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WEAKLY-INJECTIVE RINGS AND MODULES

Dedicated to Professor Manabu Harada
on his 60th birthday

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1. Introduction

The concept of weak relative-injectivity of modules was introduced in [6] in order to study rings all of whose cyclic modules are embeddable as essential submodules of projective modules. The study of weak relative-injectivity of rings and modules relates to that of quasi-Frobenius rings, QI-rings and to rings of quotients.

An R-module M is called weakly R^n-injective if every n-element generated submodule of $E(M)$, the injective hull of M, is contained in a submodule of $E(M)$ isomorphic to M. An R-module M is called weakly-injective if it is weakly R^n-injeative for all $n>0$. The ring R is called a right weakly-injective ring if R is weakly-injective as right R-module.

Lemma 3.2 shows that weak R-injectivity is not a Morita invariant. However, if R is a weakly-injective integral domain and K is a ring Morita equivalent to R, then K is a weakly-injective ring (Theorem 3.1). Furthermore, for any nonsingular ring R, it is shown that R is a weakly-injective ring, if and only if the $n\times n$ matrix ring S over R is also a weakly-injective ring (Theorem 3.3). Among other results on the weak relative injectivity of triangular matrix rings, it is proved that if V is a $(D-D)$-space over a division ring D, then $R=\begin{pmatrix} D & V \\ 0 & D \end{pmatrix}$ is weakly R-injective if and only if $V\cong D$ (Corollary 4.6).

As an application we provide an example of an artinian nonsingular QF-3 ring R which is not weakly R-injective, answering a question raised by Professor Tachikawa during S.K. Jain's visit to Japan. Recall that a ring R is said to be right QF-3 if it has a minimal faithful right module [9]. It is well known that a nonsignular ring R is right and left QF-3 if and only if R has a two-sided semi-simple artinian complete ring of quotients and both the left socle and the right socle of R are essential in R.

2. Definitions, Notation and Preliminaries

Let M and N be right R-modules and let $E(M)$ be an injective hull of M.

M is called weakly N-injective if for each homomorphism f: $N \to E(M)$, there exists a submodule X of $E(M)$ such that $f(N) \subset X \cong M$. Equivalently, for every homomorphism f: $N \to E(M)$, there exists a homomorphism g: $N \to M$ and a monomorphism h: $M \to E(M)$ such that the following diagram commutes.

$$\begin{tikzcd} N \arrow[rr, dashed, "f"] \arrow[dr, dashed, "g"'] & & E(M) \\ & M \arrow[ur, dashed, "h"'] & \end{tikzcd}.$$

For a right module M, $E(M_R)$ or simply $E(M)$ will denote the injective hull of M. The right maximal ring of quotients of R and the classical right ring of quotients of R will be denoted by $r\cdot Q(R)$ and $r\cdot Q_{cl}(R)$, respectively. The symbol $r\cdot ann_X(S)$ will denote the right annihilator of S in X, where S and X may be subsets of rings or modules. The right singular submodule and the Jacobson radical of a right R-module M will as usual be denoted by $Z(M_R)$ and $J(M_R)$ or simply $Z(M)$ and $J(M)$, respectively. If $Z(R_R)=0$, R is said to be a right nonsingular ring. The notation $N \subseteq' M$ will mean that N is essential in M. All left concepts are defined analogously. Throughout this paper all modules are right and unital unless otherwise stated.

The following Lemmas 2.1–2.7 from [6] and [7] are included here without proofs for easy reference.

Lemma 2.1. *An R-module M is weakly R^n-injective, where n is a positive integer, if and only if for all $x_1, \cdots, x_n \in E(M)$ there exists a submodule X of $E(M)$ such that $x_i \in X \cong M$, $i=1, \cdots, n$.*

In particular,

Lemma 2.2. *A ring R is weakly R^n-injective if and only if for all $q_1, \cdots, q_n \in E(R)$ there exists $q' \in E(R)$ such that $q_i \in q' R$, $i=1, \cdots, n$ and $r\cdot ann_R(q')=0$.*

Lemma 2.3. *Let M, N, P be R-modules such that $N \subset' M$ and N is weakly P-injective. Then M is weakly P-injective.*

Lemma 2.4. *The following statements are equivalent.*

(i) M is weakly N-injective.

