in the second method we characterize the generator \( u_{2m_j-2} \) and relate it to \( x_{2m_j-1} \). Moreover in Section 1 we prove a lemma which is very useful if the \( \lambda \)-ring structure of \( R(G) \) is known. Subsequent sections are devoted to practical computations. In Section 2 we treat the most elementary cases, i.e., \( G = SU(l+1), \ Sp(l) \) \( (l=2, 3, 4) \) where \( H^*(G; \mathbb{Z}) \) has no torsion. In Section 3 we consider the cases \( G = \text{Spin}(m) \) \( (m = 7, 8, 9) \) where \( H^*(G; \mathbb{Z}) \) has only 2-torsion. In Section 4 we discuss the cases \( G = G_2 \) and \( G = F_4 \).

I would like to thank my colleague H. Minami for showing me a computation of \( (a(\mu, j)) \) for the case \( G = G_2 \) and many helpful suggestions.

1. Methods

Method 1

For any group \( H \) let \( \alpha: R(H) \rightarrow K^*(BH) \) be the homomorphism of [5]. Let \( \sigma: K^i(X) \rightarrow K^{i-1}(\Omega X) \) be the suspension map. Then there is a commutative diagram
where \( \tau \) (resp. \( \tau' \)) is the cohomology transgression in the Serre spec

ence of the universal fibration \( G \to EG \to BG \) (resp. the fibration \( G \to G/T \to BT \)). For

\( j = 1, \ldots, l \) we may set (modulo decomposables)

\[
\sigma^*(y_{2m_j}) = b(m_j)x_{2m_j-1} \quad \text{for some} \quad b(m_j) \in \mathbb{Z}
\]

and

\[
\rho^*(y_{2m_j}) = c(m_j)f_{2m_j} \quad \text{for some} \quad c(m_j) \in \mathbb{Z}.
\]

Since \( \sigma^* \) and \( \tau \) are inverse to each other insofar as they are defined, it follows that

\[
\tau'(x_{2m_j}) = \frac{c(m_j)}{b(m_j)}f_{2m_j} + \text{decomposables}
\]

in \( H^*(BT; \mathbb{Q})^{*,(\ell)} = \mathbb{Q}[f_{2m_1}, \ldots, f_{2m_l}] \).

Let \( \lambda: G \to U(n) \) be a representation with weights \( \mu_1, \ldots, \mu_n \). So

\[
ch\alpha^*(\lambda) = \sum_{i=1}^{n} \exp(\mu_i) = \sum_{x \in \mathbb{Z}} \sum_{i=1}^{n} \mu_i^x / m!
\]

where \( \mu_i \in H^1(BT; \mathbb{Z}) \) (see [9]). Set

(1.1) \[ ch\beta(\lambda) = \sum_{j=1}^{l} a(\lambda, j)x_{2m_j-1} \quad \text{where} \quad a(\lambda, j) \in \mathbb{Q} \, . \]

Apply \( \tau' \) to this equation. Then the left hand side becomes

\[
\tau'ch\beta(\lambda) = \rho^*\tau ch\sigma\alpha(\lambda) = \rho^*\tau c(\lambda) \]

and the right hand side becomes

\[
\tau'(\sum_{j=1}^{l} a(\lambda, j)x_{2m_j-1}) = \sum_{j=1}^{l} a(\lambda, j)\tau'(x_{2m_j})
\]

\[
= \sum_{j=1}^{l} a(\lambda, j)c(m_j)f_{2m_j} + \text{decomposables}.
\]

Hence
This argument shows that, in order to compute $a(\lambda, j)$, it suffices to settle $f_{2m_j}$, determine $b(m_j)$, $c(m_j)$ and find the coefficients of $f_{2m_j}$ in the expression of $\text{ch}^\bullet(\lambda)$ as a polynomial of the $f_{2m_j}$. We will use this method in all cases that concern us.

Remark. In general we choose the $f_{2m_j}$ as follows. Let $\{f_{2m_1}, \ldots, f_{2m_l}\}$ be a system of generators of the ring $H^*(BT; Q)^{\Phi(G)}$. First we take

$$f_{2m_j} = b_j f_{2m_j} \in H^{2m_j}(BT; Q)^{\Phi(G)}, \quad b_j \in Q,$$

so that

(i) $f_{2m_j}$ is integral;
(ii) for any $b \in Q$ with $|b| < |b_j|$, $bf_{2m_j}$ cannot be integral.

Assume inductively that we have chosen $f_{2m_1}, \ldots, f_{2m_{j-1}}$. Then we take

$$f_{2m_j} = b_j f_{2m_j} + \text{decomposables} \in H^{2m_j}(BT; Q)^{\Phi(G)}, \quad b_j \in Q,$$

so that

(i) $f_{2m_j}$ is integral;
(ii) for any $b \in Q$ with $|b| < |b_j|$, $bf_{2m_j} + \text{decomposables} \in H^{2m_j}(BT; Q)^{\Phi(G)}$

cannot be integral.

Note that the choice of the $f_{2m_j}$ has no crucial influence on that of the $f_{2m_j}$. As will be seen in Sections 3 and 4, this settlement of the $f_{2m_j}$ is not trivial but important.

Method II

There is a commutative diagram

$$R(G) \xrightarrow{\beta} K^*(G) \xrightarrow{\text{ch}} H^*(G; Q) \xrightarrow{\sigma} K^*(\Omega G) \xrightarrow{\text{ch}} \Phi(G; Q)$$

which is natural with respect to group homomorphisms. For $j=1, \ldots, l$ we may set

$$\sigma^*(x_{2m_j-1}) = d(m_j)u_{2m_j-2} \quad \text{for some} \quad d(m_j) \in \mathbb{Z}.$$

Applying $\sigma^*$ to (1.1), we have

$$\text{ch}\sigma \beta(\lambda) = \sum_{j=1}^l a(\lambda, j)d(m_j)u_{2m_j-2}.$$

Let us now consider the case $G=SU(n+1)$; then $m_j = j+1$ for $j=1, \ldots, n$ and
where $P$ denotes the primitive module functor. Furthermore, $d(j+1)=1$ for all $j$ (e.g., see [28, Lemma 3]). Let $\lambda_i: SU(n+1)\rightarrow U(n+1)$ be the natural inclusion, and consider the case $\lambda=\lambda_i$. Then it follows from (2.2) of the next section on that

\begin{equation}
\text{char}^{\beta}(\lambda_i) = \sum_{i=1}^{\lambda} (-1)^{\frac{j}{i!}} u_{2i}.
\end{equation}

We return to the general case. Take the inclusion $k: U(n)\rightarrow SU(n+1)$ such that $SU(n+1)/U(n)=\mathbb{C}P^*$ (see [12, §3]). In [28] it was shown that for the composite

\[
\begin{array}{ccc}
PH^*(\Omega SU(n+1); Z) & \xrightarrow{(\Omega k)^*} & PH^*(\Omega U(n); Z) \\
\xrightarrow{(\Omega \lambda)^*} & & \xrightarrow{PH^*(\Omega G; Z)} = Z\{u_{2m_1-2}, \ldots, u_{2m_j-2}\},
\end{array}
\]

the following statements are equivalent:

(i) $(\Omega \lambda)^*(\Omega k)^*(u_{2m_j-2})=e(\lambda, j)u_{2m_j-2}$ for some $e(\lambda, j)\in\mathbb{Z};$

(ii) the element $\theta_j(c_m(\lambda))\in H^{2m-r-2}(G/C_i; Z)$ is exactly divisible by $e(\lambda, j)\in\mathbb{Z}$ (where $H^*(G/C_i; Z)$ has no torsion; for notations and details see [28, §2]).

Applying $(\Omega \lambda)^*(\Omega k)^*$ to (1.2), we have

\[
\text{char}^{\beta}(\lambda) = \sum_{j=1}^{\lambda} (-1)^{\frac{m_j-1}{(m_j-1)!}} e(\lambda, j) u_{2m_j-2}.
\]

Hence

\[
a(\lambda, j)d(m_j) = (-1)^{m_j-1}e(\lambda, j) (m_j-1)!.
\]

This argument shows that, in order to compute $a(\lambda, j)$, it suffices to determine $d(m_j)$ and $e(\lambda, j)$. In particular, to find $e(\lambda, j)$ one must examine the divisibility of $\theta_j(c_m(\lambda))$ in $H^{2m-r-2}(G/C_i; Z)$.

