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INVARIANTS OF THREE-MANIFOLDS DERIVED FROM LINKING MATRICES OF FRAMED LINKS

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Introduction. In [16], R. Kirby and P. Melvin study invariants of 3-manifolds $\tau_r (r \geq 3)$ introduced by E. Witten [38], N. Reshetikhin and V.G. Turaev [31], and W.B.R. Lickorish [25, 26, 27] (see also [18]). In particular, Kirby and Melvin calculated τ_3 and τ_4 explicitly. Let M be a closed, oriented 3-manifold obtained from an (integral) framed link L. Then $\tau_3(M)$ can be written as follows [16, §6].

$$\tau_3(M) = c^{-\sigma} \sqrt{2}^{-n} \sum_{S<L} \sqrt{-1}^{S\cdot S} .$$

Here n is the number of components of L, σ is the signature of its linking matrix, $c=\exp(\pi\sqrt{-1}/4)$, the sum is taken over all sublinks of L including the empty sublink, and $S\cdot S$ is the sum of all the entries in the linking matrix of S.

In this paper, we generalize τ_3 and define another series of invariants of 3-manifolds. Let q be a primitive N-th ($2N$-th, resp.) root of unity for an odd (even, resp.) positive integer N. Put

$$Z_N(M; q) = \left(\frac{G_N(q)}{|G_N(q)|}\right)^{-\sigma(A)} |G_N(q)|^{-n} \sum_{l\in(Z/NZ)^n} q^{{}^t lAl} ,$$

where $G_N(q)=\sum_{h\in Z/NZ} q^{h^2}$ (a Gaussian sum), A is the linking matrix of L, l is regarded as a column vector, and ${}^t l$ is its transposed row vector. One can easily see that $Z_2(M; \sqrt{-1})=\tau_3(M)$. We will show that these are all invariants for M (Theorem 1.3). As Kirby and Melvin proved for $\tau_3(M)$, $Z_N(M; q)$ is also invariant under homotopy equivalence. More precisely, it is determined by the first Betti number of M and the linking pairing on $\operatorname{Tor} H_1(M; \boldsymbol{Z})$ for any N and q (Proposition 2.5, Corollary 2.6).

We will express the absolute value of $Z_N(M; q)$ in terms of the cohomology ring of M with $\boldsymbol{Z}/N\boldsymbol{Z}$-coefficients (Theorem 3.2). When $|Z_N(M; q)| \neq 0$, we can also determine its phase (Theorem 4.5). It is a generalization of the Brown invariant $\beta(M)$ [16, §6] defined by the linking matrix using the signature and Brown's invariant [2] for $\boldsymbol{Z}/4\boldsymbol{Z}$-valued quadratic forms on a $\boldsymbol{Z}/2\boldsymbol{Z}$-vector space.

We can aslo calculate $Z_N(M; q)$ explicitly for 3-manifolds with linking pairings which are members of generator system of linking pairings on finite abelian groups (Theorem 5.1). We also show that when M is a cyclic covering space of an oriented link, $Z_N(M; q)$ is essentially equivalent to the link invariant introduced by E. Date, M. Jimbo, K. Miki, and T. Miwa [4] using chiral Potts models (Proposition 6.3).

Other purpose of this paper is to describe various relationship of our invariants with quantum field theory, quantum groups, and $U(1)$ gauge theory. It is known that $Z_N(M; q)$ can be obtained from solutions to the polhnomial equations associated with $\boldsymbol{Z}/N\boldsymbol{Z}$-fusion rules [20, 21, 30]. It is also defined using a quasitriangular Hopf algebra as $\tau_r(M)$ [6, 31, 16] (§7). If N is even, the absolute values of our invariants coincide with the invariants of T. Gocho [8], which is defined via $U(1)$ gauge theory with charge N (§8). We can also prove that invariants of R. Dijkgraaf and E. Witten [5] can be described using our invariants if $G=\boldsymbol{Z}/N\boldsymbol{Z}$ (§9).

For basic concepts concerning 3-manifolds and links we refer the reader to [3, 11, 32].

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1. Difinition of invariants. An oriented link in the 3-sphere S^3 is a finite collection of disjoint, smoothly embedded, oriented circles $L_1, L_2, \cdots$, and L_n in S^3. An (oriented, integral) *framed link* is an oriented link, each component L_i being provided with a framing f_i which is an isotopy class of a section of the projection $\partial N(L_i)\to L_i$. We can obtain a connected, closed, oriented 3-manifold M_L by surgery on S^3 along a framed link L. M_L is the result of gluing n copies of $D^2\times S^1$ to $S^3-\cup_{i=1}^n \operatorname{int} N(L_i)$ so that the i-th $\partial D^2\times\{*\}$ is identified with f_i. It is well known [24, 37] that each connected, closed, oriented 3-manifold can be obtained by surgery on S^3 along a certain framed link.

Let $A=(\lambda_{ij})$ $(1\leq i, j\leq n)$ be the linking matrix of L, that is, $\lambda_{ij}=\operatorname{lk}(L_i, L_j)$ and $\lambda_{ii}=\operatorname{lk}(L_i, f_i)$. Here $\operatorname{lk}(\cdot,\cdot)$ denotes the linking number in S^3. Denote by $\sigma(A)$ the signature of A (the number of positive eigenvalues — the number of negative eigenvalues). Let N and d be coprime integers $(N\geq 2, d\geq 1)$ with $N+d$ odd and put $q=\exp(d\pi\sqrt{-1})/N)$. Note that q is a primitive N-th root of unity if N is odd and a primitive $2N$-th root of unity if N is even. Now we consider the following formula:

$$Z_N(M, L; q)=\left(\frac{G_N(q)}{|G_N(q)|}\right)^{-\sigma(A)} |G_N(q)|^{-n} \sum_{l\in(\boldsymbol{Z}/N\boldsymbol{Z})^n} q^{{}^t lAl}, \tag{1.1}$$

where M is obtained by surgery on S^3 along L and $G_N(q)=\sum_{h\in\boldsymbol{Z}/N\boldsymbol{Z}} q^{h^2}$. $G_N(q)$ is called a Gaussian sum and its properties are well-known (see Lemma 4.4).

REMARK 1.2. For N odd, $q^{{}^t lAl}$ is well-defined since q is an N-th root of unity. For N even, we can also easily see that it is well-defined though q is a $2N$-th root of unity. In both case, we can regard $l \mapsto {}^t l A l$ as a quadratic form in the following sence. If N is odd, a quadratic form on $(\boldsymbol{Z}/N\boldsymbol{Z})^n$ is a function Q: $(\boldsymbol{Z}/N\boldsymbol{Z})^n \to \boldsymbol{Z}/N\boldsymbol{Z}$ satisfying $Q(ax)=a^2Q(x)$ as usual. If N is even, a $\boldsymbol{Z}/2N\boldsymbol{Z}$-valued quadratic form on $(\boldsymbol{Z}/N\boldsymbol{Z})^n$ associated to $(\cdot, \cdot)$ is a function Q: $(\boldsymbol{Z}/N\boldsymbol{Z})^n \to \boldsymbol{Z}/2N\boldsymbol{Z}$ satisfying $Q(ax)=a^2Q(x) \in \boldsymbol{Z}/2N\boldsymbol{Z}$ and $Q(x+y)=Q(x)+Q(y)+2\cdot(x, y) \in \boldsymbol{Z}/N\boldsymbol{Z}$. Here $(\cdot, \cdot)$: $(\boldsymbol{Z}/N\boldsymbol{Z})^n \times (\boldsymbol{Z}/N\boldsymbol{Z})^n \to \boldsymbol{Z}/N\boldsymbol{Z}$ is a symmetric bilinear (not assumed to be non-singular) form, and 2: $\boldsymbol{Z}/N\boldsymbol{Z} \to \boldsymbol{Z}/2N\boldsymbol{Z}$ is a homomorphism sending 1 to 2. In this case, $Q(l)={}^t\tilde{l}A\tilde{l}$ mod $2N$ with a lift $\tilde{l} \in (\boldsymbol{Z})^n$ of l and (l, l') is ${}^t lAl'$ mod N. (This definition coincides with that in [2, 10, 28] for the case that $N=2$ and $(\cdot, \cdot)$ is non-singular.) Then the sum $\sum_{l \in (\boldsymbol{Z}/N\boldsymbol{Z})^n} q^{{}^t lAl}$ is written as $\sum_{l \in (\boldsymbol{Z}/N\boldsymbol{Z})^n} q^{Q(l)}$ and is an invariant of quadratic forms.

Theorem 1.3. *$Z_N(M, L; q)$ is a topological invariant of M and does not depend on any choice of L.*

Proof. Two *unoriented* framed links L and L' determine the same closed 3-manifold if and only if L' may be obtained from L by Kirby moves; "stabilization" and "handle sliding" (see [15]). Two framed links L and L' are related by a stabilization if they are identical except for elimination or insertion of a splitted, unknotted component L'_i with framing f'_i such that $\mathrm{lk}(L'_i, f'_i)=\pm 1$. L and L' are related by a handle sliding if they are identical except for changing a component L_j by $L'_j=L_j \#_b f_i$ with framing f'_j such that $\mathrm{lk}(L'_j, f'_j)=\mathrm{lk}(L_j, f_j)+\mathrm{lk}(L_i, f_i)\pm 2\mathrm{lk}(L_i, f_j)$. Here $\#_b$ means the band connected sum with b a band connecting f_i and L_j. The sign is $+$ if the orientations of f_i and L_j are coherent and $-$ otherwise.