(ii) M is weakly N/K-injective for all $K \subset N$.

Lemma 2.5. *The following statements are equivalent for an R-module M.*

(i) M is weakly R^n-injective.

(ii) M is weakly N-injective, whenever N is an R-module generated by n

elements.

We say that an R-module M is weakly-injective if M is weakly N-injective for all finitely generated R-modules N, or equivalently, if M is weakly R^n-injective for all $n>0$.

Lemma 2.6. *A cyclic R-module M is weakly-injective if and only if it is weakly R^2-injective.*

Lemma 2.7.
(i) *A domain R is right weakly R-injective if and only if R is a right Ore-domain.*
(ii) *A domain R is right weakly-injective if and only if R is a right and left Ore-domain.*

Lemma 2.8. *If R and K are rings such that $R \subset K \subset E(K_K) \subset E(R_R)$, and if R is a weakly-injective ring, then K is a weakly-injective ring.*

Proof. Let $q_1, q_2 \in E(K_R)$. Then $q_1, q_2 \in E(R_R)$. So by Lemma 2.2, there exists $b \in E(R_R)$ such that $r\cdot ann_R(b)=0$ and $q_1, q_2 \in bR$. By hypothesis, $R \subset' K$. Thus $r\cdot ann_K(b)=0$, proving K is weakly-injective.

The following lemma is part of the folklore.

Lemma 2.9. *Let Q be nonsingular ring containing R as a subring such that $R_R \subset' Q_R$. Then $Z(Q_R)=0$, and hence $Z(R_R)=0$.*

Proof. Let $q \in Q$. Then $r\cdot ann_Q(q)$ is a closed submodule of Q and hence it is not essential in Q. Thus there exists $0 \neq K \subset Q$ such that $r\cdot ann_Q(q) \cap K=0$ which yields $r\cdot ann_R(q) \cap (K \cap R)=0$. Thus $r\cdot ann_R(q)$ is not essential in R. This proves $Z(Q_R)=0$.

Recall that for a right nonsingular ring the maximal right of quotients, $r\cdot Q(R)$ is a regular right self-injective ring and it coincides with $E(R_R)$.

Lemma 2.10. *For a right nonsingular ring R, the following statements are equivalent.*
(i) *R is a right weakly-injective ring.*
(ii) *For all $q_1, q_2 \in Q$, there exists $c \in R$ such that $q_1, q_2 \in c^{-1}R$. In particular, Q is a classical left ring of quotients of R.*

Proof. (i)$\Rightarrow$(ii). Now $Q=E(R_R)$ is a regular right self-injective ring. Let $1, q_1, q_2 \in Q$. By Lemma 2.2 there exists $b \in Q$ such that $r\cdot ann_R(b)=0$ and $1 \in bR$, $q_1 \in bR$ and $q_2 \in bR$. Since $r\cdot ann_R(b)=0$, b has a left inverse say c in Q. Also $1 \in bR$ implies b has a right inverse in R. Thus $q_i \in c^{-1}R$, where $c \in R$, $i=1, 2$. To prove that Q is a classical left ring of quotients, we need to show in addition that every regular element in R is invertible in Q. We first note that ${}_RR\subset'{}_RQ$. Next let $x\in R$ be a regular element. Then $r\cdot ann_Q(x)=l\cdot ann_Q(x)=0$ since $R_R\subset'Q_R$ and ${}_RR\subset'{}_RQ$. Hence x is invertible in Q.

(ii)$\Rightarrow$(i). Obvious.

In case R is von-Neumann regular, we get the following interesting result.

Theorem 2.11. *If R is a von-Neumann regular ring, then the following statements are equivalent.*

(i) *R is a right self-injective ring.*

(ii) *R is a right weakly-injective ring.*

Proof. This follows from Lemma 2.10.

The following example shows that a right weakly R-injective regular ring need not be right self-injective.