Define a map $\varphi: \mathbb{Z}_+\times\mathbb{Z}_+\times\mathbb{Z}_+\rightarrow\mathbb{Z}$ by

\[
\varphi(n, k, q) = \sum_{i=1}^{q} (-1)^{i-1} \binom{n}{k-i} i^{i-1}
\]

where $\mathbb{Z}_+$ denotes the set of positive integers and we use the convention that $\binom{x}{y}=0$ if $y<0$ or $x<y$. Let $\Lambda^k: R(G)\rightarrow R(G)$ be the $k$-th exterior power operation. Then we have

**Lemma 1.** If $\lambda$ is a representation of $G$ of dimension $n$, then
\[ a(\Lambda^k \lambda, j) = \varphi(n, k, m_j)a(\lambda, j) \]

for \( j = 1, \ldots, l \).

Proof. Let \( ch^k \) be the \( 2q \)-th component of \( ch \), i.e., \( ch(x) = \sum_{i \geq 0} ch^i(x) \) with \( ch^i(x) \in H^{2i}(X; \mathbb{Q}) \) for any \( x \in K^0(X) \). Consider the element \( 1_n \in R(U(n)) \) which comes from the identity \( 1_{U(n)}: U(n) \to U(n) \). Then we assert that

\[ \begin{align*}
\chi^k(\Lambda^k 1_n) &= \varphi(n, k, q)ch^k(1_n) + \text{decomposables} \\
&\quad \text{in } H^*(BU(n); \mathbb{Q}) = \mathbb{Q}[y_2, y_4, \ldots, y_{2n}] .
\end{align*} \]

This assertion implies the result. For since \( \beta = \sigma \alpha \) and \( \sigma^* \) sends a decomposable element into zero, applying \( \sigma^* \) to (1.3) yields the desired result for the case \( G = U(n) \). Then the general case follows from naturality.

To prove (1.3) we proceed by induction on \( k \). The case \( k = 1 \) is clear. Suppose that it is true for \( k \leq m-1 \), and consider the case \( k = m \). Let us recall the following relations:

\[ \psi(x) + \sum_{i=1}^{k-1} (-1)^i \psi^{k-i}(x) \Lambda^i(x) + (-1)^k k \Lambda^k(x) = 0 ; \]

\[ \chi^k(xy) = \sum_{i=0}^{k} \chi^i(x) \chi^{k-i}(y) ; \]

\[ \chi^k(x) = k^k \chi^k(x) \]

where \( x, y \in K^0(X) \) [1]. Since \( \alpha \) is a \( \lambda \)-ring homomorphism, we have

\[ \chi^k(\alpha^m(1_n)) \]

\[ = \chi^k(\alpha((-1)^{m-1} \psi^m(1_n) + \sum_{i=1}^{m-1} (-1)^{m-1-i} \psi^{m-i}(1_n) \Lambda^i(1_n))) \]

\[ = (-1)^{m-1} \chi^k \alpha \psi^m(1_n) + \sum_{i=1}^{m-1} (-1)^{m-1-i} \chi^k(\alpha \psi^{m-i}(1_n) \alpha \Lambda^i(1_n)) \]

\[ = (-1)^{m-1} \chi^k \alpha \psi^m(1_n) + \sum_{i=1}^{m-1} (-1)^{m-1-i} \left[ \sum_{r=0}^{m} \chi^r \alpha \psi^{m-i}(1_n) \chi^{s-r} \alpha \Lambda^i(1_n) \right] \]

modulo decomposables

\[ = (-1)^{m-1} \chi^k \psi^m(1_n) + \sum_{i=1}^{m-1} (-1)^{m-1-i} \left[ \binom{n}{i} \chi^i \alpha \psi^{m-i}(1_n) + n \chi^i \alpha \Lambda^i(1_n) \right] \]

\[ + n \varphi(n, i, q) \chi^i \alpha(1_n) \]

\[ = \left[ \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{n}{i} (m-i)^q + \sum_{i=1}^{m-1} (-1)^{m-1-i} n \varphi(n, i, q) \right] \chi^i \alpha(1_n) \]
\[ \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{m-j} j^{s} + n \sum_{i=1}^{m-1} (-1)^{m-1-i} \varphi(n, i, q) \text{ch}^s \alpha(1_n). \]

Thus it is sufficient to prove that

\[ \varphi(n, m, q+1) + n \sum_{i=1}^{m-1} (-1)^{m-1-i} \varphi(n, i, q) = m \varphi(n, m, q). \]

From Pascal's triangle

\[ \binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1} \]

we deduce that

\[ \sum_{j=0}^{k-1} (-1)^j \binom{n}{i} = (-1)^{k-1-j} \binom{n-1}{k-1-j}. \]

Using this, we have

\[
\begin{align*}
\varphi(n-1, m-1, q) &= \sum_{j=1}^{n-1} (-1)^{j-1} \binom{n-1}{m-1-j} j^{s-1} \\
&= \sum_{j=1}^{n-1} [(-1)^m \sum_{i=0}^{j-1} (-1)^i \binom{n}{i}] j^{s-1} \\
&= \sum_{j=1}^{n-1} (-1)^{m-1-i} \sum_{i=1}^{j} (-1)^{i-1} \binom{n}{i-j} j^{s-1} \\
&= \sum_{j=1}^{n-1} (-1)^{m-1-i} \varphi(n, i, q).
\end{align*}
\]

Therefore

\[
\begin{align*}
n \varphi(n-1, m-1, q) + \varphi(n, m, q+1) \\
&= n \sum_{j=1}^{n-1} (-1)^{j-1} \binom{n-1}{m-1-j} j^{s-1} + \sum_{j=1}^{n-1} (-1)^{j-1} \binom{n}{m-j} j^{s} \\
&= \sum_{j=1}^{n} (-1)^{j-1} n \binom{n-1}{m-1-j} j^{s-1} + \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{m-j} j^{s} \\
&= \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{m-j} (m-j) j^{s-1} + \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{m-j} j^{s} \\
&= \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{m-j} m j^{s-1} - \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{m-j} j^{s} \\
&+ \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{m-j} j^{s} \\
&= m \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{m-j} j^{s-1} + (-1)^{m-1} \binom{n}{0} m^s \\
&= m \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{m-j} j^{s-1} \\
&= m \varphi(n, m, q).
\end{align*}
\]

This proves (1.4) and completes the proof.
2. The special unitary groups and the symplectic groups

Let us first consider the case of $SU(l+1)$. In this case, $m_j = j+1$ for $j=1, \ldots, l$. As is well known we can choose elements $t_1, t_2, \ldots, t_{l+1} \in H^*(BT; \mathbb{Z})$ so that

$$H^*(BT; \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_{l+1}] / (c_1)$$

and

$$H^*(BT; \mathbb{Z})^{\Phi(SU(l+1))} = \mathbb{Z}[c_2, \ldots, c_{l+1}]$$

where $c_i = \sigma_i(t_1, \ldots, t_{l+1})$ ($\sigma_i(\cdot)$ denotes the $i$-th elementary symmetric function). It is evident that $f_{2j+2} = c_{j+1}$ for $j=1, \ldots, l$. Since $H^*(SU(l+1); \mathbb{Z})$ has no torsion, the theorem of Borel [6] assures us that $b(j+1) = c(j+1) = 1$ for all $j$. Thus we have $\tau'(x_{2j+1}) = c_{j+1}$ for $j=1, \ldots, l$.

Let us recall from [17] that

(2.1) $R(SU(l+1)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_l]$ where

(a) $\dim \lambda_k = \binom{l+1}{k}$;
(b) relations $\Lambda^k \lambda_1 = \lambda_k$ hold;
(c) the set of weights of $\lambda_i$ is given by $\{t_i | 1 \leq i \leq l+1\}$.