Now from a theorem of R. Kirby [15], it suffices to verify that a stabilization, a handle sliding, and reversing of an orientation do not change $Z_N(M, L; q)$. Assume first that two framed links L and L' are related by a stabilization. We assume that L' is obtained from L by inserting a splitted, unknotted component. Then denoting by A the linking matrix of L with n components, that of L' is given by

$$A' = \begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Since the size and the signature of A' is $n+1$ and $\sigma(A)\pm 1$ respectively, we have

$$Z_N(M, L'; q) = \left(\frac{G_N(q)}{|G_N(q)|}\right)^{-\sigma(A)\mp 1} |G_N(q)|^{-n-1} \sum_{l \in (\boldsymbol{Z}/N\boldsymbol{Z})^n} q^{{}^t lAl} \sum_{h \in \boldsymbol{Z}/N\boldsymbol{Z}} q^{\pm h^2}.$$

Since $\sum_{h \in \boldsymbol{Z}/N\boldsymbol{Z}} q^{\pm h^2}=G_N(q)$ or $\overline{G_N(q)}$ (the complex conjugate), we obtain $Z_N(M, L'; q)=Z_N(M, L; q)$.

Let L and L' be two framed links related by a handle sliding such that $L'_s=L_s \#_b f_t$. Then the linking matrix $A'=(\lambda'_{ij})$ of L' satisfies

$$\begin{aligned}
\lambda'_{ss} &= \lambda_{ss}+\lambda_{tt}\pm 2\lambda_{st}\,, \\
\lambda'_{is} &= \lambda_{is}\pm\lambda_{it} && (i\neq s)\,, \\
\lambda'_{sj} &= \lambda_{sj}\pm\lambda_{tj} && (j\neq s)\,, \\
\lambda'_{ij} &= \lambda_{ij} && (i\neq s, j\neq s)\,.
\end{aligned}$$

Hence $A'={}^tTAT$ holds with $T_{ii}=1$, $T_{ts}=\pm 1$ and $T_{ij}=0$ otherwise, where $T=(T_{ij})$. Putting $l'=T^{-1}l$, we have

$$\sum_{l'\in(Z/NZ)^n} q^{{}^tl'A'l'} = \sum_{l\in(Z/NZ)^n} q^{{}^tlAl}\,.$$

Since n and $\sigma(A)$ remain unchanged under this transformation, we have $Z_N(M, L'; q)=Z_N(M, L; q)$.

If L' is a framed link which is obtained from L by reversing orientation of a component L_k, then the linking matrix of L' is ${}^tSAS$, where $S=(S_{ij})$ with $S_{ij}=0$ $(i\neq j)$, $S_{ii}=1$ $(i\neq k)$, and $S_{kk}=-1$. So $Z_N(M, L'; q)=Z_N(M, L; q)$ by a similar way as above.

This completes the proof. ■

By Theorem 1.3 we have topological invariants of M.

DEFINITION 1.4. *Let M be a connected, closed, compact 3-manifold obtained by surgery on S^3 along a framed link L. Then we put $Z_N(M; q)=Z_N(M, L; q)$.*

2. Fundamental properties. In this section we study fundamental properties of the invariant $Z_N(M; q)$.

First of all, we note that $Z_N(S^3; q)=1$ for any N and q. If M is obtained from a framed link L, the mirror image of L gives $-M$, M with the opposite orientation. Since the linking matrix of the mirror image of L is $-A$ with A the linking matrix of L, we have

Proposition 2.1. *For a closed, oriented 3-manifold M,*

$$Z_N(-M; q) = \overline{Z_N(M; q)}\,.$$

The split union of two framed links gives the connected sum of the corresponding 3-manifolds. So we have

Proposition 2.2. *If M_1 and M_2 are closed, oriented 3-manifolds, then*

$$Z_N(M_1\# M_2; q) = Z_N(M_1; q)\, Z_N(M_2; q)\,.$$

$Z_N(M; q)$ also factors associated with a factorization of N.

Proposition 2.3. *If $N=N_1 N_2$ with coprime integers N_1 and N_2, then*

$$Z_N(M; q) = Z_{N_1}(M; q^{N_2^2})\, Z_{N_2}(M; q^{N_1^2}).$$

Proof. $l \in (\boldsymbol{Z}/N\boldsymbol{Z})^n$ is uniquely expressed as $l=N_2 l_1+N_1 l_2$ for $l_1 \in (\boldsymbol{Z}/N_1 \boldsymbol{Z})^n$ and $l_2 \in (\boldsymbol{Z}/N_2 \boldsymbol{Z})^n$. Hence we have

$$\sum_{l\in(\boldsymbol{Z}/N\boldsymbol{Z})^n} q^{{}^t lAl} = \sum_{l_1\in(\boldsymbol{Z}/N_1\boldsymbol{Z})^n,\, l_2\in(\boldsymbol{Z}/N_2\boldsymbol{Z})^n} q^{N_3^2\cdot{}^t l_1 A l_1 + N_1^2\cdot{}^t l_2 A l_2 + 2N_1 N_2\cdot{}^t l_1 A l_2}$$
$$= \sum_{l_1\in(\boldsymbol{Z}/N_1\boldsymbol{Z})^n} q^{N_2^2\cdot{}^t l_1 A l_1} \sum_{l_2\in(\boldsymbol{Z}/N_2\boldsymbol{Z})^n} q^{N_1^2\cdot{}^t l_2 A l_2},$$

where the second equality holds since $q^{2N_1N_2}=1$. In a similar way, we obtain $G_N(q)=G_{N_1}(q^{N_2^2})\, G_{N2}(q^{N_1^2})$. Therefore $Z_N(M; q)$ factors as above. ■

As R. Kirby and P. Melvin state for $\tau_3(M)$ [16, 6.2 Remark], $Z_N(M; q)$ is also a homotopy invariant (see Corollary 2.6 below) for every N and q. To prove this, we review results of M. Kneser and P. Puppe [17], A.H. Durfee [7], and R.H. Kyle [22].

Let B and B' be symmetric integral matrices. B and B' are said to be *stably equivalent* (or closely related in [22]) if they are equivalent under the equivalence relation generated by the following Q_1 and Q_2:

$$Q_1\colon B \leftrightarrow {}^t SBS \quad \text{with } S \text{ integral and unimodular,}$$
$$Q_2\colon B \leftrightarrow \begin{pmatrix} B & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

As in the previous section, let M be a 3-manifold obtained by surgery on S^3 along a framed link L and A its linking matrix. Summarizing results in [17, 7, 22], we can conclude that stable equivalence class is determined by the first Betti number of M and the linking pairing on Tor $H_1(M; \boldsymbol{Z})$. More precisely, the following proposition holds.

Proposition 2.4. *Stable equivalence class of linking matrices of framed links is determined by the first Betti number of the* 3*-manifold M obtained from it and* (Tor $H_1(M; \boldsymbol{Z}), \lambda$)*, that is, two linking matrices A and A' are stably equivalent if and only if M and M' satisfy* (1) *and* (2) *below, where M (M', resp.) are obtained from framed link L (L', resp.) with linking matrix A (A', resp.).*

(1) *The first Betti numbers of M M' are equal.*

(2) *There exists an isomorphism between* Tor $H_1(M; \boldsymbol{Z})$ *and* Tor $H_1(M'; \boldsymbol{Z})$ *which induces an isomorphism between the linking pairings λ and λ'.*

Here the linking pairing on Tor $H_1(M; \boldsymbol{Z})$ is defined as follows.
An exact sequence of coefficient groups

$$0 \to \boldsymbol{Z} \overset{i}{\to} \boldsymbol{Q} \overset{\eta}{\to} \boldsymbol{Q}/\boldsymbol{Z} \to 0$$

gives rise to a long exact sequence of homology groups of M:

$$\to H_2(M; \boldsymbol{Q}) \overset{\eta_*}{\to} H_2(M; \boldsymbol{Q}/\boldsymbol{Z}) \overset{\delta_*}{\to} H_1(M; \boldsymbol{Z}) \overset{i_*}{\to} H_1(M; \boldsymbol{Q}) \to ,$$

where δ_* is the connecting homomorphism. The *linking pairing*

$$\lambda\colon \operatorname{Tor} H_1(M; \boldsymbol{Z}) \times \operatorname{Tor} H_1(M; \boldsymbol{Z}) \to \boldsymbol{Q}/\boldsymbol{Z}$$

is defined by $\lambda(\alpha, \beta)=\alpha\cdot\hat{\beta}$ where $\delta_* \hat{\beta}=\beta$ and a dot means the intersection product

$$H_1(M; \boldsymbol{Z}) \times H_2(M; \boldsymbol{Q}/\boldsymbol{Z}) \to \boldsymbol{Q}/\boldsymbol{Z}\,.$$

One can easily check that λ is well-defined.

By the above proposition, we immediately have the following proposition.

Proposition 2.5. *If M and M' satisfy the conditions* (1) *and* (2) *in Proposition* 2.4, *then $Z_N(M; q)=Z_N(M'; q)$.*

Proof. Since the corresponding linking matrices A and A' are stably equivalent, we have $Z_N(M; q)=Z_N(M'; q)$ as in the proof of Theorem 1.4. ■

Clearly two 3-manifolds which are homotopy equivalent satisfy the conditions (1) and (2). So we have

Corollary 2.6. *If M and M' are homotopy equivalent, then $Z_N(M; q)= Z_N(M'; q)$.*

3. Absolute value.

In this section we calculate the absolute value of $Z_N(M; q)$ and give its topological meaning.

First of all we prepare a lemma which will be used frequently in this paper. A proof is an easy exercise.

Lemma 3.1. *Let z be a primitive N-th root of unity. Then*

$$\sum_{x\in(\boldsymbol{Z}/N\boldsymbol{Z})^n} z^{{}^t xy} = \begin{cases} N^n & \text{if } y=0\in(\boldsymbol{Z}/N\boldsymbol{Z})^n\,, \\ 0 & \text{if } y\neq 0\in(\boldsymbol{Z}/N\boldsymbol{Z})^n\,, \end{cases}$$

where we regard x and y as column vectors.

