There is a problem concerning the exchange property: which ring $R$ satisfies the condition that every projective right $R$-module satisfies the exchange property. A ring $R$ with the above condition is said to be a right $P$-exchange ring. $P$-exchange rings have been studied in [2], [3], [4], [6], [7], [11], [12], and recently in [9]. Among others, it is shown in [9] that semi-regular rings with right T-nilpotent Jacobson radical are right $P$-exchange rings, and the converse holds for commutative rings but not in general. It is still open to determine the structure of $P$-exchange rings. Our main object of this paper is to show that a ring is a right $P$-exchange ring if and only if all Pierce stalks $R_x$ are right $P$-exchange rings.

1. Preliminaries

Throughout this paper, all rings $R$ considered are associative and all $R$-modules are unitary. For an $R$-module $M$, $J(M)$ denotes the Jacobson radical of $M$. For a ring $R$, $B(R)$ represents the Boolean ring consisting of all central idempotents of $R$ and, as usual, $\text{Spec}(B(R))$ denotes the spectrum of all prime (=maximal) ideals of $B(R)$. For a right $R$-module $M$ and an element $a$ in $M$ and $x$ in $\text{Spec}(B(R))$ we put $M_x=M/Mx$ and $a_x=a+Mx (\in M_x)$. $M_x$ is called the Pierce stalk of $M$ for $x$ ([8]). Note that $M_x=M\otimes_R R_x$ and $R_x$ is flat as an $\Lambda$-module, hence for a submodule $N$ of $M$, $N_x=N\cap M_x$. For $e$ in $B(R)$, note that $e_x=1_x$ if and only if $e\in B(R)-x$. Let $A$ and $B$ be right $R$-modules and $x$ in $\text{Spec}(B(R))$. Then there exists a canonical homomorphism $\sigma$ from $\text{Hom}_R(A, B)$ to $\text{Hom}_{R_x}(A_x, B_x)$. We denote $f^*=\sigma(f)$ for $f$ in $\text{Hom}_R(A, B)$. We note that if $A$ is projective, then $\sigma$ is an epimorphism.

We will use later the following well known facts [8]:

a) Let $M$ and $N$ be finitely generated right $R$-modules with $M \subseteq N$. If $x\in \text{Spec}(B(R))$ and $M_x=N_x$ then $Me=Ne$ for suitable $e$ in $B(R)-x$.

b) For right $R$-modules $M$ and $N$ with $M \supseteq N$, if $N_x=M_x$ for all $x$ in $\text{Spec}(B(R))$, then $M=N$.

c) A ring $R$ is a commutative regular ring if and only if all stalks $R_x$ are fields, and similarly, a ring $R$ is a strongly regular ring if and only if all $R_x$ are division rings.
For an \( R \)-module \( M \) and a cardinal \( \alpha \), \( \alpha M \) denotes the direct sum of \( \alpha \)-copies of \( M \).

2. \( P \)-exchange ring

An \( R \)-module \( M \) is said to satisfy (or have) the exchange property if, for any direct sums

\[
X = \sum \bigoplus_i X_i = M \oplus Y
\]

of \( R \)-modules, there exist suitable submodules \( X_i \subseteq X_\alpha \) such that

\[
X = M \oplus \sum \bigoplus_i X_i.
\]

Whenever this property hold for any finite set \( I \), \( M \) is said to satisfy the finite exchange property. Recently, B. Zimmerman and W. Zimmerman pointed out an important fact that, in the definition above, we can assume that each \( X_i \) is isomorphic to \( M \). A ring \( R \) is said to be an exchange ring (or a suitable ring) if \( R \) satisfies the exchange property as a right, or equivalently left, \( R \)-module.

**Definition** (cf. ([9])). A ring \( R \) is a right \( P \)-exchange ring (resp. \( PF \)-exchange ring) if every projective right \( R \)-module satisfies the exchange (resp. finite exchange) property.

For the study of \( P \)-exchange (and \( PF \)-exchange) rings, we need the following conditions \((N_1)\) and \((N_2)\) for projective right \( R \)-modules \( P \):

\((N_1)\) For any finite sum \( P = \sum_{i=1}^s A_i \), there exist submodules \( A_i \subseteq A \) such that

\[
P = \sum_{i=1}^s \oplus A_i.
\]

\((N_2)\) For any sum \( P = \sum a_i R \), there exist suitable submodules \( a_i R \subseteq a_i R \) such that

\[
P = \sum a_i R.
\]

The following is due to Nicholson ([6]).