Put

$$s_m = s_m(t_1, \ldots, t_{l+1}) = \sum_{i=1}^{l+1} t_i^m.$$

From Newton's formula

$$s_m \left[ \sum_{i=1}^{l+1} (-1)^i s_{m-i} + (-1)^m mc_m \right] = 0$$

(where $c_m = 0$ if $m > l+1$) it follows that

$$ch\tau^*(\lambda_i) = l+1 + \sum_{m=1}^{l} \frac{(-1)^m}{m!} c_{m+1} + \text{decomposables.}$$

Therefore

(2.2) $ch\beta(\lambda_i) = \sum_{m=0}^{l} \frac{(-1)^m}{m!} x_{2m+1}$

(cf. [20, Theorem 1]). By Lemma 1, if we evaluate $\varphi(l+1, k, j+1), ch\beta(\lambda_k)$ can be calculated. Thus we have

**Theorem 2.** The Chern characters on $SU(l+1)$ for $l=2,3,4$ are given by:

- For $l=2$
  
  $$ch\beta(\lambda_1) = -x_3 + (1/2!) x_5$$
  $$ch\beta(\lambda_2) = -x_3 + (-1/2!) x_5$$

1
\[
\begin{align*}
l = 3 & \quad \text{ch}_3(\lambda_j) = -x_3 + (1/2!)x_5 + (-1/3!)x_7 \\
& \quad \text{ch}_2(\lambda_j) = -2x_3 \\
& \quad \text{ch}_1(\lambda_j) = -x_3 + (-1/2!)x_5 + (-1/3!)x_7 \\
& \quad \text{ch}_0(\lambda_j) = -x_3 + (1/2!)x_5 + (-1/3!)x_7
\end{align*}
\]

where the number on the right hand side indicates the determinant of the corresponding matrix on the left hand side.

Let us consider the case of \( Sp(l) \). In this case, \( m_j = 2j \) for \( j = 1, \ldots, l \). We can choose elements \( t_1, t_2, \ldots, t_l \in H^2(BT; Z) \) so that

\[
H^*(BT; Z) = \mathbb{Z}[t_1, \ldots, t_l]
\]

and

\[
H^*(BT; Z)^{\Phi(Sp(l))} = \mathbb{Z}[q_1, \ldots, q_l]
\]

where \( q_i = \sigma_i(t_1^2, \ldots, t_l^2) \). It is evident that \( f_{ij} = q_j \) for \( j = 1, \ldots, l \). Since \( H^*(Sp(l); Z) \) has no torsion, it follows that \( b(2j) = c(2j) = 1 \) for all \( j \). Thus we have \( \tau'(x_{j-1}) = q_j \) for \( j = 1, \ldots, l \).

Let us recall that

(2.3) \( R(Sp(l)) = \mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_l] \) where

(a) \( \text{dim} \lambda_k = \binom{2l}{k} \);
(b) \( \text{relations } \Lambda^*\lambda_k = \lambda_k \) hold;
(c) \( \text{the set of weights of } \lambda_k \) is given by \( \{ \pm t_i | 1 \leq i \leq l \} \).

Put

\[
s_{2m} = s_m(t_1^2, \ldots, t_l^2) = \sum_{i=1}^l t_i^{2m}.
\]

From Newton's formula

\[
s_{2m} + \sum_{i=1}^{m-1} (-1)^i s_{2m-2i}q_i + (-1)^m m q_m = 0
\]

it follows that

\[
\text{ch}_i(\lambda_1) = 2l + \sum_{m=1}^{l} \frac{(-1)^{m-1}}{(2m-1)!} q_m + \text{decomposables}.
\]

Therefore

\[
\text{ch}_0(\lambda_1) = \sum_{m=1}^{l} \frac{(-1)^{m-1}}{(2m-1)!} x_{4m-1}
\]
and by Lemma 1 we obtain

**Theorem 3.** The Chern characters on $Sp(l)$ for $l=2,3,4$ are given by:

\[
\begin{align*}
  l = 2 & \quad ch\beta(\lambda_1) = x_3 + (-1/3!)x_7 \\
        & \quad ch\beta(\lambda_2) = 2x_3 + (4/3!)x_7 \\
  l = 3 & \quad ch\beta(\lambda_1) = x_3 + (-1/3!)x_7 + (1/5!)x_{11} \\
        & \quad ch\beta(\lambda_2) = 4x_3 + (2/3!)x_7 + (-26/5!)x_{11} \\
        & \quad ch\beta(\lambda_3) = 6x_3 + (6/3!)x_7 + (66/5!)x_{11} \\
  l = 4 & \quad ch\beta(\lambda_1) = x_3 + (-1/3!)x_7 + (1/5!)x_{11} + (-1/7!)x_{15} \\
        & \quad ch\beta(\lambda_2) = 6x_3 + (-24/5!)x_{11} + (120/7!)x_{15} \\
        & \quad ch\beta(\lambda_3) = 15x_3 + (9/3!)x_7 + (15/5!)x_{11} + (-119/7!)x_{15} \\
        & \quad ch\beta(\lambda_4) = 20x_3 + (16/3!)x_7 + (80/5!)x_{11} + (2416/7!)x_{15}
\end{align*}
\]

where the number on the right hand side indicates the determinant of the corresponding matrix on the left hand side.

3. The spinor groups

Let us first consider the case of $Spin(7)$. In this case, $(m_1, m_2, m_3) = (2, 4, 6)$. We can choose elements $t_1, t_2, t_3, \gamma \in H^*(BT; \mathbb{Z})$ so that

\[
H^*(BT; \mathbb{Z}) = \mathbb{Z}[t_1, t_2, t_3, \gamma]/(c_1 - 2\gamma)
\]

and

\[
H^*(BT; \mathbb{Q})^{\otimes(Spin(7))} = \mathbb{Q}[p_1, p_2, p_3]
\]

where $c_i = \sigma_i(t_1, t_2, t_3)$ and $p_i = \sigma_i(t_1^i, t_2^i, t_3^i)$. In the light of the Remark in Section 1, using the formula

\[
p_i = \sum_{j=0}^{2i} (-1)^{i+j} c_{2i-j} c_j ,
\]

we have

\[
(3.1) \quad f_4 = \frac{1}{2} p_1 = -c_2 + 2\gamma^2 ,
\]

\[
f_6 = \frac{1}{4} p_2 - \frac{1}{4} f_4^2 = -c_3\gamma + c_2\gamma^2 - \gamma^4 ,
\]

\[
f_{12} = p_3 = c_3^2 .
\]

Let us determine $b(2), b(4), b(6) \in \mathbb{Z}$. To do so we use the Serre spectral sequence $\{E_r(\mathbb{Z})\}$ for the integral cohomology of the universal fibration

\[
F = Spin(7) \rightarrow E = E Spin(7) \rightarrow B = B Spin(7) .
\]
Furthermore, to investigate it, we use the Serre spectral sequence \( \{ E_r(Z/p) \} \) for the mod \( p \) cohomology of the same fibration, where \( p \) runs over all primes.

Recall that \( H^*(\text{Spin}(7); Z) \) has no \( p \)-torsion for \( p > 2 \). Let \( \Delta_{Z/p}() \) denote a \( Z/2 \)-algebra having a set in parentheses as a simple system of generators. Then it follows from [6] and [7] that

\[
H^*(\text{Spin}(7); Z/p) = \begin{cases} 
\Delta_{Z/2}(x_3, x_5, x_6, x_7) & (p = 2) \\
\Lambda_{Z/p}(x_3, x_7, x_{11}) & (p > 2)
\end{cases}
\]

and

\[
H^*(B\text{Spin}(7); Z/p) = \begin{cases} 
Z/2[\tilde{y}_6, \tilde{y}_8, \tilde{y}_9, \tilde{y}_3] & (p = 2) \\
Z/p[\tilde{y}_6, \tilde{y}_8, \tilde{y}_{12}] & (p > 2)
\end{cases}
\]

where \( \tilde{x}_i \) transgresses to \( \tilde{y}_{i+1} \) for all \( i \) and \( \beta_2(x_3) = x_6 (\beta_2 \) denotes the mod \( p \) Bockstein homomorphism). For a based space \( X \), let \( \pi_p: H^i(X; Z) \to H^i(X; Z/p) \) be the mod \( p \) reduction homomorphism. Then if \( i = 3 \) or 7, \( \pi_p(x_i) = x_i \) and \( \pi_p(y_{i+1}) = \tilde{y}_{i+1} \) for every prime \( p \). Therefore we conclude that \( \tau(x_3) = y_4 \) and \( \tau(x_7) = y_8 \). In other words, \( b(2) = b(4) = 1 \).