Now $|Z_N(M; q)|$ is given as follows. This generalizes [16, Theorem 6.3].

Theorem 3.2. *If there exists α in $H^1(M; \boldsymbol{Z}/N\boldsymbol{Z})$ with $\alpha\cup\alpha\cup\alpha\neq 0$, then $Z_N(M; q)=0$. Otherwise $|Z_N(M; q)|=|H^1(M; \boldsymbol{Z}/N\boldsymbol{Z})|^{1/2}$ where $|\cdot|$ in the right hand side is the order of the set.*

Proof. Let M be a 3-manifold obtained by surgery on S^3 along an n-

component framed link L. From (1.1), we have

$$|Z_N(M;q)|=|G_N(q)|^{-n}\cdot|\sum_{l\in(\boldsymbol{Z}/N\boldsymbol{Z})^n}q^{{}^tlAl}|\,.$$

We first calculate $|G_N(q)|^2=N$.

$$\begin{aligned}|G_N(q)|^2&=\sum_{h,h'\in\boldsymbol{Z}/N\boldsymbol{Z}}q^{h'^2-h^2}\\&=\sum_{h''}q^{h''^2}\sum_{h}q^{2h''h}\quad(h'=h''+h)\\&=N\,.\end{aligned}$$

The last equality follows from Lemma 3.1 putting $n=1$, $x=h''$, $y=h$, $z=q^2$ since q^2 is a primitive N-th root of unity.

Next we calculate the absolute value of $\sum q^{{}^tlAl}$. In a similar way as above, we have

$$\begin{aligned}|\sum_{l\in(\boldsymbol{Z}/N\boldsymbol{Z})^n}q^{{}^tlAl}|^2&=\sum_{l',l}q^{{}^tl'Al'-{}^tlAl}\\&=\sum_{l''}q^{{}^tl''Al''}\sum_{l}q^{2{}^tlAl''}\quad(l'=l''+l)\\&=N^n\sum_{l''\in\ker L_A}q^{{}^tl''Al''}\,,\end{aligned}$$

where L_A is a linear map $L_A\colon(\boldsymbol{Z}/N\boldsymbol{Z})^n\to(\boldsymbol{Z}/N\boldsymbol{Z})^n$, $l\mapsto Al$. The last equality follows from Lemma 3.1 putting $x=l$ and $y=Al''$. Therefore we have

$$|Z_N(M;q)|^2=|\sum_{l\in\ker L_A}q^{{}^tlAl}|\,.$$

Now there are two cases to consider.

Case 1: N is odd. Recall that q is an N-th root of unity. For $l\in\ker L_A$, we have ${}^tlAl=0$ in $\boldsymbol{Z}/N\boldsymbol{Z}$ and $q^{{}^tlAl}=1$. Hence $|Z_N(M;q)|^2$ is equal to the order of $\ker L_A$. By Lemma 3.3 below, we have

$$|Z_N(M;q)|=|H^1(M;\boldsymbol{Z}/N\boldsymbol{Z})|^{1/2}\,.$$

In this case $\alpha\cup\alpha\subset\alpha=0$ holds for any α in $H^1(M;\boldsymbol{Z}/NZ)$, because the cup product is skew-symmetric and the order of $H^3(M;\boldsymbol{Z}/N\boldsymbol{Z})$ is odd. Hence we obtain Theorem 4.1 for N odd.

Case 2: N is even. In this case q is a $2N$-th root of unity. As in Remark 1.3 we regard $l\mapsto{}^tlAl$ as a map $(\boldsymbol{Z}/N\boldsymbol{Z})^n\to\boldsymbol{Z}/2N\boldsymbol{Z}$. We denote the restriction of this map to $\ker L_A$ by $\varphi\colon\ker L_A\to\{0,N\}\subset\boldsymbol{Z}/2N\boldsymbol{Z}$. Then φ is a homomorphism because

$${}^t(\tilde{l}+\tilde{l}')\,A(\tilde{l}+\tilde{l}')={}^t\tilde{l}A\tilde{l}+{}^t\tilde{l}'A\tilde{l}'+2\cdot{}^t\tilde{l}A\tilde{l}'$$

and $2\cdot{}^t\tilde{l}A\tilde{l}'$ can be divided by $2N$. Therefore we have

$$|Z_N(M;q)| = \begin{cases} |\ker L_A|^{1/2} & \varphi \equiv 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemmas 3.3 and 3.4 below, we obtain

$$|Z_N(M;q)| = \begin{cases} |H^1(M;\boldsymbol{Z}/N\boldsymbol{Z})|^{1/2} & \text{if } \alpha\cup\alpha\cup\alpha = 0 \text{ for any } \alpha\in H^1(M;\boldsymbol{Z}/N\boldsymbol{Z})\,, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof. ■

Lemma 3.3. $\ker L_A$ *is isomorphic to* $H^1(M;\boldsymbol{Z}/N\boldsymbol{Z})$.

Proof. Since M is a union of S^3-int $N(L)$ and n copies of $D^2\times S^1$, we have the Mayer-Vietoris exact sequence below.

$$\cdots\xrightarrow{0} H^1(M;\boldsymbol{Z}/N\boldsymbol{Z}) \to H^1(S^3\text{-int}\,N(L);\boldsymbol{Z}/N\boldsymbol{Z})\oplus\overset{n}{\oplus}H^1(D^2\times S^1;\boldsymbol{Z}/N\boldsymbol{Z})$$
$$\xrightarrow{f} \overset{n}{\oplus}H^1(T^2;\boldsymbol{Z}/N\boldsymbol{Z})\to\cdots$$

Hence $H^1(M;\boldsymbol{Z}/N\boldsymbol{Z})$ is isomorphic to $\ker f$. Moreover f corresponds to a matrix $\begin{pmatrix} 1_n & 1_n \\ A & 0 \end{pmatrix}$ as a map $f\colon (\boldsymbol{Z}/N\boldsymbol{Z})^n\oplus(\boldsymbol{Z}/N\boldsymbol{Z})^n\to(\boldsymbol{Z}/N\boldsymbol{Z})^{2n}$, where 1_n is the $n\times n$ identity matrix. Since $\ker f$ is isomorphic to $\ker L_A$, we obtain Lemma 3.3. ■

Lemma 3.4. *Let N be even. With the isomorphism ι in Lemma 3.3, the next diagram commutes*:

$$\begin{array}{ccc} \ker L_A & \xrightarrow{\varphi} & \{0,N\}\subset\boldsymbol{Z}/2N\boldsymbol{Z} \\ \iota\downarrow & & \downarrow\times\frac{1}{2} \\ H^1(M;\boldsymbol{Z}/N\boldsymbol{Z}) & \xrightarrow{\psi} & \left\{0,\dfrac{N}{2}\right\}\subset H^3(M;\boldsymbol{Z}/N\boldsymbol{Z})=\boldsymbol{Z}/N\boldsymbol{Z}\,, \end{array}$$

where ψ is defined by $\psi(\alpha)=\alpha\cup\alpha\cup\alpha$.

Proof. Let l be an element in $\ker L_A$, and put $\alpha=\iota(l)$. We calculate $\alpha\cup\alpha\cup\alpha$ in the Poincaré dual and we will show that $\alpha\cup\alpha\cup\alpha$ is equal to $\varphi(l)/2$.

Let S be a branched surface representing the Poincaré dual modulo $\boldsymbol{Z}/N\boldsymbol{Z}$ of α in $M=(S^3-\text{int}\,N(L))\cup\bigcup_{i=1}^n D^2\times S^1$ such that branch locus of S is a union of disjoint circles in $S^3-N(L)$ and the number of sheets meeting along each circle is a multiple of N. Since $[S]$ is the Poincaré dual of $\iota(l)$, $S\cap\partial N(L_i)$ is a union of $\tilde{l}_i$ circles in $\partial N(L_i)$, each of which is parallel to the framing f_i, where $\tilde{l}_i\in\boldsymbol{Z}$ is a lift of $l_i\in\boldsymbol{Z}/N\boldsymbol{Z}$ with ${}^tl=(l_1,\cdots,l_n)$. Let m_i be a meridian of L_i in $S^3-N(L)$. Since $[m_i]$'s generate $H_1(M;\boldsymbol{Z})$, we may assume that branch locus of S is a union of m_i's. Let a_iN be the number of sheets of S meeting along m_i.