EXAMPLE 2.12. Let F be a field and K be a proper subfield. Let $S=\prod_{i=1}^{\infty} F_i$, $F_i=F$ and $R=\{(x_i)\mid \text{all but finitely many } x_i\in K\}$. Then R is regular weakly R-injective but R is not right self-injective. Incidentally, it is known ([7], Examples 1.15 (iv)) that a right continuous ring is right weakly-injective if and only if it is right self-injective. This example shows that weak R-injectivity is not equivalent to self-injectivity when the ring is continuous. (See [4], Example 13.8)

We show now that for a right nonsingular ring right and left weak-injectivity implies the coincidence of the classical ring of quotients with the maximal ring of quotients.

Theorem 2.13. *Let R be a right nonsingular ring. Then the following statements are equivalent.*

(i) *R is right and left weakly-injective.*

(ii) $E({}_RR)=l\cdot Q(R)=r\cdot Q_{cl}(R)=l\cdot Q_{cl}(R)=r\cdot Q(R)=E(R_R)$.

Proof. (i)$\Rightarrow$(ii). By Lemma 2.10 we have $Q=E(R_R)=l\cdot Q_{cl}(R)$. Therefore, considering Q as a left R-module, we have ${}_RR\subset'{}_RQ$. Since Q is von-Neumann regular, by Lemma 2.9, $Z({}_RR)=0$. Therefore, applying Lemma 2.10 to the left weakly-injective left module R, we get $r\cdot Q_{cl}(R)=l\cdot Q(R)=E({}_RR)$. Since both classical right and left quotient rings exist, they must coincide, hence

$$E({}_RR)=l\cdot Q(R)=r\cdot Q_{cl}(R)=l\cdot Q_{cl}(R)=r\cdot Q(R)=E(R_R)\,.$$

(ii)$\Rightarrow$(i). This follows by definition and Lemma 2.2.

The following result provides a method of constructing a nontrivial weakly

R-injective module over a commutative ring possessing a valuation injective module. Example 2.15 shows that the condition that R is commutative cannot be removed.

Proposition 2.14. *Let R be a commutative ring and E_R be an injective valuation module. Then for all $M \subset E$, $N = M \oplus E$ is weakly R-injective.*

Proof. Clearly $E(N) = E \oplus E$.

Let $q = (a, b) \in E(N)$, $a \in E$, $b \in E$. Now either $aR \subset bR$ or $bR \subset aR$. Without loss of generality, let $bR \subset aR$. Hence $b = ax$ for some $x \in R$. Thus we have $q = (a, b) = (a, ax) \in \{(c, cx): c \in E\} = Y \cong E$. Choose $X = Y \oplus \{(0, c): c \in M\} \cong E \oplus M$. Therefore, $N = M \oplus E$ is weakly R-injective.

EXAMPLE 2.15. Let $T = \begin{pmatrix} D & D & D \\ 0 & D & D \\ 0 & 0 & D \end{pmatrix}$ be the upper triangular 3×3 martix ring over D. T is weakly T-injective. Considering T as a right T-module, write:

$$T_T = \begin{pmatrix} D & D & D \\ 0 & 0 & D \end{pmatrix} \oplus (0 \ D \ D).$$

We show that the direct summands are not weakly T-injective. First we show that $(0\ D\ D)_T$ is not weakly T-injective. The injective hull of $(0\ D\ D)_T$ is $(D\ D\ D)_T = (1\ 0\ 0)T$, a cyclic module. If $(0\ D\ D)$ were weakly T-injective, then $(1\ 0\ 0)T$ would be embedded in $(0\ D\ D)_T$ which is impossible, since $(1\ 0\ 0)T = (D\ D\ D)_T$ has dimension 3 over D, but $(0\ D\ D)_T$ has dimension 2. Next, we assert that $\begin{pmatrix} D & D & D \\ 0 & 0 & D \end{pmatrix}_T$ is not weakly T-injective. Note that its injective hull is $\begin{pmatrix} D & D & D \\ D & D & D \end{pmatrix}$. Clearly,

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} T = \begin{pmatrix} D & D & D \\ 0 & D & D \end{pmatrix} \not\hookrightarrow \begin{pmatrix} D & D & D \\ 0 & 0 & D \end{pmatrix}$, proving our assertion.