It remains to determine \( b(6) \). Since

\[
\pi_p(x_{11}) = \begin{cases} 
x_3 \otimes x_6 & (p = 2) \\
x_{11} & (p > 2)
\end{cases}
\]

and

\[
\pi_p(y_{12}) = \begin{cases} 
\tilde{y}_6^2 & (p = 2) \\
\tilde{y}_{12} & (p > 2)
\end{cases}
\]

an analogous argument to the above yields that

\[
(0) \text{ if } p > 2, \quad \nu_p(b(6)) = 0
\]

where \( \nu_p(m) \) is the power of \( p \) in \( m \). To get \( \nu_p(b(6)) \) we consider \( \{ E_*(Z/2) \} \), which satisfies

\[
E_*^{s,t}(Z/2) \cong H^*(B; Z/2) \otimes H^t(F; Z/2)
\]

and \( E_*^{*,*}(Z/2) = 0 \) unless \( (s, t) = (0, 0) \). Then it is easy to see that

(i) \( d_q(1 \otimes x_3 x_6) = \tilde{y}_6 \otimes x_6 \).

(ii) \( d_q(\tilde{y}_6 \otimes x_6) = \tilde{y}_6^2 \otimes 1 \).

Let

\[
\beta_2^F: E_1^{s,t}(Z/2) \to E_1^{s,t+1}(Z/2)
\]

be the map induced by \( \beta_2: H^*(F; Z/2) \to H^{t+1}(F; Z/2) \) through the isomorphism

\[
E_1^{s,t}(Z/2) \cong C^*(B; H^t(F; Z/2))
\]

Then we have

(iii) \( \beta_2^F(\tilde{y}_6 \otimes x_6) = \tilde{y}_6 \otimes x_6 \).

Denote again by \( \pi_p: \{ E_*(Z) \} \to \{ E_*(Z/p) \} \) the morphism of spectral sequences induced by \( \pi_p \). By virtue of the isomorphism
we find that there exist elements \( \{x_{11}\} \in E_2^{0,11}(Z) \), \( \{v_{12}\} \in E_2^{5,6}(Z) \) and \( \{y_{12}\} \in E_2^{11,0}(Z) \) which satisfy \( \pi_2(\{x_{11}\}) = 1 \otimes x_0 \otimes x_0 \), \( \pi_2(\{v_{12}\}) = y_0 \otimes x_0 \) and \( \pi_2(\{y_{12}\}) = y_0 \otimes 1 \) respectively. Then the conditions (0), (i), (ii), (iii) imply that in \( \{E_r(Z)\} \)

(iv) \( d_6(\{x_{11}\}) = \{v_{12}\} \).

(v) \( d_6(\{x_{11}\}) = \{v_{12}\} \).

In fact, (iv) is an immediate consequence of (i). In what follows we roughly state a proof of (v). Let us begin by recalling the construction of the Serre spectral sequence \( \{E_r(R)\} \) in cohomology with \( R \)-coefficients of a fibration \( F \to E \to B \), where \( R = \mathbb{Z} \) or \( \mathbb{Z}/p \) (for details see [24]). There is a cochain complex \( \text{Hom}(C_*(E), R) \) which is filtered by its subcomplexes \( A^{s,t}_*(R) = \sum \text{A}_s^{t+1}(R) \) such that \( A^{s,t}_*(R) \subset A^{s+1,t}_*(R) \) and \( S(A_s^{t+1}(R)) \subset A_s^{t+1}(R) \) for all \( (s,t) \) (where \( \delta \) is the differential in \( \text{Hom}(C_*(E), R) \)). This filtered cochain complex gives rise to \( \{E_r(R)\} \), i.e.,

\[
Z_r^{s,t}(R) = A^{s,t}_*(R) \cap \delta^{-1}(A^{s+r,t-r+1}_*(R)),
B_r^{s,t}(R) = A^{s,t}_*(R) \cap \delta A^{s+1,t-r+1}_*(R),
E_r^{s,t}(R) = Z_r^{s,t}(R)/Z_r^{s+1,t-1}(R) + B_r^{s+1,t}(R).
\]

Note that there is an exact sequence

\[
0 \to A^{s,t}_*(Z) \to A^{s,t}_*(Z) \to A^{s,t}_*(Z/p) \to 0
\]

for all \( (s,t) \). Since \( d_r: E_r^{s,t}(R) \to E_r^{s+r,t-r+1}(R) \) is induced by \( \delta \), by (iv) we see that there exists a representative \( x \in A^{0,11}(Z) \) (resp. \( v \in A^{5,6}(Z) \)) of \( \{x_{11}\} \) (resp. \( \{v_{12}\} \)) such that

(3.2) \( \delta(x) = v \).

Let \( u \in A^{6,5}(Z) \) be a representative of \( y_0 \otimes x_0 \). Then by (iii) we observe that there exists \( u \in A^{6,5}(Z) \) such that \( \pi_2(u) = u \) and

(3.3) \( \delta(u) = 2v \)

(see [2, Chapter III, §2]). Similarly by (ii) there is a representative \( y \in A^{12,0}(Z/2) \) of \( y_0 \otimes 1 \) such that \( \delta(u) = y \). This implies that there exists a representative \( y \in A^{12,0}(Z) \) of \( \{y_{12}\} \) such that \( \pi_2(y) = y \) and

(3.4) \( \delta(u) = y \).

By (3.2), (3.3) and (3.4), we have

\[
\delta(2x) = 2v = \delta(u) = y
\]

which gives (v). It is equivalent to \( b(6) = 2 \).
We discuss the problem of determining $c(2), c(4), c(6) \in \mathbb{Z}$ in a general form. Indeed, we claim that $c(m_j) = 1$ for $j = 1, \ldots, l$ in all cases that concern us. To prove this we use the integral cohomology spectral sequence $\{E_r\}$ of the fibration

$$G/T \to BT \to BG.$$ 

Then the homomorphism $\rho^*: H^*(BG; \mathbb{Z}) \to H^*(BT; \mathbb{Z})$ can be regarded as the composite

$$H^*(BG; \mathbb{Z}) = E_1^{*,0} \to E_2^{*,0} = D_1^{*,0} \subset \cdots \subset D_\infty^{*,0} = H^*(BT; \mathbb{Z})$$

where $D_i^{*,0} = E_1^{*,0}$. According to [6], the class $\{y_{2m}\} \in E_\infty^{2*,0}$ survives to $E_\infty$. What we have to verify is to observe that no extension problems occur on the class $\{y_{2m}\} \in E_\infty^{2*,0}$. This is an essentially easy work, because all structures of $H^*(G/T; \mathbb{Z}), H^*(BT; \mathbb{Z})$ and $H^*(BG; \mathbb{Z})$ were explicitly described (for $H^*(BG; \mathbb{Z})$ see [7] and [25]; for $H^*(G/T; \mathbb{Z})$ see [27] and also [26]). For example, consider the case $G = \text{Spin}(7)$. Then it is not hard to see that if $m = 4, 8$ or $12$, $D_i^{*,0}$ is trivial or torsion free for all $i$. This assures us that $c(2) = c(4) = c(6) = 1$. In the future we omit such checks for the other cases, for our claim (except for the case $G = F_4$) has been proved in a more general setting by [13] and [15].

Let us recall from [17] that

(3.5) $R(\text{Spin}(7)) = \mathbb{Z}[\lambda, \eta, \Delta]$ where

(a) $\dim \lambda = \binom{7}{k}$ and $\dim \Delta = 8$;

(b) relations $\Lambda \lambda = \lambda$ and $\Delta = \lambda + \lambda + \lambda + 1$ hold;

(c) the set of weights of $\lambda$ is given by $\{\pm t, 0 | 1 \leq i \leq 3\}$.