**Proposition 1.** a) The following are equivalent for a ring \( R \):

1) \( R \) is right \( PF \)-exchange.
2) \( J(R) \) is right \( T \)-nilpotent (equivalently, \( J(\mathfrak{M}_R) \) is small in \( \mathfrak{M}_R \)) and \( R/J(R) \) is right \( PF \)-exchange.
3) \((N_1)\) holds for any projective right \( R \)-module \( P \).

b) If \( R \) is right \( PF \)-exchange, then so is every factor ring of \( R \).

Similar results on \( P \)-exchange ring also hold:

**Proposition 2** (Stock [9]). a) The following are equivalent for a ring \( R \):

1) \( R \) is right \( P \)-exchange.
2) \( J(R) \) is right \( T \)-nilpotent and \( R/J(R) \) is right \( P \)-exchange.
3) \((N_2)\) holds for any projective right \( R \)-module \( P \).
b) If $R$ is right $P$-exchange, then so is every factor ring of $R$.

**Lemma 1.** If $\mathfrak{s}_\lambda R$ satisfies the condition $(N_2)$ for any countable set $I$, then so does every free (hence every projective) right $R$-module.

Proof. Let $F=\sum \bigoplus R_\lambda$ be a free right $R$-module with $R_\lambda \subseteq R$. Consider a sum $F=\sum \bigoplus a_\lambda R$. For subsets $I \subseteq \Lambda$ and $J \subseteq \Gamma$, put $F(I)=\sum \bigoplus a_\lambda R$ and $A(J)=\sum a_\lambda R$. First we take a finite subset $I_1 \subseteq \Lambda$. Starting from $I_1$, we can proceed to take $J_1 \subseteq \Gamma$, $I_2 \subseteq \Lambda$, $J_2 \subseteq \Gamma$, $I_3 \subseteq \Lambda$, ... such that
1) each $I_i$ and $J_i$ are finite sets,
2) $I_1 \subseteq I_2 \subseteq \ldots$, $J_1 \subseteq J_2 \subseteq \ldots$,
3) $F(I_1) \subseteq A(J_1) \subseteq F(I_2) \subseteq A(J_2) \subseteq \ldots$.

Putting $\Lambda_1=\bigcup I_i$ and $\Gamma_1=\bigcup J_i$, we see that
4) $|\Lambda_1| \leq \aleph_0$, $|\Gamma_1| \leq \aleph_0$,
5) $F(\Lambda_1)=A(\Gamma_1)$.

Next, we take a finite subset $K_1 \subseteq \Lambda - \Lambda$. And again starting from $K_1$, we take subsets $L_1 \subseteq \Gamma - \Gamma_1$, $K_2 \subseteq \Lambda - \Lambda_1$, $K_3 \subseteq \Lambda - \Lambda_2$, ... such that
1) each $K_i$ and $L_i$ are finite sets,
2) $K_1 \subseteq K_2 \subseteq \ldots$, $L_1 \subseteq L_2 \subseteq \ldots$,
3) $F(\Lambda_2) \oplus F(K_1) \subseteq A(\Gamma_1) + A(L_1) \subseteq F(\Lambda_1) \oplus F(K_2) \subseteq A(\Gamma_1) + A(L_2) \subseteq \ldots$.

Putting $\Lambda_2=\bigcup K_i$ and $\Gamma_2=\bigcup L_i$, we see that
4) $|\Lambda_2| \leq \aleph_0$, $|\Gamma_2| \leq \aleph_0$,
5) $F(\Lambda_2)=A(\Gamma_2)$.

Proceeding this argument transfinite-inductively, we can get a well ordered set $\Omega$ and subfamilies $\{\Lambda_\alpha\}_{\alpha \in \Omega} \subseteq 2^\Lambda$ and $\{\Gamma_\alpha\}_{\alpha \in \Omega} \subseteq 2^\Gamma$ such that
a) for each $\alpha \in \Omega$, $|\Lambda_\alpha| \leq \aleph_0$ and $|\Gamma_\alpha| \leq \aleph_0$,
b) for each $\alpha \in \Omega$, $\sum \bigoplus F(\Lambda_\beta) = \sum A(\Gamma_\beta)$,
c) $F=\sum \bigoplus F(\Lambda_\alpha)=\sum A(\Gamma_\alpha)$.