Since the boundary of $S\cap(S^3-\text{int}\,N(L))$ consits of $\tilde{l}_i$ copies of f_i in $\partial N(L_i)$, we have $\sum a_i N[m_i]=\sum \tilde{l}_i[f_i]$ in $H_1(S^3-\text{int}\,N(L);\boldsymbol{Z})$. Moreover the classes $[f_i]$'s are determined by

$$\begin{pmatrix}[f_1]\\ \vdots \\ [f_n]\end{pmatrix}=A\begin{pmatrix}[m_1]\\ \vdots \\ [m_n]\end{pmatrix}.$$

Hence we obtain a relation between a_i's and $\tilde{l}_i$'s:

$$\begin{pmatrix}a_1 N\\ \vdots \\ a_n N\end{pmatrix}=A\begin{pmatrix}\tilde{l}_1\\ \vdots \\ \tilde{l}_n\end{pmatrix}.$$

Now we calculate the self-intersection of S. Since $S-\bigcup m_i$ is orientable, we can push S in a normal direction. There are self-intersections near m_i as in Figure 3.1. Hence we have

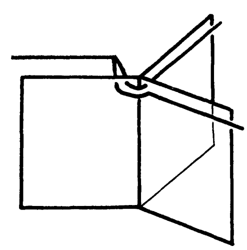

Figure 3.1.

$$\begin{aligned}[S]\cdot[S] &= \sum (1+2+\cdots+(a_iN-1))\,[m_i]\\ &= \sum \frac{a_i N}{2}\,[m_i]\in H_1(M;\boldsymbol{Z}/N\boldsymbol{Z}).\end{aligned}$$

Since $[S]\cdot[m_i]=\tilde{l}_i$, we obtain

$$\begin{aligned}[S]\cdot[S]\cdot[S] &= \sum \frac{a_i N}{2}\,\tilde{l}_i\\ &= \frac{1}{2}\,{}^t\tilde{l}A\tilde{l} = \frac{1}{2}\,\varphi(l).\end{aligned}$$

This is the required formula. ■

REMARK 3.4. The above lemma also follows algebraically from [35, Theorem I], which states that

$$\alpha\cup\alpha\cup\beta=\frac{N^2}{2}\,\lambda(\bar{\alpha},\bar{\beta})\in\boldsymbol{Z}/N\boldsymbol{Z}$$

for $\alpha, \beta \in H^1(M; \boldsymbol{Z}/N\boldsymbol{Z})$. Here $\bar{\alpha}, \bar{\beta} \in \operatorname{Tor} H_1(M; \boldsymbol{Z})$ satisfy $\lambda(\bar{\alpha}, x) = \alpha(x)/n \in \boldsymbol{Q}/\boldsymbol{Z}$ and $\lambda(\bar{\beta}, x) = \alpha(x)/n \in \boldsymbol{Q}/\boldsymbol{Z}$ for any $x \in \operatorname{Tor} H_1(M; \boldsymbol{Z})$.

4. Phase. Now we study the phase of $Z_N(M; q)$.

We use the following notations for an odd integer x: (cf. [33])

$$
\text{(4.1)} \qquad
\begin{aligned}
\varepsilon(x) &= (-1)^{(x-1)/2} = \begin{cases} 1 & x \equiv 1 \mod 4\,, \\ -1 & x \equiv 3 \mod 4\,, \end{cases} \\
\omega(x) &= (-1)^{(x^2-1)/8} = \begin{cases} 1 & x \equiv \pm 1 \mod 8\,, \\ -1 & x \equiv \pm 3 \mod 8\,. \end{cases}
\end{aligned}
$$

Note that $\varepsilon\colon (\boldsymbol{Z}/4\boldsymbol{Z})^\times \to \{1, -1\}$ and $\omega\colon (\boldsymbol{Z}/8\boldsymbol{Z})^\times \to \{1, -1\}$ are homomorphisms.

By Theorem 3.1, $Z_N(M; q) \neq 0$ if $\alpha \cup \alpha \cup \alpha = 0$ for any $\alpha \in H^1(M; \boldsymbol{Z}/N\boldsymbol{Z})$. So we assume this in the following of this section.

We put

$$
Z_N(A; q) = \left(\frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} \sqrt{N}^{-n} \sum_{l \in (\boldsymbol{Z}/N\boldsymbol{Z})^n} q^{{}^t l A l}\,,
$$

for an integral symmetric $n \times n$ matrix A. $Z_N(M; q) = Z_N(A; q)$ if M is obtained from a framed link L with linking matrix A. We note that if N is odd, A may be regarded as a matrix in $\boldsymbol{Z}/N\boldsymbol{Z}$ and if N is even, the diagonal entries in A may be regarded as integers modulo $2N$ and the off-diagonal entries modulo N. We will try to diagonalize A to calculate the phase.

From Proposition 2.3 we will restrict ourselves to the case $N = p^m$ with p prime for a while.

If p is odd, we can diagonalize A as a matrix in $\boldsymbol{Z}/N\boldsymbol{Z}$, that is, there exists a matrix $S \in SL(n, \boldsymbol{Z})$ such that

$$
\text{(4.2)} \qquad {}^t SAS \equiv \bigoplus_{j=1}^{n} (a_j) \mod p^m\,.
$$

If $p = 2$, we cannot diagonalize A itself in general, but it is proved that one can diagonalize the block sum of A and $(1) \oplus (-1) \oplus (2) \oplus (-2) \oplus \cdots \oplus (2^{m-1}) \oplus (-2^{m-1})$, that is, there exists a matrix $S \in SL(n+2m, \boldsymbol{Z})$ such that

$$
\text{(4.3)} \qquad {}^t S \big(A \oplus (1) \oplus (-1) \oplus (2) \oplus (-2) \oplus \cdots \oplus (2^{m-1}) \oplus (-2^{m-1}) \big) S \equiv \bigoplus_{j=1}^{n+2m} (a_j)\,,
$$

where the diagonal entries are considered modulo 2^{m+1} and the off-diagonal entries modulo 2^m. (In fact, it can be proved that $A \oplus (1) \oplus (2) \oplus \cdots \oplus (2^{m-1})$ is diagonalizable, using the technique to diagonalize $\begin{pmatrix} 0 & 2^j \\ 2^j & 0 \end{pmatrix} \oplus (2^j)$.) Note that the phase of $Z_N(A; q)$ remains unchanged by replacing A with $A \oplus (1) \oplus (-1) \oplus (2) \oplus (-2)$

$\oplus\cdots\oplus(2^{m-1})\oplus(-2^{m-1})$ since easy calculations show that $Z_N((2^j);q)\neq 0$ and $Z_N((2^j);q)=\overline{Z_N((-2^j);q)}$ for $j<m$.

Now the phase of $Z_{p^m}(A,q)$ is equal to that of $(G_{p^m}(q))^{-\sigma(A)}\prod_j\sum_{h\in\boldsymbol{Z}/p^m\boldsymbol{Z}}q^{a_jh^2}$. Thus we only need to calculate the sum

$$G_N(a;q)=\sum_{h\in\boldsymbol{Z}/N\boldsymbol{Z}}q^{ah^2}$$

for an integer a and a prime-power N. Note that a Gaussian sum $G_N(q)$ is equal to $G_N(1;q)$. Let $q=\exp(d\pi\sqrt{-1}/p^m)$ with $(d,p)=1$ and $d+p$ odd, and $a=p^kc$ with $(p,c)=1$. If p is odd, we also write q as $\exp(2b\pi\sqrt{-1}/p^m)$ putting $d=2b$. We can describe the above sum as follows.

Lemma 4.4.

(1) *p is odd. Let $\left(\frac{x}{p}\right)$ be Legendre's symbol, that is, $\left(\frac{x}{p}\right)=1$ if there exists an integer l such that $l^2\equiv x \bmod p$, and $\left(\frac{x}{p}\right)=-1$ otherwise. Then*

$$G_{p^m}(a;q)=\begin{cases} p^m & \text{if } k-m\geq 0\,, \\ \sqrt{p}^{\,m+k} & \text{if } k-m<0 \text{ and even}, \\ \left(\frac{c}{p}\right)\left(\frac{b}{p}\right)\sqrt{p}^{\,m+k} & \text{if } k-m<0 \text{ and odd, and } p\equiv 1 \bmod 4\,, \\ \left(\frac{c}{p}\right)\left(\frac{b}{p}\right)\sqrt{-1}\sqrt{p}^{\,m+k} & \text{if } k-m<0 \text{ and odd, and } p\equiv 3 \bmod 4\,. \end{cases}$$

(2) *$p=2$. Put $\zeta=\exp(\pi\sqrt{-1}/4)$. Then*

$$G_{2^m}(a;q)=\begin{cases} 2^m & \text{if } k-m>0\,, \\ 0 & \text{if } k-m=0\,, \\ \zeta^{cd}\sqrt{2}^{\,m+k} & \text{if } k-m<0 \text{ and even}, \\ \zeta^{\varepsilon(c)\varepsilon(e)}\sqrt{2}^{\,m+k} & \text{if } k-m<0 \text{ and odd}. \end{cases}$$

Proof. For the case that $k-m>0$ or the case that $k-m=0$ and p is odd, the formulas follow since $q^a=1$. If $k-m=0$ and p is even, $G_N(a,q)=0$ since $q^a=-1$. The case that $p=2$ and $k-m<0$, the formula follows from $G_{2^m}(a,q)=2G_{2^{m-2}}(a,q)$ $(m\geq k+3)$ and direct computations for $m=k+1$ and $k+2$. The case that p is odd and $k-m<0$ is well-known. For a proof, see for example [**23**, Chapter IV, §3]. (There are some errors in [23], which one can easily fix.) The proof is complete. ■

From this lemma, we know that the phase of $Z_N(A;q)$ takes only eight values. So we define $\phi_N(A;q)\in\boldsymbol{Z}/8\boldsymbol{Z}$ as follows.

We first consider the case that $N=p^m$ for an odd prime p. Let a_j's be diagonal entries when A is diagonalized as in (4.2). Let $a_j=p^{k_j}c_j$ with $(p,c_j)=1$

for $a_j \neq 0$. Note that we can assume $k_j - m$ is always negative. We put n_+ and n_- as follows.

$$n_+ = \#\{a_j \mid k_j - m \text{ is odd and } \left(\frac{c_j}{p}\right) = 1\}$$
$$n_- = \#\{a_j \mid k_j - m \text{ is odd and } \left(\frac{c_j}{p}\right) = -1\}$$

Here $\#\{\cdot\}$ means the number of elements in $\{\cdot\}$. Then $\phi_{p^m}(A; q) \in \boldsymbol{Z}/8\boldsymbol{Z}$ is defined as follows.

$$\phi_{p^m}(A; q) =$$
$$\begin{cases} 2\left(\left(\frac{b}{p}\right)-1\right)n_+ - 2\left(\left(\frac{b}{p}\right)+1\right)n_- & \text{if } p \equiv 1 \bmod 4 \text{ and } m \text{ is even,} \\ 2\left(\left(\frac{b}{p}\right)-1\right)n_+ - 2\left(\left(\frac{b}{p}\right)+1\right)n_- - 2\left(\left(\frac{b}{p}\right)-1\right)\sigma(A) & \text{if } p \equiv 1 \bmod 4 \text{ and } m \text{ is odd,} \\ 2\left(\frac{b}{p}\right)n_+ - 2\left(\frac{b}{p}\right)n_- & \text{if } p \equiv 3 \bmod 4 \text{ and } m \text{ is even,} \\ 2\left(\frac{b}{p}\right)n_+ - 2\left(\frac{b}{p}\right)n_- - 2\left(\frac{b}{p}\right)\sigma(A) & \text{if } p \equiv 3 \bmod 4 \text{ and } m \text{ is odd.} \end{cases}$$

Then from Lemma 4.4 and $\left(\frac{x}{p}\right) = \zeta^{2((\frac{x}{p})-1)}$, it follows that $\phi_{p^m}(A; q)\,\pi\sqrt{-1}/4$ is the phase of $Z_{p^m}(A; q)$.