By the same calculation as in the case of $Sp(l)$, we have

$$ch^\alpha i^*(\lambda_i) = p_1,$$

$$ch^\alpha i^*(\lambda_i) = -\frac{1}{6} p_2 + \text{decomposables},$$

$$ch^\alpha i^*(\lambda_i) = \frac{1}{120} p_3 + \text{decomposables}.$$ 

On the other hand, from (3.1) and the results on $b(m_j)$ and $c(m_j)$ it follows that

$$\tau'(x_3) = f_s = \frac{1}{2} p_1,$$

$$\tau'(x_7) = f_s = \frac{1}{4} p_2 + \text{decomposables},$$
Combining these, we have
\[ \text{ch}_\beta(\lambda_i) = \frac{1}{2} f_{12} = \frac{1}{2} p_8. \]

Therefore by Lemma 1,
\[ \text{ch}_\beta(\lambda_i) = 10 x_3 + \frac{1}{3} x_7 - \frac{5}{12} x_{11} \]
and
\[ \text{ch}_\beta(\lambda_i + \lambda_i + \lambda_i + 1) = 32 x_3 + \frac{16}{3} x_7 + \frac{4}{15} x_{11}. \]

On the other hand, by the formula (2) of [16, p. 8],
\[ \beta(\Delta^2) = 8 \beta(\Delta) + 8 \beta(\Delta) = 16 \beta(\Delta). \]
Thus from the relation \( \Delta^2 = \lambda^2 + \lambda^2 + \lambda^2 + 1 \) we deduce that
\[ \text{ch}_\beta(\Delta_i) = 2 x_3 + \frac{1}{3} x_7 + \frac{1}{60} x_{11}. \]

**Theorem 4.** The Chern characters on Spin(7) are given by:
\[
\begin{align*}
\text{ch}_\beta(\lambda_i) &= 2 x_3 + (\theta/3)x_7 + (2/51)x_{11} \\
\text{ch}_\beta(\lambda_i) &= 10 x_3 + (4/3)x_7 + (50/1)x_{11} \\
\text{ch}_\beta(\Delta_i) &= 2 x_3 + (2/3)x_7 + (50/1)x_{11}
\end{align*}
\]
and the determinant of the corresponding matrix is 1.

Let us next consider the case of Spin(8). In this case, \((m_1, m_2, m_3, m_4) = (2, 4, 4, 6)\). We can choose elements \(t_1, t_2, t_3, t_4, \gamma \in H^2(BT; Z)\) so that
\[ H^*(BT; Z) = Z[t_1, \ldots, t_4, \gamma]/(c_1 - 2\gamma) \]
and
\[ H^*(BT; Q)^{\Phi(Spin(8))} = Q[p_1, c_4, p_2, p_3] \]
where \(c_i = \sigma_i(t_1, \ldots, t_4)\) and \(p_i = \sigma_i(t_1^2, \ldots, t_4^2)\). By a similar calculation to the before, we have
\[
\begin{align*}
f &= \frac{1}{2} p_1 = -c_2 + 2\gamma^2, \\
f' &= c_4, \\
f' &= \frac{1}{4} p_2 - \frac{1}{2} f^2 - \frac{1}{4} f^2 = -c_3 + c_4^2 - 4, \end{align*}
\]
Let us determine \( b(2), b(4)', b(4), b(6) \in \mathbb{Z} \). But, since \( H^*({\text{Spin}}(8); \mathbb{Z}) \) has no \( p \)-torsion for \( p > 2 \) and
\[
H^*({\text{Spin}}(8); \mathbb{Z}/2) = \Delta_{\mathbb{Z}/2}(x_5, x_5, x_6, x_7, x_7)
\]
where all the \( x_i \) are universally transgressive and \( \beta_2(x_8) = x_8 \) [7], the situation is quite similar to that for \( G = \text{Spin}(7) \), and so we get a similar result, i.e., \( b(2) = b(4)' = b(4) = 1 \) and \( b(6) = 2 \). On the other hand, as mentioned earlier, \( c(2) = c(4)' = c(4) = c(6) = 1 \). Thus we have
\[
(3.6) \quad \tau'(x_2) = f_4 = \frac{1}{2} p_1 , \\
\tau'(x_i) = f'_8 = c_4 , \\
\tau'(x_7) = f_8 = \frac{1}{4} p_2 - \frac{1}{2} c_4 + \text{decomposables} , \\
\tau'(x_{11}) = \frac{1}{2} f_{12} = \frac{1}{2} p_3 .
\]
Let us recall from [17] that
\[
(3.7) \quad R(\text{Spin}(8)) = \mathbb{Z}[\lambda_1, \lambda_2, \Delta_8^+, \Delta_8^-] \quad \text{where}
\]
(a) \( \dim \lambda_k = \binom{8}{k} \) and \( \dim \Delta_8^+ = \dim \Delta_8^- = 8 \);  
(b) relations \( \Lambda^* \lambda_1 = \lambda_8 \) and \( \Delta_8^+ \Delta_8^- = \lambda_8 + \lambda_1 \) hold;  
(c) the set of weights of \( \lambda_1 \) is given by \( \{ \pm t_i | 1 \leq i \leq 4 \} \) and that of \( \Delta_8^+ \) is given by \( \{ \pm \gamma, \gamma - t_i - t_j | 1 \leq i < j \leq 4 \} \).  
By direct calculations we have
\[
(3.8) \quad ch^2\alpha_i^*(\lambda_1) = p_1 , \\
ch^4\alpha_i^*(\lambda_1) = \frac{1}{12} (-2p_2 + p_4^\pm) , \\
ch^6\alpha_i^*(\lambda_1) = \frac{1}{360} (3p_3 - 3p_2p_1 + p_4^\pm)
\]
and
\[
(3.9) \quad ch^2\alpha_i^*(\Delta_8^+) = p_1 , \\
ch^4\alpha_i^*(\Delta_8^+) = \frac{1}{48} (4p_4 + 24c_4 + p_6^\pm) .
\]
There are involutive automorphisms \( \kappa \) and \( \bar{\kappa} \) of \( T \) and \( \text{Spin}(8) \) respectively, which make the diagram
commute, such that the automorphism \((B\kappa)^*\) of \(H^*(BT; Z)\) satisfies
\[
(B\kappa)^*(t_i) = \begin{cases} 
  t_i & (1 \leq i \leq 3) \\
  -t_4 & (i = 4) 
\end{cases}
\]
Therefore \((B\kappa)^*(p_i) = p_i\), \((B\kappa)^*(c_i) = -c_i\) and the automorphism \(\kappa^*\) of \(R(\text{Spin}(8))\) satisfies \(\kappa^*(\Delta_r^+) = \Delta_r^-\). Applying \((B\kappa)^*\) to (3.9), it follows that
\[
\begin{align}
ch^*\alpha^*(\Delta_r^+) &= p_1, \\
ch^*\alpha^*(\Delta_r^-) &= \frac{1}{48}(4p_2 - 24c_4 + p_7).
\end{align}
\]
Combining (3.8), (3.9), (3.10) with (3.6), we have
\[
\begin{align}
ch\beta(\lambda_i) &= 2x_3 - \frac{1}{3}x_7^2 - \frac{2}{3}x_7 + \frac{1}{60}x_{11}, \\
ch\beta(\Delta_r^+) &= 2x_3 + \frac{2}{3}x_7^2 + \frac{1}{3}x_7 + ax_{11}, \\
ch\beta(\Delta_r^-) &= 2x_3 - \frac{1}{3}x_7^2 + \frac{1}{3}x_7 + ax_{11}
\end{align}
\]
for some \(a \in \mathbb{Q}\). From Lemma 1 and the relation \(\Delta_r^+\Delta_r^- = \lambda_3 + \lambda_4\) we deduce that \(a = 1/60\).

**Theorem 5.** The Chern characters on \(\text{Spin}(8)\) are given by:
\[
\begin{align}
ch\beta(\lambda_1) &= 2x_3 + (-2/31)x_7^2 + (-4/31)x_7 + (2/51)x_{11}, \\
ch\beta(\lambda_2) &= 12x_3 + (-48/51)x_{11} \\
ch\beta(\Delta_r^+) &= 2x_3 + (4/31)x_7^2 + (2/31)x_7 + (2/51)x_{11} \\
ch\beta(\Delta_r^-) &= 2x_3 + (-2/31)x_7^2 + (2/31)x_7 + (2/51)x_{11}
\end{align}
\]
and the determinant of the corresponding matrix is \(-1\).

**Remark.** The equation \(ch\beta(\Delta_r^+ - \Delta_r^-) = x_7^2\) confirms the fact that \(\text{Spin}(8)/\text{Spin}(7) = S^7\) (see \([22, \text{Proposition 6.2}]\)).