For each $\alpha \in \Omega$, let $\psi_\alpha: F=\sum \bigoplus F(\Lambda_\alpha) \to F(\Lambda_\alpha)$ be the projection. By b) we see that

$$F(\Lambda_\alpha) = \psi_\alpha(A(\Gamma_\alpha)).$$

and

$$F = \sum \bigoplus F(\Lambda_\alpha) = \sum \psi_\alpha(A(\Gamma_\alpha)).$$

Since $F(\Lambda_\alpha) = \sum \bigoplus R_\psi = \sum \psi_\alpha(a_\lambda R) = \psi_\alpha(A(\Gamma_\alpha))$, we can take $a_\lambda \leq a_\lambda R$ for all $\lambda \in \Gamma_\alpha$ such that

$$\sum_{\lambda \in \Gamma_\alpha} \bigoplus \psi_\alpha(a_\lambda R) = \sum_{\psi \in \Lambda_\alpha} \bigoplus R_\psi.$$
Since $\psi_{\lambda}(a_\alpha R)$ is projective, we can take $a_\alpha \in a_\lambda R$ such that the restriction map $\psi_{\alpha}|a_\alpha R$ is an isomorphism for each $\lambda \in \Gamma_\alpha$ and $\alpha \in \Omega$. Then we see that

$$F = \sum_{\alpha \in \Omega} \bigoplus \left( \sum_{\lambda \in \Gamma_\alpha} \oplus a_\lambda R \right)$$

as desired.

**Lemma 2.** If $R_0 R$ satisfies the exchange property, then $R_0 R$ satisfies the condition $(N_2)$.

Proof. Let $F = \sum_{i=1}^{\infty} \bigoplus m_i R$ be a free right $R$-module $R = m_i R$ by $r \mapsto m_i r$.

Consider a sum $F = \sum_{i=1}^{\infty} a_i R$, and let $\psi: \sum_{i=1}^{\infty} m_i R \to \sum_{i=1}^{\infty} a_i R$ be the canonical epimorphism from Lemma 1. Since $F = \sum_{i=1}^{\infty} a_i R$ is projective, $\ker \psi \psi F$; say $F = B \oplus \ker \psi$. Let $\pi: F = B \oplus \ker \psi \to B$ be the projection and put $b_i = \pi(m_i)$ for all $i$. Then $\psi(b_i) = a_i$ for all $i$. By assumption, there exist a decomposition $m_i R = n_i R \oplus t_i R$ for each $i$ such that

$$F = \left( \sum_{i=1}^{\infty} b_i R \right) \oplus \ker \psi$$

$$= \left( \sum_{i=1}^{\infty} n_i R \right) \oplus \ker \psi.$$ 

Since $\pi(n_i R) \subseteq b_i R$ and $\sum_{i=1}^{\infty} \pi(n_i R) = \sum_{i=1}^{\infty} b_i R$, we have that $\sum_{i=1}^{\infty} \psi \pi(n_i R) = \sum_{i=1}^{\infty} a_i R$ and $\psi \pi(n_i R) \subseteq a_i R$ for each $i$. Thus $F$ satisfies the condition $(N_2)$.

**Theorem 1.** The following conditions are equivalent for a given ring $R$:

1) $R$ is a right $P$-exchange ring.
2) Every projective right $R$-module satisfies the condition $(N_2)$.
3) $R_0 R$ has the exchange property.
4) $R_0 R$ satisfies the condition $(N_2)$.

Proof. The implications 1)$\Rightarrow$3) and 2)$\Rightarrow$4) are trivial. 1)$\Leftrightarrow$2) is Proposition 2. The implication 4)$\Rightarrow$2) is Lemma 1 and 3)$\Rightarrow$4) is Lemma 2.

3. Commutative $P$-exchange ring

In this section, we study the rings whose Pierce stalks are local right perfect rings. Such rings are right $P$-exchange rings and for commutative rings the converse also holds (Theorem 2 and Corollary 1)

**Lemma 3.** If $R$ is a ring such that all $R_x$ are local right perfect rings, then so is every factor ring of $R$.

Proof. Let $I$ be an ideal of $R$, and put $R = R/I$. Let $y$ be in $\text{Spec}(B(R))$ and put $x = \{ e \in B(R) | e + I \in y \}$. Then $x \in \text{Spec}(B(R))$ and there is a ring
epimorphism from \( R_x \) to \( R_y \), as a result, \( R_y \) is also a local right perfect ring.