Next we consider the case $N = 2^m$. Let a_j's be diagonal entries when A is diagonalized as in (4.3). Let $a_j = 2^{k_j} c_j$ with c_j odd for $a_j \neq 0$. Here we assume $k_j - m < 0$ as before. Then $\phi_{2^m}(A; q)$ is defined by

$$\phi_{2^m}(A; q) = \begin{cases} d \sum_{k_j - m\,:\,\text{even}} c_j + \mathcal{E}(d) \sum_{k_j - m\,:\,\text{odd}} \mathcal{E}(c_j) - d\sigma(A) & \text{if } m \text{ is even,} \\ d \sum_{k_j - m\,:\,\text{even}} c_j + \mathcal{E}(d) \sum_{k_j - m\,:\,\text{odd}} \mathcal{E}(c_j) - \mathcal{E}(d)\,\sigma(A) & \text{if } m \text{ is odd.} \end{cases}$$

From Lemma 4.4, the phase of $Z_{2^m}(A; q)$ is $\phi_{2^m}(A; q)\,\pi\sqrt{-1}/4$.

According to Proposition 2.3, we define $\phi_N(A; q)$ for an arbitrary N by using

$$\phi_N(A; q) = \phi_{N_1}(A; q^{N_2^2}) + \phi_{N_2}(A; q^{N_1^2}) \in \boldsymbol{Z}/8\boldsymbol{Z},$$

where $N = N_1 N_2$ with coprime integers N_1 and N_2.

For a closed, oriented 3-manifold M, we define $\phi_N(M; q) = \phi_N(A; q)$ for the linking matrix A of a framed link which gives M. Summarizing the above argument we have the next proposition.

Theorem 4.5. *If $\alpha \cup \alpha \cup \alpha = 0$ for any $\alpha \in H^1(M; \boldsymbol{Z}/N\boldsymbol{Z})$, then*

$$Z_N(M, q) = \exp\left(\frac{\pi\sqrt{-1}}{4}\,\phi_N(M; q)\right) |H^1(M; \boldsymbol{Z}/N\boldsymbol{Z})|^{1/2},$$

where $\phi_N(M; q) \in \boldsymbol{Z}/8\boldsymbol{Z}$ is defined above. In particular $\phi_N(M; q)$ is a topological *invariant of M.*

REMARK 4.6. By defintion, $\beta(M)=-\phi_2(M;\sqrt{-1})$ is the Brown invariant [16, §6]. See [2, 10, 28] for Brown's invariant of $\boldsymbol{Z}/4\boldsymbol{Z}$-valued quadratic forms on a $\boldsymbol{Z}/2\boldsymbol{Z}$-vector space.

As applications of Theorem 4.5, we calculate $Z_N(M;q)$ for $\boldsymbol{Z}/p\boldsymbol{Z}$-homology spheres. (A closed, oriented 3-manifold M is called a $\boldsymbol{Z}/p\boldsymbol{Z}$-homology sphere if $H_i(M;\boldsymbol{Z}/p\boldsymbol{Z})=H_i(S^3;\boldsymbol{Z}/p\boldsymbol{Z})$ for all i.)

Corollary 4.7. *Let $N=2^m$ and $q=\exp(d\pi\sqrt{-1}/N)$. If M is a $\boldsymbol{Z}/2\boldsymbol{Z}$-homology sphere, then the value of $Z_N(M;q)$ is as follows.*

$$Z_N(M;q)=\begin{cases}\zeta^{-d\mu(M)} & \text{if } m \text{ is even,}\\ \omega(|H_1(M;\boldsymbol{Z})|)\,\zeta^{-\varepsilon(d)\mu(M)} & \text{if } m \text{ is odd.}\end{cases}$$

where $\zeta=\exp(\pi\sqrt{-1}/4)$ and $\mu(M)$ is the μ-(or Rochlin) invariant of M (the signature modulo 16 of a spin 4-manifold with boundary M).

Proof. Since $H^1(M;\boldsymbol{Z}/N\boldsymbol{Z})=0$, we calculate the phase. After a change of basis we may assume that A is diagonal (mod $2N$) with diagonal entries a_j. Since M is a $\boldsymbol{Z}/2\boldsymbol{Z}$-homology sphere, a_j is always odd. We also assume that $a_j=1, 3, 5$, or 7 because there exists an odd integer l such that $cl^2=1, 3, 5$, or 7 mod $2N$ for any odd integer c. Let n_c be the number of c's in these diagonal entries ($c=1, 3, 5$, or 7).

For m even, by the definition of $\phi_N(M;q)$, we have

$$\phi_N(M;q)\equiv d(n_1+3n_3+5n_5+7n_7-\sigma(A)) \mod 8.$$

Since $\mu(M)\equiv\sigma(A)-(n_1+3n_3+5n_5+7n_7)$ mod 8 (see [16, Appendix C]), we obtain the required formula.

For m odd, we have

$$\phi_N(M;q)=\varepsilon(d)(n_1-n_3+n_5-n_7-\sigma(A))$$

Thus $\phi_N(M;q)+\varepsilon(d)\,\mu(M)\equiv-4\varepsilon(d)(n_3+n_5)$ mod 8. Since $\varepsilon(d)=\pm1$, we have

$$\phi_N(M;q)\equiv-\varepsilon(d)\,\mu(M)+4(n_3+n_5) \mod 8.$$

Moreover since

$$|H_1(M;\boldsymbol{Z})|=\pm\det A\equiv\pm3^{n_3}5^{n_5}7^{n_7}\equiv\pm3^{n_3}(-3)^{n_5}(-1)^{n_7} \mod 8,$$

we obtain $\omega(|H_1(M;\boldsymbol{Z})|)=(-1)^{n_3+n_5}$. Therefore we obtain the required formula. ■

Corollary 4.8. *Let $N=p^m$ with odd prime p and q an N-th root of unity.*

If M is a $\boldsymbol{Z}/p\boldsymbol{Z}$-homology sphere, then

$$Z_N(M;q)=\left(\frac{r}{p}\right)^m$$

where $r=|H_1(M;\boldsymbol{Z})|$ *and* $\left(\frac{r}{p}\right)$ *is Legendre's symbol.*

Proof. Adding a splitted, unknotted component if necessary, we assume that $\det A$ is positive so that $r=\det A$. Let b, a_j's, n_+, and n_- be as in the notation of the definition of $\phi_{p^m}(A;q)$. Since M is a $\boldsymbol{Z}/p\boldsymbol{Z}$-homology sphere, $(p,a_j)=1$ and so $k_j=0$ for any j. We also note that $r=\det A\equiv\prod a_j \bmod p$. Thus we have

$$\left(\frac{r}{p}\right)=\left(\frac{\prod a_j}{p}\right)=\prod\left(\frac{a_j}{p}\right)=\begin{cases}1 & \text{if } n_- \text{ is even,}\\ -1 & \text{if } n_- \text{ is odd.}\end{cases}$$

For m even, we have $n_+=n_-=0$. Hence $Z_N(M;q)=1$.

Next we consider the case that m is odd. In this case, $n_++n_-=n$, the size of A. So $n_+=n-n_-$. We also have $\sigma(A)\equiv n \bmod 4$ since $\det A>0$.

If $p\equiv 1 \bmod 4$, then by definition, we have

$$\begin{aligned}\phi_N(M;q)&=2\left(\left(\frac{b}{p}\right)-1\right)n_+-2\left(\left(\frac{b}{p}\right)+1\right)n_--\left(\left(\frac{b}{p}\right)-1\right)\sigma(A)\\&=2\left(\left(\frac{b}{p}\right)-1\right)(n-\sigma(A))-4n_-\\&\equiv 4n_- \mod 8.\end{aligned}$$

If $p\equiv 3 \bmod 4$, then we also have

$$\begin{aligned}\phi_N(M;q)&=2\left(\frac{b}{p}\right)n_+-2\left(\frac{b}{p}\right)n_--2\left(\frac{b}{p}\right)\sigma(A)\\&=2\left(\frac{b}{p}\right)(n-\sigma(A))-4\left(\frac{b}{p}\right)n_-\\&\equiv 4n_- \mod 8.\end{aligned}$$

Therefore we obtain the value of $Z_N(M;q)$ as above, completing the proof. ■

5. Calculation for generators of linking pairings

Any linking pairing is a direct sum of the following linking pairings [**36**, 14]:

$$(p^{-k}r)\,(k\geq 1),\quad A^k(n)\,(k\geq 1),\quad E_0^k(k\geq 1),\quad \text{and}\quad E_1^k(k\geq 2),$$

where p is odd, prime integer, r is 1 or a fixed quadratic non-residue modulo p, and $n=1(k=1)$, $\pm 1(k=2)$, ± 1 or $\pm 3(k\geq 3)$. Here we use the notation of

[14].