Let us lastly consider the case of \(\text{Spin}(9)\). In this case, \((m_1, m_2, m_3, m_4) = (2, 4, 6, 8)\). We can choose \(t_1, t_2, t_3, t_4, \gamma \in H^*(BT; Z)\) so that
\[
H^*(BT; Z) = Z[t_1, \ldots, t_4, \gamma]/(c_1 - 2\gamma)
\]
and

\[ H^*(BT; Q) = Q[p_1, p_2, p_3, p_4] \]

where \( c_i = \sigma_i(t_1, \ldots, t_n) \) and \( p_i = \sigma_i(t_1^2, \ldots, t_n^2) \). By a straightforward calculation we have

\[
\begin{align*}
  f_4 &= \frac{1}{2} p_1 = -c_2 + 2\gamma^2, \\
  f_8 &= \frac{1}{2} p_2 - \frac{1}{2} f_8^2 = c_4 + 2(-c_2\gamma + c_2\gamma^2 - \gamma^4), \\
  f_{12} &= p_3 = -2c_2^2 + c_3^2, \\
  f_{16} &= \frac{1}{4} p_4 - \frac{1}{4} f_8^2 = c_6\gamma - c_4\gamma^2 - c_2\gamma^4 + 2c_2\gamma^6 \\
  &\phantom{= \frac{1}{4} p_4 - \frac{1}{4} f_8^2} + c_4\gamma^4 - c_2\gamma^4 - 2c_2\gamma^4 + 2c_2\gamma^6 - \gamma^8.
\end{align*}
\]

Since \( H^*(\text{Spin}(9); Z) \) has no \( p \)-torsion for \( p > 2 \) and

\[ H^*(\text{Spin}(9); Z/2) = \Delta \times Z/2 \]

where all the \( x_i \) are universally transgressive and \( \beta_2(x_6) = x_6 \) \([7]\), as in the case of \( \text{Spin}(7) \), it follows that \( b(2) = b(4) = 1, b(6) = 2 \) and \( b(8) = 1 \). On the other hand, \( c(2) = c(4) = c(6) = c(8) = 1 \). Thus we have

(3.11)

\[
\begin{align*}
  \tau'(x_3) &= f_4 = \frac{1}{2} p_1, \\
  \tau'(x_7) &= f_8 = \frac{1}{2} p_2 + \text{decomposables}, \\
  \tau'(x_{11}) &= \frac{1}{2} f_{12} = \frac{1}{2} p_3, \\
  \tau'(x_{15}) &= f_{16} = \frac{1}{4} p_4 + \text{decomposables}.
\end{align*}
\]

**Remark.** Let \( j : \text{Spin}(8) \rightarrow \text{Spin}(9) \) be the natural inclusion. Then by (3.6) and (3.11) we see that the homomorphism \( j^* : H^*(\text{Spin}(9); Z) \rightarrow H^*(\text{Spin}(8); Z) \) satisfies

\[
\begin{align*}
  j^*(x_i) &= \begin{cases} x_i & (i = 3, 11) \\
  x_i^2 + 2x_7 & (i = 7) \\
  0 & (i = 15)
\end{cases}
\end{align*}
\]

Let us recall that

(3.12) \[ R(\text{Spin}(9)) = Z[\lambda_1, \lambda_2, \lambda_3, \Delta_9] \quad \text{where} \]

(a) \( \dim \lambda_i = \binom{9}{k} \) and \( \dim \Delta_9 = 16 \);

(b) \( \text{relations } \Lambda^k \lambda_i = \lambda_i \) and \( \Delta_9^2 = \lambda_4 + \lambda_6 + \lambda_8 + \lambda_2 + \lambda_1 + 1 \) hold;
(c) the set of weights of $\lambda_i$ is given by $\{ \pm t_i, 0 | 1 \leq i \leq 4 \}$.

The rest of the argument is parallel to that for $G = \text{Spin}(7)$. We only exhibit the result.

**Theorem 6.** The Chern characters on $\text{Spin}(9)$ are given by:

\[
\begin{align*}
ch_\beta(\lambda_1) &= 2x_3 + (-2/3!x_7 + (2/5!x_{11} + (-4/7!x_{15}) \\
ch_\beta(\lambda_2) &= 14x_3 + (-2/3!x_7 + (-46/5!x_{11} + (476/7!x_{15}) \\
ch_\beta(\lambda_3) &= 4x_3 + (18/3!x_7 + (-18/5!x_{11} + (-4284/7!x_{15}) \\
ch_\beta(\Delta_3) &= 4x_3 + (2/3!x_7 + (4/5!x_{11} + (34/7!x_{15})
\end{align*}
\]

and the determinant of the corresponding matrix is 1.

4. The exceptional Lie groups $G_2$ and $F_4$

Let us first consider the case of $G_2$. In this case, $(m_1, m_2)=(2, 6)$. We use the root system $\{\alpha_1, \alpha_2\}$ of [11]. Let $\omega_1, \omega_2$ be the fundamental weights. If we put

\[
t_1 = \omega_1, \quad t_2 = \omega_1 - \omega_2, \quad t_3 = -2\omega_1 + \omega_2,
\]

then

\[
H^*(BT; \mathbb{Z}) = \mathbb{Z}[t_1, t_2, t_3]/(c_1)
\]

where $c_1 = \sigma_I(t_1, t_2, t_3)$, on which $\Phi(G_2)$ acts as follows:

<table>
<thead>
<tr>
<th></th>
<th>$R_1$</th>
<th>$R_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>$-t_2$</td>
<td>$t_1$</td>
</tr>
<tr>
<td>$t_2$</td>
<td>$-t_1$</td>
<td>$t_3$</td>
</tr>
<tr>
<td>$t_3$</td>
<td>$-t_3$</td>
<td>$t_2$</td>
</tr>
</tbody>
</table>

where $R_j$ $(j=1, 2)$ is the reflection to the hyperplane $\alpha_j = 0$, and $\{R_1, R_3\}$ generates $\Phi(G_2)$. Therefore

\[
H^*(BT; \mathbb{Q})^{\Phi(G_2)} = \mathbb{Q}[p_1, p_3].
\]

where $p_i = \sigma_I(t_i^2, t_i^3, t_i^3)$, and it follows that

\[
f_4 = \frac{1}{2} p_1 = -c_2,
\]

\[
f_{12} = p_3 = c_3^2.
\]

Since $H^*(G_2; \mathbb{Z})$ has no $p$-torsion for $p > 2$ and

\[
H^*(G_2; \mathbb{Z}/2) = \Delta_{Z/2}(x_3, x_5, x_8)
\]
where all the $\alpha_i$ are universally transgressive and $\beta_3(x_5) = x_6$ [7], as in the case of Spin(7), it follows that $b(2) = 1$ and $b(6) = 2$. On the other hand, $c(2) = c(6) = 1$. Thus we have

$$\tau'(x_3) = f_1 = \frac{1}{2} p_1,$$

$$\tau'(x_{11}) = \frac{1}{2} f_{12} = \frac{1}{2} p_3.$$

Let us recall that

(4.1) $R(G_2) = Z[\rho_1, \Lambda^2 \rho_1]$ where

(a) $\dim \Lambda^2 \rho_1 = \binom{7}{2}$ (and $\dim \rho_2 = 14$);

(b) a relation $\Lambda^2 \rho_1 = \rho_1 + \rho_2$ holds;

(c) the set of weights of $\rho_1$ is given by $\{ \pm t_i (1 \leq i \leq 3), 0 \}$.

By a calculation we have

$$ch^2 \alpha^* (\rho_1) = p_1,$$

$$ch^6 \alpha^* (\rho_1) = \frac{1}{120} p_3 + \text{decomposables}.$$

Therefore

$$ch^6 (\rho_1) = 2x_3 + \frac{1}{60} x_{11}$$

and by Lemma 1 we get

**Theorem 7.** The Chern characters on $G_2$ are given by:

$$ch^6 (\rho_1) = 2x_3 + (2/51)x_{11}$$

$$ch^6 (\Lambda^2 \rho_1) = 10x_3 + (-50/51)x_{11}$$

and the determinant of the corresponding matrix is $-1$.

**Remark.** Consider the following fibration

$$k: G_2 \to Spin(7) \to Spin(7)/G_2 = S^7.$$
(see [31]). Using these facts, we find that Theorem 7 follows from Theorem 4.

\[ H^*(\Omega G_2; \mathbb{Z}) \] (for degrees \( \leq 10 \)) was calculated implicitly by Bott [10]. Using it and the cohomology spectral sequence of the path fibration \( \Omega G_2 \to PG_2 \to G_2 \), we can show that
\[ d(2) = 1 \quad \text{and} \quad d(6) = 2 \]
(see [12] and [28, p. 474]).