**Proposition 3.** Let \( R \) be a ring whose Pierce stalks are local right perfect rings. Then

1) \( J(R) \) is right T-nilpotent,
2) \( J(E) \) coincides with the set of all nilpotent elements of \( R \).

**Proof.** 1) Let \( \{a_i| i=1, 2, \ldots \} \) be a subset of \( J(R) \) and let \( x \in \text{Spec}(B(R)) \). Since \( J(R)_x \subseteq J(R_x) \), \( \{(a_i)_x| i=1, 2, \ldots \} \subseteq J(R_x) \). Hence there exists \( n \) such that \((a_n)_x(a_{n-1})_x \cdots (a_1)_x = 0 \). So there exists a neighborhood \( N(x) \) of \( x \) such that \((a_nz \cdots a_1z)_x = 0 \) for all \( z \) in \( N(x) \). Hence by the partition property of \( \text{Spec}(B(R)) \), we can have neighborhoods \( N_{i_1}, N_{i_2}, \ldots, N_{i_k} \) such that \((a_{i_1}a_{i_2} \cdots a_{i_k})_{x_i} = O_x \) for all \( x \) in \( N_{i_i} \) for \( i = 1, \ldots, k \). Hence if we put \( m = \max\{m_i\} \), then \((a_ma_{m-1} \cdots a_1)_x = O_x \) for all \( x \) in \( \text{Spec}(B(R)) \), hence \( a_m a_{m-1} \cdots a_1 = 0 \).

2) By 1) \( J(R) \) is nil. For \( x \in \text{Spec}(B(R)) \), we denote by \( M(x) \) the unique maximal (right) ideal of \( R \) containing \( R_x \). Then we see that \( \{M(x) \mid x \in \text{Spec}(B(R))\} \) is just the family of all maximal right ideals of \( R \). For, if \( M \) is a maximal right ideal of \( R \), then \( \{e \in B(R) \mid e \in M\} \subseteq \text{Spec}(B(R)) \). As a result, we have \( J(R) = \bigcap \{M(x) \mid x \in \text{Spec}(B(R))\} \). Now, let \( a \) be a nilpotent element of \( R \). Since \( M(x) \mid R_x = J(R_x) \), we see that \( a \in M(x) \). (Note that \( R_x \) is local). Hence \( a \in \bigcap \{M(x) \mid x \in \text{Spec}(B(R))\} = J(R) \). Accordingly \( J(R) \) coincides with the set of all nilpotent elements of \( R \).

**Lemma 4.** Let \( R \) be a ring such that \( J(R) = O \) and all stalks \( R_x \) are local right perfect rings. Then \( R \) is a strongly regular ring.

**Proof.** We may show that all stalks are division rings. Let \( x \in \text{Spec}(B(R)) \). Let \( a \) be in \( R \) such that \( a_x \in J(R_x) \). Then there exists \( n \) such that \((a_x)^n = (a_x^n) = 0 \), so \( a^n e = O \) for a suitable \( e \) in \( B(R) - x \). Since \((ae)^n = d^n e = O \), Proposition 4 shows that \( ae \in J(R) = O \), so \( a_x = O_x \). Thus \( J(R_x) = O \). Since \( R_x \) is a right perfect ring, it follows that \( R_x \) is a division ring.

**Notation.** For a ring \( R \), we denote by \( I(R) \) the set of all idempotents of \( R \). Of course \( B(R) \subseteq I(R) \).

**Lemma 5.** For a ring \( R \), the following are equivalent:

1) \( I(R) = B(R) \).
2) \( I(R_x) = \{1_x, O_x\} \) for all \( x \in \text{Spec}(B(R)) \).

**Proof.** 1) \( \Rightarrow \) 2): Let \( a \in R \) such that \( a_x \in I(R_x) \) (where \( x \in \text{Spec}(B(R)) \). Since \((a^2)_x = a_x^2, d^2 e = ae \) for some \( e \) in \( B(R) - x \). Then \( ae \in I(R) = B(R) \), we see that \( a_x = (ae)_x \) is either \( 1_x \) or \( O_x \). 2) \( \Rightarrow \) 1): Let \( a \in I(R) \) and \( x \in \text{Spec}(B(R)) \). Then \( a_x = 1_x \) or \( a_x = 0_x \) since \( a_x \in I(R_x) \). Here using the partition property of
Spec($B(R)$), we can take a suitable $e$ in $B(R)$ such that $ae=e$ and $a(1-e)=0$, whence $a=e \in B(R)$. Thus $I(R)=B(R)$.

