ON A RELATION BETWEEN HIGHER ORDER ASYMPTOTIC RISK SUFFICIENCY AND HIGHER ORDER ASYMPTOTIC SUFFICIENCY IN A LOCAL SENSE

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1. Introduction. In Takeuchi [4] higher order asymptotic risk sufficiency of maximum likelihood estimator has been discussed. In this paper we try to find some relations between asymptotic risk sufficiency with a special loss function and asymptotic sufficiency in a local sense.

Let $\mathcal{P}_n = \{P_{\theta, n}; \theta \in \Theta\}$ be a family of probability distributions on measurable space $(\mathcal{X}, \mathcal{A}_n)$ with an index set $\Theta$ which is a subset of an Euclidean space with the usual norm $| \cdot |$. For a sub $\sigma$-field $\mathcal{C}$ of $\mathcal{A}_n$, real number $c \geq 0$ and $\theta, \theta' \in \Theta$ let

$$r_n^C(c; \theta, \theta') = \inf \{ (1+c)^{-1} \{ 1 - E_{P_{\theta, n}}(\phi) + c E_{P_{\theta', n}}(\phi) \}; \phi \text{ are } C\text{-measurable statistical test functions on } \mathcal{X} \}. $$

We note that $r_n^C(c; \theta, \theta')$ means the Bayes risk of statistical problem of testing a hypothesis 'P_{\theta, n} is true' against an alternative 'P_{\theta', n} is true' with experiment $(\mathcal{X}, C, \{P_{\theta, n}, P_{\theta', n}\})$ relative to a prior probability distribution $(c/(1+c), 1/(1+c))$ on $\{\theta', \theta\}$ provided that the loss function is simple.

Let $\{\mathcal{B}_n; n=1, 2, \ldots\}$ be a sequence of sub $\sigma$-fields of $\{\mathcal{A}_n; \mathcal{B}_n \subset \mathcal{A}_n\}$. In this paper we give a sufficient condition about the Bayes risk $r_n^B$ for $\{\mathcal{B}_n\}$ to be higher order locally asymptotically sufficient sequence of $\sigma$-fields. More precisely our main result in this paper is the following: Under some conditions if for some positive number $\alpha$ and $\beta$

$$r_n^B(c; \theta, \theta^*) = o(n^{-\beta})$$

for every $b>0$ and every compact subset $K$ of $\Theta$, then for every $\beta$ satisfying $0<\beta<3^{-1} \alpha$ $\{\mathcal{B}_n\}$ is locally asymptotically sufficient for $\{\mathcal{P}_n\}$ with order $o(n^{-\beta})$ in the sense that for each $n=1, 2, \ldots$ and each $\theta_0 \in \Theta$ there exists a family $\{Q_{\theta_0, n}; \theta \in \Theta\}$ of probability distributions on $(\mathcal{X}, \mathcal{A}_n)$ for which $\mathcal{P}_n$ is sufficient $\sigma$-field and that for every $b>0$

$$\sup_{\theta : n^{1/2} | \theta - \theta_0 | \leq b} ||P_{\theta, n} - Q_{\theta_0, n}||_{\mathcal{A}_n} = o(n^{-\beta})$$

uniformly in $\theta_0$ over every compact subsets of $\Theta$. Here $|| \cdot ||_{\mathcal{A}_n}$ means the total variation norm over $\mathcal{A}_n$.

We have discussed such a problem in the case $\alpha=\beta=0$ in Suzuki [3] under
non-local situation. In LeCam [1], Chap. 5 he discusses some relations between
insufficiency and deficiency in his terminology.

In Section 2 some auxiliary results about the order of asymptotic suffi-
ciency are proved. The main theorem is stated and followed by some discus-
sions about the asymptotic sufficiency in non-local sense in Section 3.

2. Auxiliary results. For each \( n \in \mathbb{N} = \{1, 2, \ldots\} \) let \( \mathcal{P}_n = \{P_{\theta,n}; \theta \in \Theta\} \)
be a family of probability distributions on a measurable space \((\mathcal{X}, \mathcal{A}_n)\) with an
index set \( \Theta \). For a subset \( U(\neq \emptyset) \) of \( \Theta \) we shall denote by \( \mathcal{P}_U^n \) the totality of
\( P_{\theta,n} \)'s satisfying \( \theta \in U \). We assume that for each \( n \in \mathbb{N} \) \( \mathcal{P}_n \) is dominated by a \( \sigma-
finite measure \( \mu_n \) on \((\mathcal{X}, \mathcal{A}_n)\). The probability density function of \( P_{\theta,n} \) relative
to \( \mu_n \) will be denoted by \( p_{\theta,n}(x, \theta) \). Without loss of generality we assume in the
following that \( \mu_n \) is a probability measure on \((\mathcal{X}, \mathcal{A}_n)\).