Let us now consider the case of \( F_4 \). In this case, \((m_1, m_2, m_3, m_4) = (2, 6, 8, 12)\). We can choose elements \( t_1, t_2, t_3, t_4, \gamma \in H^2(BT; \mathbb{Z}) \) so that
\[ H^*(BT; \mathbb{Z}) = \mathbb{Z}[t_1, \cdots, t_4, \gamma]/(\gamma - 2\gamma) \]
and the action of \( \Phi(F_4) \) on it is as described in [9, § 19] (see [18] and [29]). Let \( c_i = \sigma_i(t_1, \cdots, t_4) \) and \( \bar{p}_i = \sigma_i(t_1^2, \cdots, t_4^2) \). If we put
\[
\begin{align*}
I_4 &= p_1, \\
I_{12} &= -6p_3 + p_2p_1, \\
I_{18} &= 12p_4 - 3p_2p_1 + p_2^2, \\
I_{24} &= -72p_4p_2 + 27p_2^2 + 27p_3^2 - 9p_2p_1 + 2p_3^2,
\end{align*}
\]
then we have
\[ H^*(BT; \mathbb{Z})^\Phi(F_4) = \mathbb{Q}[I_4, I_{12}, I_{18}, I_{24}] \].

For a proof see [27, Lemma 5.1], however, its main part is accomplished by a pure calculation; see (4.7) and (4.8) below. By a troublesome calculation we obtain
\[
\begin{align*}
f_4 &= \frac{1}{2} I_4 = -c_2 + 2\gamma^2, \\
f_{12} &= -\frac{1}{2} I_{12} \\
&= -4c_1c_2^2 + 3c_3^2 + c_2^3 - 4c_1c_2^2\gamma - 4c_1\gamma^2 - 2c_2\gamma^2 + 8c_3\gamma^2, \\
f_{18} &= \frac{1}{16} (I_{18} + 2f_{12}f_4 + f_4^2) \\
&= c_1^2 - c_1c_2\gamma + c_1c_2^2\gamma + c_2^3\gamma^2 - 2c_1c_2\gamma^3 - c_1\gamma^4 + c_2\gamma^4 + 2c_1\gamma^5 - 2c_2\gamma^6 + \gamma^8, \\
f_{24} &= -\frac{1}{64} (f_{24} + 16f_{12}f_4^2 - 3f_{12}^2 + f_4^4) \\
&= 2c_1^2 - c_1c_2^2 - 3c_2^2 + c_1c_2^2\gamma + c_1c_2^2\gamma - 7c_2^2\gamma^2 - 3c_1c_2^2\gamma^2 - c_2^3\gamma^2 - c_2^3\gamma^2 + 2c_1c_2c_2\gamma^3 \\
&+ 2c_1\gamma^3 + 2c_1c_2^2\gamma - 7c_2c_2\gamma^4 + 2c_1c_2\gamma^4 - 2c_2c_2\gamma^4 - c_2\gamma^4 - 2c_1c_2\gamma^5 - 4c_2c_2\gamma^6 \\
&- 2c_1c_2\gamma^8 - c_2^3\gamma^8 + 4c_2\gamma^9 + 4c_2c_2\gamma^7 + c_1\gamma^8 - 7c_2^2\gamma^8 - 2c_2\gamma^9 + 6c_2\gamma^{10} - 2\gamma^{12}.
\end{align*}
\]
Let us determine \( b(2), b(6), b(8), b(12) \in \mathbb{Z} \). Recall that \( H^*(F_4; \mathbb{Z}) \) has no \( p \)-torsion for \( p > 3 \). Since

\[
H^*(F_4; \mathbb{Z}/2) = \Delta_{\mathbb{Z}/2}(x_3, x_5, x_6, x_{15}, x_{23})
\]

where all the \( x_i \) are universally transgressive and \( \beta_2(x_5) = x_5 \) [7], it follows that \( \nu_2(b(2)) = 0, \nu_2(b(6)) = 1, \nu_2(b(8)) = 0 \) and \( \nu_2(b(12)) = 0 \). Consider the case \( p = 3 \).

Recall from [7] and [25] that

\[
H^*(F_4; \mathbb{Z}/3) = \Delta_{\mathbb{Z}/3}(x_3) \otimes \Delta_{\mathbb{Z}/3}(x_5, x_7, x_{11}, x_{13})
\]

\[
H^*(BF_4; \mathbb{Z}/3) = \mathbb{Z}/3[y_3, y_{13}] \otimes C,
\]

\[
C = \mathbb{Z}/3[y_4, y_8] \otimes \{1, y_{19}, y_{20}, y_{21}, y_{25}\} + \Lambda_{\mathbb{Z}/3}(y_9) \otimes \mathbb{Z}/3[y_{26}] \otimes \{1, y_{20}, y_{21}, y_{25}\}
\]

where \( \tau(x_i) = y_{i+1} \) for \( i = 3, 7, 8 \) and \( \beta_3(x_i) = x_i \). Here we may suppose that

\[
\pi_3(x_3) = x_3, \quad \pi_3(y_4) = y_4,
\]

\[
\pi_3(x_{11}) = x_{11}, \quad \pi_3(y_{13}) = y_{13},
\]

\[
\pi_3(x_{15}) = x_{15}, \quad \pi_3(y_{19}) = y_{19},
\]

\[
\pi_3(x_{23}) = x_{23}, \quad \pi_3(y_{21}) = y_{21}.
\]

In the mod 3 cohomology spectral sequence \( \{E_r(Z/3)\} \) of the universal fibration

\[
F = F_4 \rightarrow E = EF_4 \rightarrow B = BF_4,
\]

if

\[
\beta_3^3 : E_2^{s,t}(Z/3) \rightarrow E_2^{s+1,t}(Z/3)
\]

is the map induced by \( \beta_3 : H^*(B; Z/3) \rightarrow H^{s+1}(B; Z/3) \) through the isomorphism

\[
E_2^{s,t}(Z/3) \approx H^*(B; H^*(F; Z/3)),
\]

then we have

\[
\begin{align*}
& d_9(1 \otimes x_{11}) = y_9 \otimes x_3 \quad \cdots \quad (*) \\
& \beta_3^3(y_9 \otimes x_3) = y_9 \otimes x_3 \\
& d_4(y_9 \otimes x_3) = y_9 \otimes y_3 \otimes 1 \quad \cdots \quad (*)
\end{align*}
\]

\[
\begin{align*}
& d_9(1 \otimes x_{15}) = y_9 \otimes x_7 \quad \cdots \quad (*) \\
& \beta_3^3(y_9 \otimes x_7) = y_9 \otimes x_7 \\
& d_4(y_9 \otimes x_7) = y_9 \otimes y_3 \otimes 1 \\
& d_9(y_9 \otimes x_{23}) = y_9 \otimes x_{23} \\
& \beta_3^3(y_9 \otimes x_{23}) = y_9 \otimes x_{23} \\
& d_4(y_9 \otimes x_{23}) = y_9 \otimes y_3 \otimes 1
\end{align*}
\]
where the asterisks are due to [3]. Generally, with the obvious notation, since 
\(\delta: \mathcal{C}^{t+1}(\mathbb{Z}/3) \rightarrow \mathcal{C}^{t+1}(\mathbb{Z}/3)\) can be identified with the differential 
\(\delta: \mathcal{C}(\mathbb{Z}/3) \rightarrow \mathcal{C}(\mathbb{Z}/3)\), if \(\beta^{\mathbb{Z}/3}(\{u\}) = \{v\}\), then there exist \(u, v \in A^*(\mathbb{Z})\) such that 
\(\pi_{d}(u) = u, \pi_{d}(v) = v\) and \(\delta(u) = 3v\). In this way the same argument as in the 
case of \(\text{Spin}(7)\) is valid. Therefore the conditions (4.2), (4.3) and (4.4) imply 
that \(\nu_{3}(b(6)) = 1, \nu_{3}(b(8)) = 1 \) and \(\nu_{3}(b(12)) = 2\) respectively. Summarizing these, we have

\[
b(2) = 1, \quad b(6) = 6, \quad b(8) = 3 \quad \text{and} \quad b(12) = 9.
\]