We are now ready to show the following.

**Theorem 2.** The following conditions are equivalent for a given ring $R$:
1) $R$ is a right $P$-exchange ring and $I(R)=B(R)$.
2) $R/J(R)$ is a strongly regular ring, $J(R)$ is right $T$-nilpotent and $I(R)=B(R)$.
3) All stalks are local right perfect rings.

Proof. 1)$\Rightarrow$3): By Proposition 2 (b) and Lemmas 3 and 5, each $R_x$ is a $P$-exchange ring with $I(R_x)=B(R_x)$, whence $R_x$ is a right perfect ring by [11, Theorem 8]. The implication 2)$\Rightarrow$1) follows from Proposition 2. The implication 3)$\Rightarrow$2) follows from Proposition 3 and Lemmas 3 and 4.

**Corollary 1.** The following conditions are equivalent for a commutative ring $R$.
1) $R$ is $P$-exchange ring.
2*) $R/J(R)$ is a regular ring and $J(R)$ is $T$-nilpotent.
3) All stalks are lodal perfect rings.

**Remark 3.** The equivalence of 1) and 2) in Theorem 2 above is shown in [9]. It should be noted that an exchange ring with $T$-nilpotent Jacobson radical need not be a $P$-exchange ring, because there exist a non-regular commutative exchange ring $R$ with $J(R)=0$ ([5]).

**4. Main Theorem**

As we see later, or by [9] the equivalence of 1) and 2) in Corollary 1 does not hold in general. However we show that 1) and 3) are equivalent, that is, the following holds:

**Theorem 3.** A ring $R$ is a right $P$-exchange ring if and only if all Pierce stalks $R_x$ are $P$-exchange rings.

**Lemma 6.** Let $P$ be a projective right $R$-module and let $x \in \text{Spec}(B(R))$. 1) If $A$ is a finitely generated submodule with $A_x \subseteq P_x$, then $Ae \subseteq P$ for a suitable $e$ in $B(R)-x$. 2) If $P$ is finitely generated and $A_1$ and $A_2$ are finitely generated submodules of $P$ with $P_x=(A_1)_x \oplus (A_2)_x$, then $Pe=A_1e \oplus A_2e$ for a suitable $e$ in $B(R)-x$.

Proof. As 1) follows from 2), we may only show 2). Let $\tau_1$ be the

*) Prof. Y. Kurata informed the authors that commutative rings $R$ which satisfy the condition 2) in Corollary 2 are studied in [1].
inclusion mapping: $A_i \to P$ for $i = 1, 2$. Since $P$ is projective, there exist $\pi_i: P \to A_i$ and $\pi'_i: P \to A_i$ such that $(\pi_i)^*\pi'_i$ is the projection: $P = (A_i) \oplus (A_i)$ for $i = 1, 2$. Noting that $P$, $A_1$ and $A_2$ are finitely generated, we can take a suitable $e$ in $B(R) - x$ such that

$$(1 - (\tau_1\pi_i + \tau_2\pi'_i))(P e) = 0,$$

$$(\tau_1\pi_i - (\tau_1\pi'_i))(P e) = 0 \quad \text{for} \quad i \neq j.$$  

Then it follows that $P e = A_1 e \oplus A_2 e$.

**Lemma 7.** Let $P$ be a projective right $R$-module with a sum $P = \sum_{i=1}^n a_i R$, and let $x \in \text{Spec}(B(R))$. If $P_x = \sum_{i=1}^n (a_i R)_x$, then there exists $\{e_i\}_{i=1}^n \subseteq B(R) - x$ such that $\sum_{i=1}^n a_i e_i R = \sum_{i=1}^n (a_i e_i R) \oplus P$ for all $n$.