Since $Z_N(M; q)$ is an invariant of first Betti numbers and linking pairings (Proposition 2.5), and linking pairings split as above, we can calculate $Z_N(M; q)$ if we know them for 3-manifolds with the linking pairings above from Proposition 2.3. Note that the free part of the first homology affects $Z_N(M; q)$ only by absolute values (Theorem 3.2).

In the following, we denote $Z_N(M; q)$ by $Z_N(\Lambda; q)$ if the linking pairing on $H_1(M; \boldsymbol{Z})$ is isomprphic to Λ in the above.

Theorem 5.1. *Let p and p' be odd, prime integers ($p \neq p'$), band b' integers with $(p, b)=1$ and $(p', b')=1$, and d an odd integer. Put $q=\exp(2b\pi\sqrt{-1}/p^m)$, $q'=\exp(2b'\pi\sqrt{-1}/p'^m)$, $q''=\exp(d\pi\sqrt{-1}/2^m)$, and $\zeta=\exp(\pi\sqrt{-1}/4)$. We also use the notations* (4.1).

(1) *The case $\Lambda=(p^{-k}r)$.*

$$\boldsymbol{Z}_{2^m}((p^{-k}r); q'') = \begin{cases} 1 & \textit{for } (*, 0, *, *), \\ -\omega(p)\left(\frac{r}{p}\right) & \textit{for } (*, 1, 0, 1), \\ -\varepsilon(d)\,\omega(p)\left(\frac{r}{p}\right)\sqrt{-1} & \textit{for } (*, 1, 0, 3), \\ -\left(\frac{r}{p}\right) & \textit{for } (*, 1, 1, 1), \\ -\left(\frac{r}{p}\right)\sqrt{-1} & \textit{for } (*, 1, 1, 3). \end{cases}$$

$$Z_{p^m}((p^{-k}r); q) = \begin{cases} \sqrt{p}^m & \textit{for } (+\textit{or } 0, *, 0, *), \\ \left(\frac{r}{p}\right)\left(\frac{b}{p}\right)\sqrt{p}^m & \textit{for } (+\textit{or } 0, *, 1, 1), \\ -\left(\frac{r}{p}\right)\left(\frac{b}{p}\right)\sqrt{-1}\,\sqrt{p}^m & \textit{for } (+\textit{or } 0, *, 1, 3), \\ \sqrt{p}^k & \textit{for } (-, 0, *, *), \\ \left(\frac{r}{p}\right)\left(\frac{b}{p}\right)\sqrt{p}^k & \textit{for } (-, 1, *, 1), \\ \left(\frac{r}{p}\right)\left(\frac{b}{p}\right)\sqrt{-1}\,\sqrt{p}^k & \textit{for } (-, 1, 0, 3), \\ -\left(\frac{r}{p}\right)\left(\frac{b}{p}\right)\sqrt{-1}\,\sqrt{p}^k & \textit{for } (-, 1, 1, 3). \end{cases}$$

Here $(\cdot, \cdot, \cdot, \cdot)$ *is* (*sign of* $k-m$, k mod 2, m mod 2, p mod4).

$$Z_{p'^m}((p^{-k}r); q') = \left(\frac{p'}{p}\right)^{mk}.$$

(2) *The case $\Lambda=A^k(n)$, E_0^k, or E_1^k.*

$$Z_{2^m}(A^k(1);q'')=\begin{cases}\zeta^{-d}\sqrt{2}^m & \textit{for } (+,*,0),\\ \zeta^{-\varepsilon(d)}\sqrt{2}^m & \textit{for } (+,*,1),\\ 0 & \textit{for } (0,*,*),\\ \sqrt{2}^k & \textit{for } (-,0,*),\\ \omega(d)\sqrt{2}^k & \textit{for } (-,1,*).\end{cases}$$

$$Z_{2^m}(A^k(3);q'')=\begin{cases}\zeta^{5d}\sqrt{2}^m & \textit{for } (+,0,0),\\ \zeta^{d}\sqrt{2}^m & \textit{for } (+,1,0),\\ \zeta^{\varepsilon(d)}\sqrt{2}^m & \textit{for } (+,0,1),\\ \zeta^{-3\varepsilon(d)}\sqrt{2}^m & \textit{for } (+,1,1),\\ 0 & \textit{for } (0,*,*),\\ \sqrt{2}^k & \textit{for } (-,0,*),\\ \omega(d)\sqrt{2}^k & \textit{for } (-,1,*).\end{cases}$$

Here $(\cdot,\cdot,\cdot)$ *is* (*sign of* $k-m$, k mod 2, m mod 2).

$$Z_{2^m}(A^k(-1);q'')=\overline{Z_{2^m}(A^k(1);q'')} \quad (\textit{complex conjugate}).$$

$$Z_{2^m}(A^k(-3);q'')=\overline{Z_{2^m}(A^k(3);q'')}\,.$$

$$Z_{2^m}(E_0^k;q'')=\begin{cases}2^m & \textit{if } k\geq m,\\ 2^k & \textit{if } k<m.\end{cases}$$

$$Z_{2^m}(E_1^k;q'')=\begin{cases}(-1)^{m+k}\,2^m & \textit{if } k\geq m,\\ 2^k & \textit{if } k<m.\end{cases}$$

$$Z_{p^m}(A^k(n);q)=\begin{cases}-1 & \textit{if } m \textit{ and } k \textit{ are odd, and } p\equiv\pm 3 \bmod 8,\\ \ \ 1 & \textit{otherwise.}\end{cases}$$

$$Z_{p^m}(E_0^k;q)=Z_{p^m}(E_1^k;q)=1$$

Proof. For $(p^{-k}r)$, we consider the lens space $L(p^k, r)$. It can be obtained from a framed link with linking matrix of the form

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & 0\\ 1 & a_2 & 1 & \cdots & 0\\ 0 & 1 & \ddots & & \vdots\\ \vdots & \vdots & & \ddots & 1\\ 0 & 0 & \cdots & 1 & a_n\end{pmatrix}.$$

Here the continued fraction

$$a_1-\cfrac{1}{a_2-\cfrac{1}{\cdots-\cfrac{1}{a_n}}}$$

is equal to p^k/r. See for example [32]. So we can calculate $Z_{p^m}((p^{-k}r); q)$ using Theorem 4.5. The value $Z_{p'^m}((p^{-k}r); q')$ can be calculated using Corollary 4.8. $Z_{2^m}((p^{-k}r); q'')$ can be obtained from Corollary 4.7 and the fact

$$2\mu(L(\alpha, \beta)) \equiv 2(\alpha+1)-4(\beta|\alpha) \mod 16,$$

where $(\beta|\alpha)$ is the Jacobi symbol [12, Theorem 8.14]. Note that our definition of the μ-invariant differs from that in [12].

For $A^k(1)$ and $A^k(3)$, we choose linking matrices of the form

$$(2^k) \quad \text{and} \quad (-1)^{k+1}\begin{pmatrix} (4^{m+1}-(-2)^k)/3 & 2^{m+1} \\ 2^{m+1} & 3 \end{pmatrix},$$

respectively. (Note that they are diagonal in $\boldsymbol{Z}/2^{2m+1}\boldsymbol{Z}$.) Then we can calculate $Z_{2^m}(A^k(n); q'')$ $(n=1, 3)$ using Theorem 4.5. Since if the linking pairing for a 3-manifold M is $A^k(n)$, then that for $-M$ is $A^k(-n)$, the values $Z_{2^m}(A^k(n); q'')$ $(n=-1, -3)$ are obtained from Proposition 2.1.

To calculate $Z_{2^m}(E_0^k; q'')$ $(m\neq k)$, we use the relation (see [14])

$$A^k(1)\oplus 2A^k(-1) = A^k(-1)\oplus E_0^k .$$

Since $Z_{2^m}(A^k(-1); q'')\neq 0$ for $m\neq k$, we obtain $Z_{2^m}(E_0^k; q'')$ from Proposition 2.2. For $m=k$, we can directly calculate it choosing $\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}$ as a linking matrix for E_0^k.

Using the relations (see [14] again)

$$3A^k(1) = A^k(3)\oplus E_1^k \quad \text{and} \quad E_1^k\oplus A^{k+1}(1) = E_0^k\oplus A^{k+1}(-3),$$

we can obtain $Z_{2^m}(E_1^k; q'')$ for any m.

The values $Z_{p_m}(A^k(n); q)$, $Z_{p^m}(E_0^k; q)$ and $Z_{p_m}(E_1^k; q)$ are easily obtained from Corollary 4.7.

The proof is complete. ■

REMARK 5.2. The series $\{Z_N(\cdot\,; q)\}$ is *not* a complete invariant of linking pairings. For example $Z_N(32A^1(1)\oplus 16A^2(1); q)=Z_N(16A^1(1)\oplus 24A^2(1); q)$ for any N and q but $32A^1(1)\oplus 16A^2(1)$ is not equivalent to $16A^1(1)\oplus 24A^2(1)$.

From Theorem 5.1, we have another condition for $Z_N(M; q)$ to be zero.

Corollary 5.3. *$Z_N(M; q)=0$ if and only if there exists $x\in H_1(M; \boldsymbol{Z})$ of order 2^m with $\lambda(x, x)=c/2^m$, where $N=2^m b$ with b odd, c is an odd integer, and λ is the linking pairing on* Tor $H_1(M; \boldsymbol{Z})$.