We now produce a simple T-module S such that $S \oplus E(S)$ is not weakly T-injective. Consider $\begin{pmatrix} D & D & D \\ 0 & 0 & D \end{pmatrix}_T = (D\ D\ D)_T \oplus (0\ 0\ D)_T$. Here $(D\ D\ D)_T$ is valuation and injective and $(0\ 0\ D)$ is simple. Therefore, we have shown that if S is a simple T-module, then $S \oplus E(S)$ need not be weakly T-injective answering a question raised by L. Fuchs in a private conversation.

We conclude this section with an example of a weakly R-injective module which is a direct sum of copies of a module A although A is not weakly R-injective itself.

EXAMPLE 2.16. Let $A = \begin{pmatrix} D & D & D & D \\ 0 & 0 & D & D \end{pmatrix}$. A is a right T-module, where T is

4×4 upper triangular matrix ring over D. Here $(A\oplus A)_T$ is weakly T-injective but A_T is not weakly T-injective as sketched below:

$$(A\oplus A)_T \cong \begin{pmatrix} D & D & D & D \\ D & D & D & D \\ 0 & 0 & D & D \\ 0 & 0 & D & D \end{pmatrix}_T .$$

Clearly $T_T \subset' (A\oplus A)_T$. Since T_T is weakly T-injective, therefore, $(A\oplus A)_T$ is weakly T-injective. Next we show A_T is not weakly T-injective. $E(A_T)=\begin{pmatrix} D & D & D & D \\ D & D & D & D \end{pmatrix}$. Clearly $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} T \cong \begin{pmatrix} D & D & D & D \\ 0 & D & D & D \end{pmatrix}$ which cannot be embedded in A. Hence, A_T is not weakly T-injective.

3. Matrix Rings Over Weakly-Injective Rings

We first show that if R is a weakly-injective domain and K is a ring Morita equivalent to R, then K is a weakly-injective ring. However, weak R-injectivity is not, in general, a Morita invariant.

Theorem 3.1. *Let R be a domain and let the ring K be Morita equivalent to R. If R is a right weakly-injective ring, then K is also a right weakly-injective ring.*

Proof. Suppose that R is a right weakly-injective domain. By Lemma 2.7 (ii), R is a two-sided Ore-domain. Further, by ([5], Theorem 1.2), A ring K is Morita equivalent to a right Ore-domain R if and only if K is a prime right Goldie ring with a projective uniform right ideal U such that U_K is a generator and $R\cong End\,(U_K)$. It follows that K is prime and right and left Goldie. This yields by ([3], Theorem 3.37) and Theorem 2.13 that K is weakly-injective.

The existence of a right Ore-domain which is not left Ore and Lemma 3.2 proved below show that if R is weakly R-injective and K is Morita equivalent to R, K need not be weakly K-injective.

Lemma 3.2. *Let R be a domain. If R is right Ore and $S=M_n(R)$ is right weakly S-injective, $n>1$, then R is left Ore.*

Proof. Suppose R is not a left Ore-domain. Therefore, there exist nonzero elements a, b in R such that $Ra\cap Rb=0$. Furthermore, since R is a right Ore-domain, $E(R_R)=Q$ is a division ring. Consider the element $q=\begin{pmatrix} a^{-1} & b^{-1} & 0 & 0\cdots 0 \\ 0 & 0 & 0 & 0\cdots 0 \\ \vdots & \vdots & \vdots & \vdots\cdots\vdots \\ 0 & 0 & 0 & 0\cdots 0 \end{pmatrix}$ in $E(S)=M_n(Q)$. Because S is right weakly S-injective, there exists y in $E(S)$ such that $r\cdot ann_S(y)=0$ and $q\in yS$. It follows easily that $r\cdot ann_{E(S)}(y)=0$ and, therefore, y has an inverse $y^{-1}=(q_{ij})\in E(S)$. It follows that $y^{-1}q$ belongs to S. Clearly not all entries in the first column of y^{-1} are zeros. Without loss of generality, we may assume $q_{11}\neq 0$. Then, since $y^{-1}q\in S$, we get that both $q_{11}a^{-1}$ and $q_{11}b^{-1}$ are in R.