For each \( \theta, \theta' \in \Theta \) let
\[
S_n(\theta, \theta') = \{x; p_{\theta,n}(x, \theta)/p_{\theta',n}(x, \theta') > 0\}
\]
and let \( h_n(\theta, \theta') = p_{\theta,n}(x, \theta)/p_{\theta',n}(x, \theta') \) if \( x \in S_n(\theta, \theta') \), \(+\infty\) \( \text{if } x \in S_n(\theta) \cap S_n(\theta') \), \(-1\) \( \text{if } x \in S_n(\theta') \cap S_n(\theta)^c \). We put \( \beta_n(\theta, \theta') = \mathbb{P}_n \{S_n(\theta')\} \).

For each \( \theta, \theta' \in \Theta \) and real number \( s \geq 1 \) we define
\[
J_n(s; \theta, \theta') = E_{p_{\theta,n}}[\{h_n(x; \theta, \theta')\}^s].
\]

We note that \( \beta_n(\theta, \theta') = 1 - J_n(1; \theta, \theta') \).

Let \( \{U_n\} \) be a sequence of nonempty subsets of \( \Theta \). For \( \{U_n\} \) we consider
the following assumption.

ASSUMPTION 1. There exist a sequence \( \{\theta^*_n\}_n \in \mathbb{N} \) and a positive
number \( \gamma \) such that

(a) For every \( s \geq 1 \)
\[
\lim_{b \to \infty} \sup_{\theta \in U_n} J_n(s; \theta, \theta^*_n) < \infty,
\]
(b) \( \sup_{\theta \in U_n} \beta_n(\theta, \theta^*_n) = o(n^{-\gamma}) \).

For a sub \( \sigma \)-field \( \mathcal{C} \) of \( \mathcal{A}_n \) we denote by \( \Phi(\mathcal{C}) \) the family of \( \mathcal{C} \)-measurable statistical test functions on \( \mathcal{X} \). For each \( \theta, \theta' \in \Theta \) and each real number \( c \geq 0 \) we define
\[
r_c(\mathcal{C}; \theta, \theta') = \inf (1 + c)^{-1}\{1 - E_{p_{\theta,n}}(\phi) + c E_{p_{\theta',n}}(\phi); \phi \in \Phi(\mathcal{C})\}.
\]

Let \( \{\mathcal{B}_n\} \) be a sequence of sub \( \sigma \)-fields of \( \{\mathcal{A}_n\} \) \( (\mathcal{B}_n \subset \mathcal{A}_n) \). For each \( \theta \in \Theta \)
define \( p_{\theta,n}(x, \theta) = E_{p_{\theta,n}}[p_{\theta}(x, \theta) \mid \mathcal{B}_n] \) the conditional expectation of \( p_{\theta}(x, \theta) \) given \( \mathcal{B}_n \)
with respect to \( \mu_n \) and put \( S_n^*(\theta) = \{x; g_{\theta,n}(x, \theta) > 0\} \). For \( \theta, \theta' \in \Theta \) define \( g_{\theta,n}(x, \theta, \theta') = p_{\theta,n}(x, \theta)/p_{\theta,n}(x, \theta') \) if \( x \in S_n^*(\theta') \), \(+\infty\) \( \text{if } x \in S_n^*(\theta) \cap S_n^*(\theta') \), \(-1\) \( \text{if } x \in S_n^*(\theta') \cap S_n^*(\theta)^c \). For \( c > 0 \) and \( \delta > 0 \) let \( E_n(c, \theta, \delta) = \{x; g_{\theta,n}(x, \theta, \theta^*_n) < c + \delta \leq h_{\theta,n}(x, \theta, \theta^*_n)\} \) and \( E_n^*(c, \theta, \delta) = \{x; g_{\theta,n}(x, \theta, \theta^*_n) > c - \delta > h_{\theta,n}(x, \theta, \theta^*_n)\} \).
Proposition. Suppose that for some positive number $\alpha$ and a sequence $\{\theta^*_n\}_{n \in \mathbb{N}}$ ($\theta^*_n \in U_n$)

\begin{equation}
\sup_{c > 0} \sup_{\theta \in U_n} \{r_n(c, \theta, \theta^*_n) - r_n^*(c, \theta, \theta^*_n)\} = o(n^{-\alpha}).
\end{equation}