Proof. From the above theorem and Proposition 2.2, $Z_N(M; q)=0$ if and only if the linking pairing has a direct summand of the form $A^k(n)$, If $Z_N(M; q)=0$ then the existence of an element x as in the statement of the corollary follows easily. Conversely, suppose that there exists x as above. Then since the linking pairing restricted to the cyclic group generated by x is non-singular, it has $A^k(n)$ as a direct summand with $n \equiv c$ mod 8 (see [36, Lemma (1)]). The proof is complete. ∎

6. Invariants for links. For an oriented link L in S^3 (without framing) and an integer $s(\geq 2)$, one can construct the s-fold cyclic branched covering space branched along L associated with the kernel of a map $H_1(S^3-L; \boldsymbol{Z}) \to \boldsymbol{Z}/s\boldsymbol{Z}$ sending each meridian to 1. Since it is a closed, oriented 3-manifold, we can define $Z_N(L; q, s)$ to be $Z_N(M(L, s); q)$, where $M(L, s)$ is the s-fold cyclic branched covering space as above. $Z_N(L; q, s)$ is an invariant of L for every s since $M(L, s)$ is uniquely determined by L and s.

A framed link description for $M(L, s)$ is given by S. Akubult and R. Kirby [1]. Denoting a Seifert matrix for L constructed from a connected Seifert surface by V, its linking matrix is given by $V \otimes B + {}^tV \otimes {}^tB$, where $B=(B_{ij})$ $(1 \leq i, j \leq s-1)$ with $B_{ij}=1$ for $1 \leq i \leq j \leq s-1$ and $B_{ij}=0$ otherwise. So we have

Lemma 6.1.

$$Z_N(L; q, s) = \left(\frac{G_N(q)}{|G_N(q)|}\right)^{-\sigma(A)} \sqrt{N}^{-g(t-1)} \sum_{l \in (\boldsymbol{Z}/N\boldsymbol{Z})^{g(s-1)}} q^{{}^tlAl},$$

where $A = V \otimes B + {}^tV \otimes {}^tB$ *and* g *is the size of* V.

Note that if $s=2$, $\sigma(A)$ is just $\sigma(L)$, the signature of L [29, 34].

In [4], E. Date, M, Jimbo, K. Miki, and T. Miwa define link invariants using generalized chiral Potts models. They are give as follows.

DEFINITION 6.2. [4]. *Let N be a positive odd integer, q a primitive N-th root of unity, and C an $(s-1)\times(s-1)$ integral matrix $(s>1)$. For an oriented link L with Seifert matrix V of size g, we put*

$$\tau(L; N, q, s, C) = \sqrt{N}^{-g(s-1)} \sum_{l \in (\boldsymbol{Z}/N\boldsymbol{Z})^{g(s-1)}} q^{{}^tl(V \otimes C)l}.$$

Since ${}^tl(V \otimes C + {}^tV \otimes {}^tC)\, l = 2({}^tl(V \otimes C)\, l)$, we have

Proposition 6.3. *Let* $q = \exp(2b\pi\sqrt{-1}/N)$ *and* $q' = \exp((N+1)\, b\pi \sqrt{-1}/N)$ *with* $(b, N)=1$. *Then*

$$Z_N(L; q', s) = \left(\frac{G_N(q')}{|G_N(q')|}\right)^{-\sigma(A)} \tau(L; N, q, s, B),$$

where $A = V \otimes B + {}^tV \otimes {}^tB$ *and B is as above. Note that q' is also a primitive N-th root of unity because N is odd.*

REMARK 6.4. For a positive even integer N and a primitive N-th root of unity q,

$$\tau(L; N, q, s, C) = \sqrt{N}^{-g(s-1)} \sum_{l \in (Z/NZ)^{g(s-1)}} q^{{}^t l (V \otimes C) l}$$

is also an invariant of a link L. This follows from the fact that the above formula is invariant of S-equivalence class [3, 29, 34] of Seifert matrices for links. Proposition 6.3 also holds in this case. (q' is now a primitive $2N$-th root of unity.)

The cyclotomic invariant $T_N(L)$ [19] is given by $\tau(L; N, \exp(2\pi\sqrt{-1}/N), 2, (1))$ for an integer greater than 1. (See also [9, 13].) So we have

Proposition 6.5. *Put* $q = \exp((N+1)\pi\sqrt{-1}/N)$. *Then*

$$T_N(L) = \left(\frac{G_N(q)}{|G_N(q)|}\right)^{\sigma(L)} Z_N(L; q, 2).$$

For relations of the cyclotomic invariants to the polynomial invariants for links, see [9, 19].

7. A family of quasitriangular Hopf algebras. We will give another description for $Z_N(M; q)$ using representations of some algegras. A *Hopf algebra* $\mathcal{A}$ is an algebra over a field k with comultiplication $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, counit $\varepsilon: \mathcal{A} \to k$ and antipode $\gamma: \mathcal{A} \to \mathcal{A}$. Let R be an element in $\mathcal{A} \otimes \mathcal{A}$. The pair (A, R) is called a *quasitriangular Hopf algebra* [6] if R is invertible in $\mathcal{A} \otimes \mathcal{A}$, $P \circ \Delta(a) = R\Delta(a) R^{-1}$ for any $a \in \mathcal{A}$, where P is the permutation operator ($P(x \otimes y) = y \otimes x$), and

$$(\Delta \otimes \mathrm{id})(R) = R_{13} R_{23}$$
$$(\mathrm{id} \otimes \Delta)(R) = R_{13} R_{12}$$

where $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, and $R_{13} = \sum \alpha_i \otimes 1 \otimes \beta_i$ for $R = \sum \alpha_i \otimes \beta_i$.

Let r be a positive integer and q a primitive r-th root of unity. We define a quasitriangular Hopf algebra A_q over the field $\boldsymbol{Q}(q)$. The algebra A_q is generated by 1, K, and K^{-1} with relation $K^r = 1$. A comultiplication, counit and antipode are defined by $\Delta(K) = K \otimes K$, $\varepsilon(K) = 1$ and $\gamma(K) = K^{-1}$, respectively. Let R be $r^{-1} \sum_{i,j=0}^{r-1} q^{-ij} K^i \otimes K^j$. Then we have

Lemma 7.1. (A_q, R) *is a quasitriangular Hopf algebra.*

Proof. The inverse element of R is given by $r^{-1} \sum_{i,j=0}^{r-1} q^{ij} K^i \otimes K^j$ because

$$\begin{aligned}
& R \cdot r^{-1} \sum_{i',j'} q^{i'j'} K^{i'} \otimes K^{j'} \\
& = r^{-2} \sum q^{i'j'-ij} K^{i+i'} \otimes K^{j+j'} \\
& = r^{-2} \sum_{i,i',k} q^{ik} \left(\sum_j q^{-(i+i')j}\right) K^{i+i'} \otimes K^k \quad (k = j+j') \\
& = r^{-1} \sum_k \left(\sum_i q^{ik}\right) \cdot 1 \otimes K^k \\
& = 1.
\end{aligned}$$

Here the third and fourth equalities follow from Lemma 3.1.

Since A_q is commutative, we have $R\Delta(a)R^{-1}=\Delta(a)=P\circ\Delta(a)$. Moreover

$$\begin{aligned} R_{13}R_{23} &= r^{-2}\sum q^{-ij-i'j'}K^i\otimes K^{i'}\otimes K^{j+j'} \\ &= r^{-2}\sum q^{-i'k}(\sum q^{(i'-i)j})K^i\otimes K^{i'}\otimes K^k, \quad (k=j+j') \\ &= r^{-1}\sum q^{-ik}K^i\otimes K^i\otimes K^k \\ &= (\Delta\otimes\mathrm{id})(R). \end{aligned}$$

A similar calculation shows $(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12}$. ■

Since A_q is commutative, all irreducible representation spaces are one-dimensional. We denote these representations by $\{V_j\}_{j=0,1,\cdots,r-1}$, with the action $\rho_j(K)$ given by the multiplication by q^j. For representations $\rho_i: A_q\to\mathrm{End}(V_i)$ and $\rho_j: A_q\to\mathrm{End}(V_j)$, a tensor product representation is defined by $(\rho_i\otimes\rho_j)\circ\Delta$: $A_q\to\mathrm{End}(V_i\otimes V_j)$. The action ρ_j^* on the dual space V_j^* induced from the antipode γ is given by the multiplication by q^{-j}. We can easily see that $(A_q, R, v, \{V_j\})$ is a modular Hopf algebra [31] putting $v=r^{-1}\sum_{i,j=0}^{r-1}q^{j(i-j)}K^i$. With this algebra $(A_q, R, v, \{V_j\})$, we can construct invariants of 3-manifolds according to [31]. We survey an outline of the procedure for constructing them.

Let L be a framed link and consider its diagram. We assume that its framing f_i of a component L_i is parallel to L_i in the plane. A *coloring* of L is an assignment of V_j to each component of L. Now we associate an operator Ω with each crossing of a colored framed link as follows.

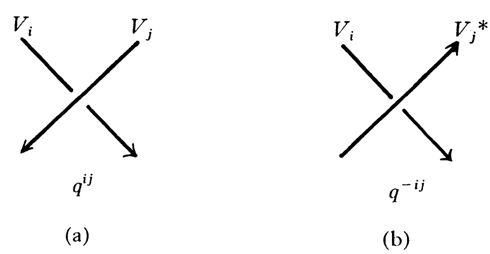

Figure 7.1.