On the other hand, \(c(2) = c(6) = c(8) = c(12) = 1\). Thus we obtain

\[
(4.5) \quad \tau'(x_2) = f_4 = \frac{1}{2} I_4,
\]

\[
\tau'(x_{11}) = \frac{1}{6} f_{12} = -\frac{1}{12} I_{12},
\]

\[
\tau'(x_{15}) = \frac{1}{3} f_{16} = \frac{1}{48} I_{16}+\text{decomposables},
\]

\[
\tau'(x_{23}) = \frac{1}{9} f_{24} = -\frac{1}{576} I_{24}+\text{decomposables}.
\]

Let us recall from [30] that

\[
(4.6) \quad R(F_4) = \mathbb{Z}[^{\rho_4}, \Lambda^2 \rho_4, \Lambda^3 \rho_4, \rho_4] \quad \text{where}
\]

(a) \(\dim \Lambda^k \rho_4 = \binom{26}{k}\) and \(\dim \rho_4 = 52\);

(b) \(\text{the set of weights of } \rho_4 \text{ is given by}
\]

\[
\{\pm t_i (1 \leq i \leq 4), \frac{1}{2} (\pm t_1 \pm t_2 \pm t_3 \pm t_4), 0, 0\}
\]

and that of \(\rho_1 \) is given by

\[
\{\pm t_i (1 \leq i < j \leq 4), \pm t_i (1 \leq i \leq 4), \frac{1}{2} (\pm t_1 \pm t_2 \pm t_3 \pm t_4), 0, 0, 0, 0\}
\]

We have to calculate \(\text{ch}^*(\rho_4)\) and \(\text{ch}^*(\rho_1)\). Consider the inclusion \(k: \text{Spin}(9) \rightarrow F_4\) such that \(F_4/\text{Spin}(9) = \Pi\), the Cayley projective plane (see, e.g., [9, §19]). Then \(k^*: R(F_4) \rightarrow R(\text{Spin}(9))\) satisfies \(k^*(\rho_4) = \lambda_7 + \Delta_9 + 1\) and \(k^*(\rho_1) = \lambda_2 + \Delta_9\); see (4.6) (b). Let us calculate \(\text{ch}^*(\Delta_9)\), where the set of weights of \(\Delta_9\) is

\[
\{1/2(\pm t_1 + t_2 + t_3 + t_4)\}.
\]

To do so we first calculate \(\text{ch}^*(\Delta_9)\), where the set of weights of \(\Delta_9\) is

\[
\{1/2(\pm t_1 + t_2)\}; \text{ using it, we calculate } \text{ch}^*(\Delta_7); \text{ and using it, we calculate } \text{ch}^*(\Delta_9)\). Our final result is
\[ \text{ch}^2 \alpha^*(\Delta_9) = 2p_1, \]
\[ \text{ch}^6 \alpha^*(\Delta_9) = \frac{1}{2880} (48p_3 + 12p_2p_1 + p_1^3), \]
\[ \text{ch}^8 \alpha^*(\Delta_9) = \frac{1}{645120} (1088p_4 + 256p_3p_1 + 16p_2^2 + 24p_2p_1^3 + p_1^4), \]
\[ \text{ch}^{12} \alpha^*(\Delta_9) = \frac{1}{122624409600} (31488p_4p_2 + 42432p_4p_1^2 + 3072p_3^2 + 4608p_3p_2p_1 \\
+ 1920p_2p_1^3 + 64p_2^3 + 240p_2^2p_1^2 + 60p_2p_1^4 + p_1^6). \]

By a similar calculation to the before, we have

\[ \text{ch}^2 \alpha^*(\lambda_1) = p_3, \]
\[ \text{ch}^6 \alpha^*(\lambda_1) = \frac{1}{360} (3p_3 - 3p_2p_1 + p_1^3), \]
\[ \text{ch}^8 \alpha^*(\lambda_1) = \frac{1}{20160} (-4p_4 + 4p_3p_1 + 2p_2^2 - 4p_2p_1^3 + p_1^4), \]
\[ \text{ch}^{12} \alpha^*(\lambda_1) = \frac{1}{239500800} (6p_4p_2 - 6p_4p_1^2 + 3p_3^2 - 12p_3p_2p_1 + 6p_3p_1^3 - 2p_1^3 \\
+ 9p_2^2p_1^2 - 6p_2p_1^4 + p_1^6). \]

Thus we have

(4.7) \[ \text{ch}^2 \alpha^*(\rho_4) = 3p_1, \]
\[ \text{ch}^6 \alpha^*(\rho_4) = \frac{1}{960} (24p_3 - 4p_2p_1 + 3p_1^3), \]
\[ \text{ch}^8 \alpha^*(\rho_4) = \frac{1}{645120} (960p_4 + 384p_3p_1 + 80p_2^2 - 104p_2p_1^3 + 33p_1^4), \]
\[ \text{ch}^{12} \alpha^*(\rho_4) = \frac{1}{40874803200} (11520p_4p_2 + 13120p_4p_1^2 + 1536p_3^2 - 512p_3p_2p_1 \\
+ 1664p_3p_1^3 - 320p_2^3 + 1616p_2^2p_1^2 - 1004p_2p_1^4 + 171p_1^6). \]

On the other hand, \( \text{ch} \alpha^*(\rho_4 - \rho_4) \) was calculated in [27, §5] (with certain indeterminacy). Following it, we have

(4.8) \[ \text{ch}^2 \alpha^*(\rho_4 - \rho_4) = 6p_1, \]
\[ \text{ch}^6 \alpha^*(\rho_4 - \rho_4) = \frac{1}{60} (-12p_3 + 2p_2p_1 - p_1^3), \]
\[ \text{ch}^8 \alpha^*(\rho_4 - \rho_4) = \frac{1}{10080} (240p_4 - 156p_3p_1 + 20p_2^2 + 16p_2p_1^3 + 3p_1^4), \]
Thus we get
\[ ch^{12} \alpha \beta (p_1 - p_4) = \frac{1}{39916800} \left( -720 p_4 p_2 + 1270 p_4 p_2^3 + 366 p_2^3 - 122 p_3 p_2 p_1 \right. \\
\left. -346 p_3 p_1^3 + 20 p_1^3 + 86 p_1^3 p_1 + 16 p_2 p_1^3 + p_1^7 \right). \]

Combining these with (4.5), it follows that
\[ ch^{12} \alpha \beta (p_1) = 9 I_4, \]
\[ ch^8 \alpha \beta (p_1) = \frac{7}{240} I_{13} \text{ plus decomposables}, \]
\[ ch^8 \alpha \beta (p_1) = \frac{17}{8064} I_{18} \text{ plus decomposables}, \]
\[ ch^{12} \alpha \beta (p_1) = \frac{1}{4055040} I_{24} \text{ plus decomposables}. \]

Combining these with (4.5), it follows that
\[ ch \beta (p_4) = 6 x_2 + \frac{1}{20} x_1 + \frac{1}{168} x_{15} + \frac{1}{44320} x_{23}, \]
\[ ch \beta (p_1) = 18 x_2 + \frac{7}{20} x_1 + \frac{17}{168} x_{15} - \frac{1}{7040} x_{23} \]
and by Lemma 1 we obtain

**Theorem 8.** The Chern characters on \( F_4 \) are given by:
\[ ch \beta (p_4) = 6 x_2 + (6/5!) x_1 + (30/7!) x_{15} + (90/11!) x_{23} \]
\[ ch \beta (\Lambda^2 p_4) = 144 x_2 + (-36/5!) x_1 + (-3060/7!) x_{15} + (-181980/11!) x_{23} \]
\[ ch \beta (\Lambda^3 p_4) = 1656 x_2 + (-1584/5!) x_1 + (-24480/7!) x_{15} + (11180160/11!) x_{23} \]
\[ ch \beta (p_1) = 18 x_2 + (-42/5!) x_1 + (510/7!) x_{15} + (-5670/11!) x_{23} \]
and the determinant of the corresponding matrix is 1.

\( H^*(\Omega F_4; Z) \) (for degrees \( \leq 22 \)) was calculated implicitly in [28]. Using it and the cohomology spectral sequence of the path fibration \( \Omega F_4 \to PF_4 \to F_4 \), we can show that
\[ d(2) = 1, \quad d(6) = 2, \quad d(8) = 1 \quad \text{and} \quad d(12) = 3. \]

**References**


[29] T. Watanabe: Cohomology operations in the loop space of the compact exceptional


Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558, Japan