Since $q_{11}\in Q=r\cdot Q_{cl}(R)$, $q_{11}=cd^{-1}$. Hence, we have $cd^{-1}a^{-1}=r_1$ and $cd^{-1}b^{-1}=r_2$. This implies $cd^{-1}=r_1a$ and $cd^{-1}=r_2b$. Therefore, $q_{11}=cd^{-1}=0$ because $Ra\cap Rb=0$. This contradicts our choice that $q_{11}\neq 0$. Thus R is a left Ore-domain

We now proceed to show that over right nonsingular rings weak-injectivity goes up to and comes down from rings of matrices.

Theorem 3.3. *Let R be a right nonsingular ring. Then the following statements are equivalent*:

(i) *R is a right weakly-injective ring.*

(ii) *$S=M_n(R)$ is a right weakly-injective ring.*

Proof. Let us start with the implication (i)$\Rightarrow$(ii). Let $Q=E(R_R)$. By Lemma 2.10 we have $Q=l\cdot Q_{cl}(R)$. Also, we have that $l\cdot Q_{cl}(M_n(R))\cong M_n(l\cdot Q_{cl}(R))$. (See, for example [8], Exercise 9(i)). Hence

$$E(S_S)=M_n(Q)=M_n(l\cdot Q_{cl}(R))\cong l\cdot Q_{cl}(M_n(R))=l\cdot Q_{cl}(S)\,.$$

Therefore, by Lemma 2.10 once again, $S=M_n(R)$ is a right weakly-injective ring. Conversely, suppose $S=M_n(R)$ is a right weakly-injective ring. One more application of Lemma 2.10 yields

$$E(S_S)=E(M_n(R))=l\cdot Q_{cl}(M_n(R))\cong M_n(l\cdot Q_{cl}(R))\,.$$

Therefore, $M_n(l\cdot Q_{cl}(R))$ is a right self-injective ring. Hence $l\cdot Q_{cl}(R)$ is a right self-injective ring. Consider the following diagiam:

$$\begin{tikzcd}
E(R_R)=Q \arrow[d, dash, "\lambda"'] \arrow[dr, dashed, "\phi"] & \\
R \arrow[r, dashed, "\varphi"'] & l\cdot Q_{cl}(R)
\end{tikzcd}$$

φ and λ are the inclusion inclusion R-homomorphisms. Because $l\cdot Q_{cl}(R)$ is a right self-injective ring, φ can be extended to ϕ which is a monomorphism, because φ is a monomorphism and $R_R\subset' Q_R$. Now define $f\colon Q\to \operatorname{Im}\phi$. Clearly f is an R-isomorphism, therefore, the inverse of f, say g, exists. Pick $q_1, q_2\in Q$. Hence $f(q_1), f(q_2)\in l\cdot Q_{cl}(R)$. Thus $f(q_1)=a^{-1}b_1$ and $f(q_2)=a^{-1}b_2$ where b_1, b_2 and a in R with a is a regular element. Applying g we get

$$q_1 = gf(q_1) = g(a^{-1}b_1) = g(a^{-1})\, b_1$$

and

$$q_2 = gf(q_2) = g(a^{-1}b_2) = g(a^{-1})\, b_2 .$$

$g(a^{-1})$ is a regular element because a^{-1} is a regular element and g is an R-isomorphism. Let $q'=g(a^{-1})$. Therefore,

$$q_1, q_2 \in q'R \cong R .$$

Hence R is a right weakly-injeative ring.