Then we have

\begin{equation}
\sup_{c > 0, \delta > 0} \delta (1 + c)^{-1} \lambda_n(c, \delta) = o(n^{-\alpha}), \quad \text{and}
\end{equation}

\begin{equation}
\sup_{c > 0, \delta > 0} \delta (1 + c)^{-1} \lambda'_n(c, \delta) = o(n^{-\alpha})
\end{equation}

where $\lambda_n(c, \delta) = \sup_{\theta \in U_n} \mathbb{P}(n_n(c, \theta, \delta))$ and $\lambda'_n(c, \delta) = \sup_{\theta \in U_n} \mathbb{P}(n'_n(c, \theta, \delta))$.

This proposition can be proved in the same way as the proof of the first and second steps of Theorem 1 in Suzuki [3]. So we shall omit the proof of the proposition.

Theorem 1. Suppose that Assumption 1 is satisfied with a sequence $\{\theta^*_n\}_{n \in \mathbb{N}}$ and $\gamma > 0$, and that $\{\mathcal{B}_n\}$ has the property (2.1) with $\beta > 0$. Then for every $\beta$ satisfying $0 < \beta < 3^{-1} \alpha$ and $\beta \leq \gamma$, $\{\mathcal{B}_n\}$ is asymptotically sufficient for $\{\mathcal{P}_n\}$ with order $o(n^{-\beta})$ in the following sense: For each $n \in \mathbb{N}$ there exists a family $\{q_n(x; \theta, \theta^*_n); \theta \in \Theta\}$ of probability density functions on $(\mathcal{X}, \mathcal{A}_n)$ relative to $\mu_n$ such that

(i) each $q_n$ can be factorized as follows:

$$q_n(x; \theta, \theta^*_n) = r_n(x; \theta, \theta^*_n) p_n(x, \theta^*_n)$$

where $r_n$ is a $\mathcal{B}_n$-measurable function, and

(ii) $\sup_{\theta \in U_n} \int_{\mathcal{X}} |p_n(x, \theta) - q_n(x; \theta, \theta^*_n)| d\mu_n = o(n^{-\beta})$.

Proof. We shall divide the proof into several steps.

The first step. Suppose that Assumption 1 is satisfied with a sequence $\{\theta^*_n\}_{n \in \mathbb{N}}$ and $\gamma > 0$, and that $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ has the property (2.1) with $\alpha > 0$. Let $\beta$ be any number satisfying $0 < \beta < 3^{-1} \alpha$ and $\beta \leq \gamma$. Take $\varepsilon_1$ be any number satisfying $0 < \varepsilon_1 < 3^{-1} \cdot (\alpha - 3 \beta)$. Let $\alpha_n = n^{-\beta} \log n$, $m_n = n^{a_n}$ and $i_n = [m_n \alpha^{-1}] + 1$ where $[a]$ means the maximum integer not exceeding $a$. Put $(\gamma_n = ) \gamma_n(x; \theta, \theta^*_n) = |h_n(x; \theta, \theta^*_n) - g_n(x; \theta, \theta^*_n)|$ and $(\gamma'_n = ) \gamma'_n(x; \theta, \theta^*_n) = |h_n(x; \theta, \theta^*_n) - I_{W_n}(x) g_n(x; \theta, \theta^*_n)|$ and

$$r_n(\theta, \theta^*_n) = \int_{\mathcal{X}} \gamma'_n(x; \theta, \theta^*_n) \mu_{n,n} d\mu_n$$

where $W_n = W_n(\theta, \theta^*_n) = \{x; g_n(x; \theta, \theta^*_n) \leq m_n\}$ and $I_{W_n}$ means the indicator function of $W_n$.