**Proof.** Since $(a_i R)_x \ominus P_x$, there exists $e_i \in B(R) - x$ such that $a_i e_i R \ominus P$ by Lemma 6. Since $(a_i R \ominus a_i R)_x \ominus P_x$, there exists $e'_i \in B(R) - x$ such that $a_i e'_i R \ominus a_i e'_i R \ominus P$. Put $e_2 = e_1 e'_2$. Then we see that

$$a_i e_i R + a_i e'_2 R = a_i e_i R \oplus a_i e'_2 R \ominus P.$$  

By similar argument, we can take $\{e_i\}_{i=1}^n \subseteq B(R) - x$ such that $e_n e_{n+1} = e_{n+1}$ for $n = 1, 2, \ldots$ and

$$a_i e_i R + \cdots + a_n e_n R = a_i e_i R \oplus \cdots \oplus a_n e_n R \ominus P$$  

for $n = 1, 2, \ldots$.

**Lemma 8.** Let $P$ be a finitely generated projective right $R$-module such that all stalks $P_x$ have the exchange property. Then $P$ has the exchange property.

**Proof.** Since $P$ is finitely generated, we may show that $P$ satisfies the condition $(N_1)$ (Proposition 1). So, let $P = A \oplus B$, where $A$ and $B$ are finitely generated submodules. Let $x \in \text{Spec}(B(R))$. Since $P_x$ satisfies $(N_1)$, we can take finitely generated submodules $A^x \subseteq A$ and $B^x \subseteq B$ such that $P_x = (A^x) \oplus (B^x)$.

Then, by Lemma 6, $P e = A^x e \oplus B^x e$ for a suitable $e$ in $B(R) - x$. Using the partition property of $\text{Spec}(B(R))$, we can take orthogonal idempotents $e_1, \ldots, e_n$ in $B(R)$ and finitely generated submodules $A^x_1, \ldots, A^x_n$ of $A$ and $B^x_1, \ldots, B^x_n$ of $B$ such that

$$P = A^x_1 e_1 \oplus \cdots \oplus A^x_n e_n \oplus B^x_1 e_1 \oplus \cdots \oplus B^x_n e_n.$$  

Hence putting $A^* = A^x_1 e_1 \oplus \cdots \oplus A^x_n e_n$ and $B^* = B^x_1 e_1 \oplus \cdots \oplus B^x_n e_n$, we have that $P = A^* \oplus B^*$.  

**Proof of Theorem 3.** If $R$ is a right $P$-exchange ring, then all $R_x$ are right $P$-exchange rings by Proposition 2. Conversely, assume that all $R_x$ are right $P$-exchange rings. We may show that $e_0 R$ satisfies the condition $(N_2)$. Let
$F = \bigoplus_{i=1}^{\infty} R_i$ be a free right $R$-module with $R_i \cong R$ for all $i$, so $F = \mathbb{R}$. We put $F(s) = R_1 \oplus \cdots \oplus R_s$ for $s = 1, 2, \cdots$. Now, consider a sum $F = \bigoplus_{i=1}^{\infty} a_i R$. For any $x$ in Spec$(B(R))$, as $F_x$ satisfies $(N_2)$, we can take by Lemma 7 \{b_i \in a_i R | i = 1, 2, \cdots\}$ such that $F_x = \bigoplus_{i=1}^{\infty} (b_i R)_x$ and

$$F \oplus b_1^* R \oplus \cdots \oplus b_s^* R = b_1^* R \oplus \cdots \oplus b_s^* R$$

for all $n$.

Let $x \in \text{Spec}(B(R))$ and take any $s_1 \geq 1$. Then there exists $n(x)$ such that

$$F(s_1) \subseteq \bigoplus_{i=1}^{n(x)} (b_i R)_x$$

and so there exists $e(x)$ in $B(R) - x$ such that

$$F(s_1) e(x) \subseteq \bigoplus_{i=1}^{n(x)} b_i e(x) R \oplus F$$

Using the partition property, we have $x_1, \ldots, x_s$ in Spec$(B(R))$, orthogonal idempotents $\{e(x_1), \ldots, e(x_s)\} \subseteq B(R)$ and $m_1$ such that $1 = \sum_{i=1}^{m_1} e(x_i)$ and

$$F(s_1) \subseteq \bigoplus_{i=1}^{m_1} b_i e(x_i) R \oplus \cdots \oplus \bigoplus_{i=1}^{m_1} b_i e(x_s) R \oplus F$$