As a consequence of the Lemma 3.2 and Theorem 3.3, we have

Theorem 3.4. *Let R be a domain. Then the following statements are equivalent.*

(i) *R is a right weakly-injective ring.*
(ii) *$S=M_n(R)$ is a right weakly-injective ring, $n>1$.*
(iii) *$S=M_n(R)$ is a right weakly S-injective ring, $n>1$.*

Proof. That (i) implies (ii) follows directly from either Theorem 3.1 or Theorem 3.3. The implication (ii) implies (iii) is clear. In order to show that (iii) implies (i), all we need to show is that R is right weakly R-injective. Then the result follows by Lemma 3.2 and Lemma 2.7 (ii). Suppose $S=M_n(R)$ is right weakly S-injective. Let $Q=E(R_R)$ which is regular and right self-injective. Thus $E(S_S)=M_n(Q)$ is also regular and right selt-injective. Let $q\in Q$. We need $q'\in Q$ such that $r\cdot ann_R(q')=0$ and $q\in q'R$. Consider the element

$$\begin{pmatrix} q & 1 & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \quad\ \vdots \\ 0 & 0 & 0 \cdots 0 \end{pmatrix} \in E(S) .$$

There exists $y=(q_{ij})$ in $E(S)$ such that $r\cdot ann_S(y)=0$ and

$$\begin{pmatrix} q & 1 & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \quad\ \vdots \\ 0 & 0 & 0 \cdots 0 \end{pmatrix} \in yS .$$

Since $E(S)$ is a regular ring, y has a left inverse in $E(S)$, say (p_{ij}). We have

$$\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} q & 1 & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \quad\ \vdots \\ 0 & 0 & 0 \cdots 0 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & & & \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix},$$

which yields $p_{11}\, q=r_{11}\in R$ and $p_{11}=r_{12}\in R$. Therefore $r\cdot ann_Q(p_{11})=r\cdot ann_R(p_{11})$

$=0$.

We note $p_{11}\, q \neq 0$, and so ${}_R R \subset' {}_R Q$. Therefore, $l \cdot ann_Q(p_{11}) = 0 = r \cdot ann_Q(p_{11})$ and thus p_{11} is invertible in Q because Q is a regular ring. For $p_{11}\, q = r$, we get $q = p_{11}^{-1}\, r_{11}$. Thus by Lemma 2.2, R is weakly R-injective.

4. Weakly-Injective Triangular Matrix Rings

Let $T(R)$ (or simply T if there is no ambiguity) denote the $n \times n$ upper triangular matrix ring over the ring R. We will show that if R is a domain the ring T is right weakly T-injective if and only if R is a right weakly-injective ring. Also an example is provided to show that T need not be weakly T^2-injective.

Theorem 4.1. *If R is a domain and $T = T(R)$, then the following statements are equivalent.*

(i) *R is a right weakly-injective ring.*

(ii) *T is a right weakly T-injective ring.*

Proof. (i)$\Rightarrow$(ii). By hypothesis, it follows that R is a two-sided Ore-domain. Therefore, $E(R) = D$ is a division ring. Now the injective hull $E(T)$ of T_T is the full $n \times n$ matrix ring $M_n(D)$ over D. Let $A \in E(T)$. From elementary matrix theory we know that there exists $B \in E(T)$ such that $B^{-1}A \in T$. Thus T is a right weakly T-injective ring.

(ii)$\Rightarrow$(i). Let $S = M_n(R)$. Since $T \subset S \subset E(S_S) \subset E(T_T)$, by Lemma 2.8, S is right weakly S-injective. Therefore, by Theorem 3.5, R is a right weakly-injective ring.

As a summary, the results obtained thus far for a domain R and related rings may be exhibited in the following diagram.

$$
\begin{tikzcd}
S \text{ is right weakly } S\text{-injective} & & T \text{ is right weakly } T\text{-injective} \\
& R \text{ is right weakly } R^2\text{-injective} \arrow[ul, Leftrightarrow, "\text{(Theorem 3.4)}"] \arrow[ur, Leftrightarrow, "\text{(Theorem 4.1)}"'] \arrow[dl, Leftrightarrow, "\text{(Theorem 3.1 or Theorem 3.4)}"'] \arrow[dr, Leftrightarrow, "\text{(Lemma 2.7 (ii))}"] & \\
S \text{ is right weakly } S^2\text{-injective} & & R \text{ is two-sided Ore}
\end{tikzcd}
$$

EXAMPLE 4.2. Let R be a right nonsingular weakly-injective ring. Then $T=T(R)=\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is not weakly T^2-injective.