We have
\[
\sup_{\theta \in U_n} \rho_n(\theta, \theta^*_n) \leq \sup_{\theta \in U_n} \int_{w_n} \gamma_n \, dP_{\theta^*_n} + \sup_{\theta \in U_n} \int_{w_n} h_n(x; \theta, \theta^*_n) \, dP_{\theta^*_n} = J^*_n + J^{**}_n,
\]
(2.3) and
\[
J^*_n = \sup_{\theta \in U_n} \int_{w_n} \gamma_n \, dP_{\theta^*_n} \leq \alpha_n + \sup_{\theta \in U_n} \int_{D_n \cap w_n} \gamma_n \, dP_{\theta^*_n} = \alpha_n + I_n
\]
\[
(D_n = \{x; \gamma_n \leq \alpha_n\}).
\]
Furthermore we have
\[
I_n = \sup_{\theta \in U_n} \int_{D_n \cap w_n} \gamma_n \, dP_{\theta^*_n} \leq \sup_{\theta \in U_n} \int_{D_n \cap W_n} \gamma_n \, dP_{\theta^*_n} + \sup_{\theta \in U_n} \int_{W_n} \gamma_n \, dP_{\theta^*_n}
\]
where \( W'_n = \{x; h_n(x; \theta, \theta^*_n) \leq m_n\} \), \( \tilde{W}_n = W_n \cap W'_n \) and \( W^*_n = W_n \cap (W'_n)^c \).

The second step. It holds that
\[
I_n = \sup_{\theta \in U_n} \int_{D_n \cap W_n} \gamma_n \, dP_{\theta^*_n} \leq \sum_{i=1}^{2m_n^2} \sup_{\theta \in U_n} \int_{B_i} \gamma_n \, dP_{\theta^*_n}
\]
\[
= I_{n,1} + I_{n,2} \tag{2.4}
\]
where \( B_i = \tilde{W}_n \cap \{x; h_n(x; \theta, \theta^*_n) \leq 2^{-1}(i+1)\alpha_n, g_n(x; \theta, \theta^*_n) < 2^{-1}i\alpha_n\} \) and \( C_i = \tilde{W}_n \cap \{x; h_n(x; \theta, \theta^*_n) < 2^{-1}(i+1)\alpha_n, g_n(x; \theta, \theta^*_n) \geq 2^{-1}(i+2)\alpha_n\} \). Using the property (2.2) in Proposition we can evaluate \( I_{n,i} (i=1, 2) \) as follows. Taking account of \( 3\varepsilon_1 < \alpha - 3\beta \) we have
\[
I_{n,1} = \sum_{i=1}^{2m_n^2} \sup_{\theta \in U_n} \int_{B_i} \gamma_n \, dP_{\theta^*_n}
\]
\[
\leq 2i_n m_n \left[ \sup_{1 \leq i \leq 2m_n^2} \sup_{\theta \in U_n} P_{\theta^*_n}(x; h_n(x; \theta, \theta^*_n) \geq 2^{-1}(i+1)\alpha_n, g_n(x; \theta, \theta^*_n) < 2^{-1}i\alpha_n) \right]
\]
\[
\leq 2i_n m_n \left[ \sup_{1 \leq i \leq 2m_n^2} \lambda_n(2^{-1}i\alpha_n, 2^{-1}\alpha_n) \right]
\]
\[
\leq 4i_n m_n \left[ \sup_{1 \leq i \leq 2m_n^2} \alpha_n^{-1}(1+2^{-1}i\alpha_n)n^{-\gamma'}(\gamma' = o(1)) \right]
\]
\[
\leq 4i_n^2 m_n n^{-\gamma} \gamma'
\]
\[
\leq A_1 n^{-(\alpha-2\beta-3\varepsilon_1)/(3\varepsilon_1)} (\log n)^2 \eta' (A_1 \text{ is a constant})
\]
\[
= o(n^{-\theta}) \tag{2.5}
\]
Similarly we have
\[
I_{n,2} = o(n^{-\theta}) \tag{2.6}
\]
Thus from (2.4) and (2.5) we have
\[
I'_n = o(n^{-\theta}) \tag{2.7}
\]
The third step. Next we evaluate $I'_n$ as follows. For every $s > 1$ we have

$$I'_n = \sup_{\theta \in U_n} \int_{W_n} \gamma_n dP_{\theta_{n,s,n}} \leq \sup_{\theta \in U_n} \int_{(\hat{\eta}_n > m_n)} h_n(x; \theta, \theta^*_n) dP_{\theta_{n,s,n}}$$

$$\leq (m_n)^{-1} \sup_{\theta \in U_n} J_n(s; \theta, \theta^*_n).$$

Hence we have

$$I'_n \leq A_2(s) (m_n)^{1-s} = A_2(s) n^{1-s},$$

where $A_2(s)$ is some constant depending only on $s$. We can choose $s > 1$ large enough so that