Put $b_i^* = \sum_{i=1}^{m_1} b_i e(x_i)$ for $i = 1, \ldots, m_1$. Then $b_i^* \in a_i R$ and

$$F(s_1) \subseteq \bigoplus_{i=1}^{m_1} b_i^* R \oplus F$$

Put $G_1 = \sum_{i=1}^{m_1} b_i^* R$. Then $G_1 \subseteq F(s_1)$ for a suitable $s_2 > s_1$. By the same argument as above, we can take $m_2$ and $b_i^* \in a_i R$ for $i = 1, \ldots, m_2$ such that

$$F(s_2) \subseteq \bigoplus_{i=1}^{m_2} b_i^* R$$

Put $G_2 = \sum_{i=1}^{m_2} b_i^* R$. Then $G_2 \subseteq F(s_3)$ for some $s_3 > s_2 > s_1$. Continuing this argument, we can take $s_1 < s_3 < s_2 < \cdots$ and $G_1 = \sum_{i=1}^{m_1} b_i^* R$, $G_2 = \sum_{i=1}^{m_2} b_i^* R$, $\cdots$ such that $b_i^* \in a_i R$ for all $i, k$, each $G_i$ is a direct summand of $F$ and

$$F(s_1) \subseteq G_1 \subseteq F(s_2) \subseteq G_2 \subseteq F(s_3) \subseteq \cdots$$

Since $\bigcup_{i=1}^{\infty} G_i \subseteq \bigcup_{i=1}^{\infty} F(s_i)$, we see $F = \bigoplus_{i=1}^{\infty} G_i$. Since $G_{s-1}$ has the exchange property by Lemma 8, there exists $\{c_i^* \in b_i^* R | i = 1, \ldots, m_3\}$ such that

$$G_n = G_{n-1} \oplus \bigoplus_{i=1}^{m_3} c_i R$$

In particular, put $c_i^* = b_i^*$ for $i = 1, \ldots, m_3$. Then we see that

$$F = \bigoplus_{i=1}^{m_1} c_1 R \oplus \bigoplus_{i=1}^{m_2} c_i R \oplus \bigoplus_{i=1}^{m_3} c_i R \oplus \cdots$$

We put
\[ A_k = \sum_{i=1}^{n} \bigoplus c_i R \quad \text{for} \quad k = 1, 2, \ldots. \]

Then \( A_i \subseteq a_i R \) for all \( i \) and \( F = \sum_{i=1}^{n} A_i \). This completes the proof.

**Corollary 2.** If \( R \) is a ring such that all Pierce stalks are right perfect rings, then \( R \) is a right \( P \)-exchange ring.

By making use of the corollary, we shall give a right \( P \)-exchange ring.

**Example.** Let \( P \) be an indecomposable right perfect ring and \( Q \) an indecomposable right perfect subring of \( P \) with the same identity. Consider the rings \( W = \prod_{i} P_{i} \) and \( V = \prod_{i} Q_{i} \), where \( Q_{i} \subseteq Q \) and \( P_{i} \subseteq P \) for all \( \alpha \in I \). Then the ring \( W \) is an extension ring of \( V \) and becomes a right \( V \)-module. Put \( R = \sum_{i} \bigoplus P_{i} + 1Q \), where 1 is the identity of \( W \). Then \( R \) is a ring such that \( B(R) = \sum_{i} \bigoplus B(P_{i}) + 1B(Q) \). We can easily see that \( \text{Spec}(B(R)) = \{ x_{0} \} \cup \{ x_{\alpha} | \alpha \in I \} \), where \( x_{0} = \sum_{i} \bigoplus B(P_{i}) \) and \( x_{\alpha} = \sum_{i=0}^{\alpha} \bigoplus B(P_{i}) + 1B(Q) \). Further we see that \( R_{x_{0}} \subseteq Q \) and \( R_{x_{\alpha}} \subseteq P \) for all \( \alpha \in I \). Hence Corollary 3 says that \( R \) is a right \( P \)-exchange ring. In particular, if we take \( \begin{pmatrix} F & F \\ F & F \end{pmatrix} \) and \( \begin{pmatrix} F & F \\ F & 0 \end{pmatrix} \) as \( P \) and \( Q \), respectively, where \( F \) is a division ring, then \( R \) is a non-singular, right \( P \)-exchange ring with \( J(R) = 0 \).

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**References**


*) Note that \( B(P) = B(Q) = GF(2) \)