Proof. By Theorem 2.13, $Q=E(R_R)=E({}_RR)$. Therefore, Q is a regular right and left self-injective ring. If T were weakly T^2-injective, then given I and $a=\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}\in E(T)$ with $q_3\neq 0$, there must exist $b\in E(T)$ such that $a\in bT$, $I\in bT$, where $r\cdot ann_T(b)=0$. Since $E(T)$ is regular and $r\cdot ann_T(b)=0$, has a left inverse in $E(T)$. Now $I=bt$, $t\in T$ implies b a has a right inverse in T. Therefore, b^{-1} exists in T. We claim $b\in T(Q)$. Suppose $b=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\in E(T)$. Then $I=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$, implies $\alpha x=1$ and $\gamma x=0$, therefore, $x\alpha=1$ because Q is regular right and left self-injective, and so x^{-1} exists. This implies $\gamma=0$, proving our claim. Furthermore, from $a\in bT$, it follows $a\in T(Q)$ and so $q_3=0$, a contradiction. Thus T is not weakly T^2-injective.

The following well-known result will be used in the proof of our next theorem.

Lemma 4.3. *Let R be a right nonsingular ring and F be free $(R-R)$-bimodule. Then the injective hull of $S=\begin{pmatrix} R & F \\ 0 & R \end{pmatrix}$ is*

$$E(S)=\begin{pmatrix} \mathrm{End}_R(E(F)) & E(F) \\ \mathrm{Hom}_R(E(F),\, E(R)) & E(R) \end{pmatrix}.$$

Proof. See ([3], Proposition 4.4).

Theorem 4.4. *Let R be a domain and $F=\oplus\sum_{i\in I} {}_R(R_i)_R$, where for all $i\in I$ ${}_R(R_i)_R\cong {}_RR_R$. Then the following statements are equivalent:*

(i) $S=\begin{pmatrix} R & F \\ 0 & R \end{pmatrix}$ *is right weakly S-injective.*

(ii) ${}_RF_R\cong {}_RR_R$ *and R is a right and left Ore-domain.*

Proof. For the implication (i) implies (ii), we use Lemma 4.3 to get

$$E(S)=\begin{pmatrix} \mathrm{End}_R(E(F) & E(F) \\ \mathrm{Hom}_R(E(F),\, E(R)) & E(R) \end{pmatrix}.$$

We show ${}_RF_R\cong {}_RR_R$.

Suppose not and assume that F_R has a basis with at least two elements. Pick $0\neq \varphi_0\in \mathrm{End}_R(E(F))$ such that $Ker\ \varphi_0\neq 0$. Clearly, $q=\begin{pmatrix} \varphi_0 & 0 \\ 0 & 0 \end{pmatrix}\in E(S)$.

Therefore, there exists $q'=\begin{pmatrix}\varphi & v\\ \psi & d\end{pmatrix}\in E(S)$ such that $r\cdot ann_S(q')=0$ and

$$(1)\qquad \begin{pmatrix}\varphi_0 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}\varphi & v\\ \psi & d\end{pmatrix}\begin{pmatrix}x & y\\ 0 & z\end{pmatrix},\ x, z\in R \text{ and } y\in F.$$

We note that the element $x\in R$ may be identified with the mapping $\varphi_x\in \mathrm{End}_R(E(F))$ such that $\varphi_x(\omega)=x\omega$ under the canonical embedding $R\hookrightarrow \mathrm{End}_R(E(F))$. From (1), we obtain

$$\begin{pmatrix}\varphi_0 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}\varphi\varphi_x & \varphi(y)+vz\\ \psi\varphi_x & \psi(y)+dz\end{pmatrix},$$

and so $\varphi_0=\varphi\varphi_x$ and $0=\psi\varphi_x$.