$$I'_n = o(n^{-s}).$$

From (2.7) and (2.8) we have

$$I_n = o(n^{-s}).$$

Hence from (2.3) we have

$$J^*_n = o(n^{-s}).$$

Put $W'_n = \{ x : h_n(x; \theta, \theta^*_n) < 2^{-1} m_n \}$. Then we have

$$J^*_n = \sup_{\theta \in U_n, W'_n} \int_{W'_n} h_n(x; \theta, \theta^*_n) dP_{\theta_{n}}$$

$$\leq \sup_{\theta \in U_n, W'_n} \int_{W'_n} h_n(x; \theta, \theta^*_n) dP_{\theta_{n,s,n}} + \sup_{\theta \in U_n, W'_n} \int_{W'_n} h_n(x; \theta, \theta^*_n) dP_{\theta_{n,s,n}}$$

$$\leq 2^{-1} m_n \chi_n(m_n, m_n/2) + (m_n/2)^{1-s} \sup_{\theta \in U_n} J_n(s; \theta, \theta^*_n).$$

The first term on the right hand side is of order $o(n^{-s})$ by Proposition. The similar consideration as the evaluation of $I'_n$ implies that the second term of (2.9) is also of order $o(n^{-s})$ for sufficiently large number $s$. Thus we have

$$J^*_n = o(n^{-s}).$$

Hence it follows from (2.3) that

$$\sup_{\theta \in U_n} \rho_n(\theta, \theta^*_n) = o(n^{-s}).$$

The fourth step. Let $a_n(\theta, \theta^*_n) = \left[ \int_{X} I_{W_n}(x) g_n(x; \theta, \theta^*_n) dP_{\theta_{n,s,n}} \right]^{-1}$ and let $r_n(x; \theta, \theta^*_n) = a_n(\theta, \theta^*_n) I_{W_n}(x) g_n(x; \theta, \theta^*_n)$ if $a_n(\theta, \theta^*_n) < \infty$, $= 1$ otherwise. Define $q_n(x; \theta, \theta^*_n) = r_n(x; \theta, \theta^*_n) p_n(x; \theta^*_n)$ and let $Q_n$ be the probability distribution on $(\mathcal{X}, \mathcal{A}_n)$ with density $q_n(x; \theta, \theta^*_n)$ relative to $\mu_n$. We note that $\mathcal{B}_n$ is suf-
cient \(\sigma\)-field for the family \(\{Q_{\theta}^n; \theta \in \Theta\}\) by the factorization theorem. It follows from (2.10) that there exists \(n_0\) such that \(a_n(\theta, \theta^*) < \infty\) for every \(n \geq n_0\) and every \(\theta \in U_n\). Therefore we can assume without loss of generality that \(a_n(\theta, \theta^*) < \infty\) for every \(\theta \in U_n\) and every \(n \geq 1\).

Under this circumstances we have

\[
\|P_{\theta, n} - Q_{\theta}^n\|_{\mathcal{A}_n} = \int_{\mathcal{X}} |p_n(x, \theta) - q_n(x; \theta, \theta^*)| d\mu_n
\]

\[
\leq \int_{S_n(\Theta^*)} |h_n(x; \theta, \theta^*) - a_n(\theta, \theta^*) I_{w_n}(x) g_n(x; \theta, \theta^*)| d\mu_n + \beta_n(\theta, \theta^*)
\]

\[
= \rho_n(\theta, \theta^*) + |1 - a_n(\theta, \theta^*)| + \beta_n(\theta, \theta^*)
\]

\[
\leq 2 \rho_n(\theta, \theta^*) + 2 \beta_n(\theta, \theta^*).
\]

Here \(\|\nu\|_{\mathcal{A}_n}\) means the total variation norm of a signed meausur \(\nu\) on \((\mathcal{X}, \mathcal{A}_n)\).

From Assumption 1, (b) and (2.10) we have

\[
\sup_{\theta \in U_n} \|P_{\theta, n} - Q_{\theta}^n\|_{\mathcal{A}_n} = o(n^{-\beta}).
\]

This completes the proof of the theorem.