Since $Ker\ \varphi_0\neq 0$, $Ker\ \varphi\varphi_x\neq 0$. Therefore, there exists a nonzero $\omega\in E(F)$ such that $\varphi\varphi_x(\omega)=0$ which implies $\varphi(x\omega)=0$, that is,

$$(2)\qquad x\omega\in Ker\ \varphi\,.$$

Now $\psi\varphi_x=0$ implies $\psi\varphi_x(\omega)=0$, for all $\omega\in E(F)$ and, therefore, $\psi(x\omega)=0$, that is,

$$(3)\qquad x\omega\in Ker\ \psi\,.$$

Clearly, $Ker\ \varphi\cap Ker\ \psi=0$, because $r\cdot ann_S(q')=0$. By (2) and (3) we get $x\omega=0$ and hence $x=0$. Thus $\varphi_0=\varphi\varphi_x=0$, a contradiction.

Therefore, $F_R\cong R_R$. Similarly ${}_RF\cong {}_RR$. Thus $S=\begin{pmatrix}R & R\\ 0 & R\end{pmatrix}$ which is right weakly S-injective, and so by Theorem 4.1 and Lemma 2.7(ii), R is a two-sided Ore-domain. The converse follows by Theorem 4.1 and Lemma 2.7 (ii).

Corollary 4.5. *Let V be $(D-D)$-space over a division ring D. Then the following statements are equivalent.*

i) $S=\begin{pmatrix}D & V\\ 0 & D\end{pmatrix}$ *is right weakly S-injective.*

ii) $V\cong D$.

Proposition 4.6. *Let R be a domain. Then the following statements are equivalent.*

(i) *R is right and left Ore.*

(ii) *There exists an ideal I of R such that $S=\begin{pmatrix}R & I\\ 0 & R\end{pmatrix}$ is right weakly S-injective.*

(iii) *For all ideals I in R, $S=\begin{pmatrix}R & I\\ 0 & R\end{pmatrix}$ is right weakly S-injective.*

Proof. (i) implies (ii) follows by choosing $I=R$. For the implication (ii) implies (iii), R is a domain implies every ideal I in R is essential. Hence $E(I_R)=E(R_R)$. Thus the conclusion is clear. Finally for (iii) implies (i), set $I=R$. Thus we have $S=\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is right weakly S-injective. Therefore, R is two-sided Ore.

EXAMPLE 4.7. We construct a nonsingular artinian ring R which is QF-3 but not weakly R-injective. Let $S=\begin{pmatrix} \boldsymbol{R} & \boldsymbol{R}^3 \\ 0 & \boldsymbol{R} \end{pmatrix}$ where $\boldsymbol{R}$ denotes reals. Then the map

$$\begin{pmatrix} a & (b_1\, b_2\, b_3) \\ 0 & b \end{pmatrix} \xrightarrow{\varphi_1} \begin{pmatrix} a & 0 & 0 & b_1 \\ 0 & a & 0 & b_2 \\ 0 & 0 & a & b_3 \\ 0 & 0 & 0 & b \end{pmatrix}$$

embeds S in its maximal right ring of quotients $r\cdot Q(S)=M_4(\boldsymbol{R})$. On the other hand, S is embedded in its maximal left ring of quotients $l\cdot Q(\mathrm{S})=M_4(\boldsymbol{R})$ via map

$$\begin{pmatrix} a & (b_1\, b_2\, b_3) \\ 0 & b \end{pmatrix} \xrightarrow{\varphi_2} \begin{pmatrix} a & b_1 & b_2 & b_3 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}.$$

Now let R be the ring

$$R=\left\{\begin{pmatrix} a & b & c & d \\ 0 & \alpha & 0 & e \\ 0 & 0 & \alpha & f \\ 0 & 0 & 0 & g \end{pmatrix} : a, b, c, d, e, f, g, \alpha\in\boldsymbol{R}\right\}.$$

Since $\varphi_i(S)\subset R$, $i=1, 2$, it follows that $E(R_R)=r\cdot Q(R)=M_4(\boldsymbol{R})=l\cdot Q(R)=E({}_RR)$. Therefore, R is nonsingular and right and left QF–3. R is not weakly R-injective. For $x=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in $E(R)$. If y in $E(R)$ such that $yx\in R$, then y is of the form $y=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43}, & a_{44} \end{pmatrix}$ which is not invertible. Therefore, R is not weakly R-injective.

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