3. The order of local asymptotic sufficiency. In this section the index set \(\Theta\) is assumed to be a subset of \(p\)-dimensional Euclidean space \(R^p\). We denote by \(|\cdot|\) the usual Euclidean norm in \(R^p\). For \(\theta \in \Theta\) and \(b > 0\) let \(U_n(\theta, b) = \{\theta' \in \Theta; n^{1/2} |\theta' - \theta| \leq b\}\).

Let \(\{\mathcal{B}_n\}_{n \in \mathbb{N}}\) be the sequence of sub \(\sigma\)-fields \(\mathcal{B}_n \subset \mathcal{A}_n\) as in the previous section. We consider the following assumption which will be used to prove our main theorem, Theorem 2.

**Assumption 2.** For every compact subset \(K\) of \(\Theta\) and \(b > 0\)

(a) \(\limsup_{n \to \infty} \sup_{\theta \in K} \sup_{\theta^* \in U_n(\theta^*, b)} J_n(s; \theta, \theta^*) < \infty\) \((\forall s > 1)\), and

(b) \(\sup_{\theta^* \in K} \sup_{\theta \in U_n(\theta^*, b)} \beta_n(\theta, \theta^*) = o(n^{-\gamma})\).

Let \(\alpha\) be a given positive number. We state a result about higher order locally asymptotic sufficiency of \(\{\mathcal{B}_n\}\) for \(\{\mathcal{D}_n\}\).

**Theorem 2.** Suppose that Assumption 2 is satisfied with \(\gamma > 0\), and that for every compact subset \(K\) of \(\Theta\) and every \(b > 0\)

\[
\sup_{c > 0} \sup_{\theta^* \in K} \sup_{\theta \in U_n(\theta^*, b)} \{r_n(c; \theta^*) - r_n(c; \theta, \theta^*)\} = o(n^{-\alpha}).
\]

Then for every positive number \(\beta\) satisfying \(\beta < 3^{-1}\alpha\) and \(\beta \leq \gamma\) \(\{\mathcal{B}_n\}_{n \in \mathbb{N}}\) is locally asymptotically sufficient for \(\{\mathcal{D}_n\}\) with order \(o(n^{-\beta})\) in the following sense: For each
n \in N \text{ and each } \theta_0 \in \Theta \text{ there exists a family } Q_0^n = \{Q_{\theta_0}^n; \theta \in \Theta\} \text{ of probability distributions on } (\mathcal{X}, \mathcal{A}_n) \text{ such that}

(i) \mathcal{B}_n \text{ is sufficient for } Q_{\theta_0}^n, \text{ and}

(ii) for every compact subset } K \text{ of } \Theta \text{ and every } b > 0

\sup_{\theta \in K} \sup_{\theta \in U_n(\theta, b)} ||P_{\theta,n} - Q_{\theta_0}^n||_{\mathcal{A}_n} = o(n^{-b}).

Since the above result follows directly from Theorem 1 we shall omit the proof.

It is open problem whether non-local version of Theorem 2 still holds or not, i.e., whether any conditions such as in Theorem 2 imply the followings or not: There exists a sequence } Q_a = \{Q_{\theta,a}; \theta \in \Theta\} \text{ of probability distributions on } (\mathcal{X}, \mathcal{A}_a) \text{ such that } \mathcal{B}_a \text{ is sufficient for } Q_a, \text{ and that for every compact subset } K \text{ of } \Theta

(3.2) \sup_{\theta \in K} ||P_{\theta,a} - Q_{\theta,a}||_{\mathcal{A}_a} = o(n^{-b}).

The case of } \alpha = \beta = 0 \text{ has been discussed in Suzuki [3] in such a non-local situation.

It is well known that under some regularity conditions there exist a sequence } \{v_n\} \_{n \in N} \text{ of estimators of } \theta, \text{ a positive number } \gamma \text{ and a number } v \geq 1 \text{ having the following property: For every compact subset } K \text{ of } \Theta \text{ there corresponds } a(K) \text{ such that}

\sup_{\theta \in K} P_{\theta,n} \{n^{1/2} | \hat{\theta}_n(x) - \theta | \geq a(K) (\log n)^{1/2} \} = o(n^{-\gamma})

(c.f. Matsuda [2], Chap. 3).

Using such an estimator } \{\hat{\theta}_n\} \text{ we may be able to construct } \{Q_{\theta,a}; \theta \in \Theta\} \text{ satisfying the property (3.2), and for which } \mathcal{B}_a \text{ is sufficient.

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