If the crossing is as in Figure 7.1(a), then Ω is a homomorphism from $V_i\otimes V_j$ to $V_j\otimes V_i$ given by $x\otimes y\mapsto(P\circ((\rho_i\otimes\rho_j)R))(x\otimes y)$. It follows that $\Omega(x\otimes y)=q^{ij}(y\otimes x)$ because

$$\begin{aligned} ((\rho_i\otimes\rho_j)R)(x\otimes y) &= r^{-1}\sum_{i',j'}q^{-i'j'}(\rho_i(K^{i'})x\otimes\rho_j(K^{j'})y) \\ &= r^{-1}\sum_{i'}q^{ii'}\sum_{j'}q^{(j-i')j'}(x\otimes y) \\ &= q^{ij}(x\otimes y), \end{aligned}$$

where the last equality follows from Lemma 3.1 again. If the crossing is as in Figure 7.1(b), then Ω is a homomorphism from $V_i \otimes V_j^*$ to $V_j^* \otimes V_i$ given by $P \circ ((\rho_i \otimes \rho_j^*))R)$ and we see that $\Omega(x \otimes y^*) = q^{-ij}(y^* \otimes x)$. Similar calculations show that if the crossing is positive, then Ω is the multiplication by q^{ij} (and the interchanging of the coordinate) and if the crossing is negative, then Ω is the multiplication by q^{-ij}.

Then we can obtain an invariant of a 3-manifold as the sum of the products $\prod_{\text{positive crossings}} q^{ij} \prod_{\text{negative crossings}} q^{-i'j'}$ for all colorings after some normalization.

$Z_N(M; q)$ corresponds to this invariant putting $r=2N$ for N even and $r=N$ for N odd.

8. Operator invariants for 3-dimensional cobordism and invariants of Gocho

As in [31] we can extend the invariants $Z_N(M; q)$ to operator invariants of 3-dimensional cobordisms with non-empty parametrized boundaries, using the modular Hopf algebra structure in A_q described in §7. In this section, we define them by using linking matrices, and prove that invariants of T. Gocho [8] are essentially the absolute values of our invariants. See [31, §4] for the precise definition of 3-dimensional cobordisms with parametrized boundaries.

We denote by G_g^T (G_g^B, resp.) a horizontal line segment with g arcs glued to the top (bottom, resp.), which is embedded in S^3 as described in Figures 8.1 and 8.2. Each arc has a framing (or parametrization) indicated by a thin line parallel to it in the plane.

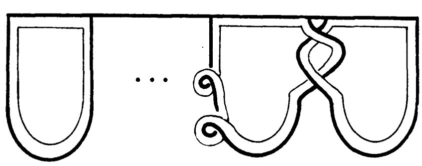

Figure 8.1.

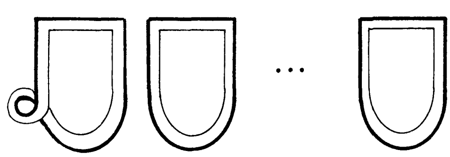

Figure 8.2.

Let $\hat{G}_g^T$ ($\hat{G}_g^B$, resp.) be a farmed link obtained by eliminating short segments between arcs from G_g^T (G_g^B, resp.) as in Figures 8.3 and 8.4.

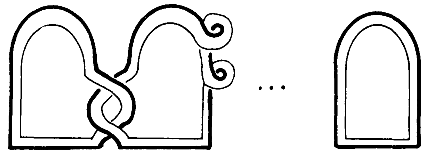

Figure 8.3.

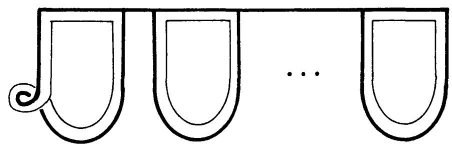

Figure 8.4.

Let (M, F', F) be a 3-dimensional cobordism with connected M whose parametrized boundaries are F' and F. For simplicity we assume that F' and F are connected surfaces of genus g' and g respectively. We can represent M by Dehn surgery on S^3 as follows. We consider graphs $G^B_{g'}$ and G^T_g, and a framed link L in S^3, where L is located between $G^B_{g'}$ and G^T_g as shown in Figure 8.5. With suitably chosen L, we can put $M = M_L$-$(\text{int}\, N(G^B_{g'}) \cup \text{int}\, N(G^T_g))$, where M_L is a 3-manifold obtained by Dehn surgery in S^3 along L, and $N(G^B_{g'})$ and $N(G^T_g)$ are tubular neighborhoods of $G^B_{g'}$ and G^T_g respectively.

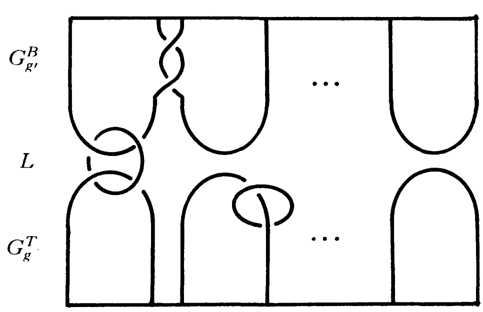

Figure 8.5.

Let V_g be an N^g-dimensional complex vector space with basis $\{e_h\}$, where N is an integer greater than 1 and $h \in (\boldsymbol{Z}/N\boldsymbol{Z})^g$. V^*_g is its dual with dual basis $\{e^*_h\}$. We define an operator invariant of M in $V^*_{g'} \otimes V_g \cong \text{Hom}(V_{g'}, V_g)$ by

$$Z_N(M; q)$$
$$= \left(\frac{G_N(q)}{|G_N(q)|}\right)^{-\sigma(A)} |G_N(q)|^{-n-(g'/2)-(g/2)} \sum_{\substack{h' \in (\boldsymbol{Z}/N\boldsymbol{Z})^{g'} \\ h \in (\boldsymbol{Z}/N\boldsymbol{Z})^{g}}} \Big(\sum_{l \in (\boldsymbol{Z}/N\boldsymbol{Z})^{n}} q^{{}^t\begin{pmatrix} h' \\ l \\ h \end{pmatrix} A \begin{pmatrix} h' \\ l \\ h \end{pmatrix}} \Big) e^*_{h'} \otimes e_h,$$

where q and $G_N(q)$ are as in §1, A is the linking matrix of $\hat{G}^B_{g'} \cup L \cup \hat{G}^T_g$, and n

is the number of components of L. In a similar way as the proof of Theorem 1.3, we can show that this is a topological invariant of M as a 3-dimensional cobordism with parametrized boundary.

The following proposition is a corollary to [31, Theorem 4.5]. We give a direct proof using the formula above.

Proposition 8.1. *If a 3-dimensional cobordism (M, F_1, F_3) is a composition of two cobordisms (M_1, F_1, F_2) and (M_2, F_2, F_3), then for some integer c*

$$Z_N(M; q) = \zeta^c Z_N(M_2; q) \circ Z_N(M_1; q)\,,$$

where $Z_N(M_1; q) \in V_{g_1}^* \otimes V_{g_2} = \mathrm{Hom}(V_{g_1}, V_{g_2})$, $Z_N(M_2; q) \in \mathrm{Hom}(V_{g_2}, V_{g_3})$, g_i is the genus of F_i, and $\zeta = \exp(\pi\sqrt{-1}/4)$.

Proof. For simplicity, we assume that $F_1 = F_3 = \emptyset$. We present M_1 and M_2 by $L_1 \cup G_{g_2}^T$ and $G_{g_2}^B \cup L_2$ respectively, where $M_1 = M_{L_1} - \mathrm{int}\, N(G_{g_2}^T)$ and $M_2 = M_{L_2} - \mathrm{int}\, N(G_{g_2}^B)$. Then M is presented by a framed link $L_1 \cup L_0 \cup L_2$, where L_0 is a framed link obtained from $G_{g_2}^T$ and $G_{g_2}^B$ by gluing arcs as shown in Figure 8.6.

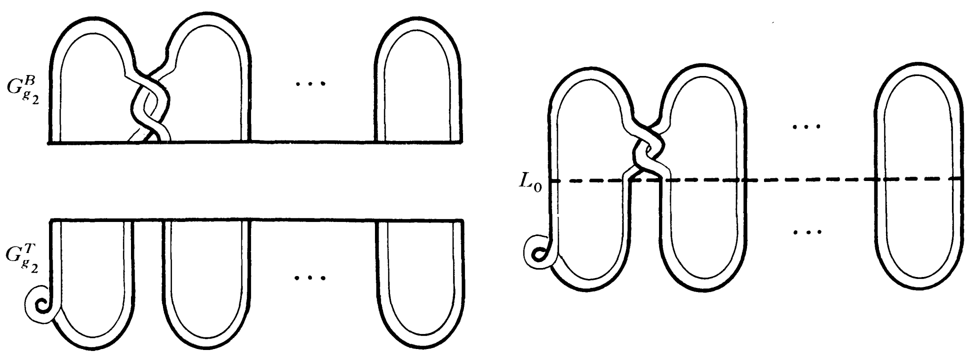

Figure 8.6.

Let A, A_1, and A_2 be the linking matrices of $L_1 \cup L_0 \cup L_2$, $L_1 \cup \hat{G}_{g_2}^T$, and $\hat{G}_{g_2}^B \cup L_2$ respectively. We have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix},$$

where 0's are zero matrices with suitable sizes. Hence we have

$${}^t\begin{pmatrix} l_1 \\ h \\ l_2 \end{pmatrix} A \begin{pmatrix} l_1 \\ h \\ l_2 \end{pmatrix} = {}^t\begin{pmatrix} l_1 \\ h \end{pmatrix} A_1 \begin{pmatrix} l_1 \\ h \end{pmatrix} + {}^t\begin{pmatrix} h \\ l_2 \end{pmatrix} A_2 \begin{pmatrix} h \\ l_2 \end{pmatrix}.$$

It follows that $Z_N(M; q)$ is equal to $Z_N(M_2; q) \circ Z_N(M_1; q)$ with a scalar multiple

$(G_N(q)/|G_N(q)|)^{\sigma(A)-\sigma(A_1)-\sigma(A_2)}$. Since the phase of a Gaussian sum has a value of eighth root of unity, we obtain the required formula. ■

Let $\mathfrak{M}_g$ be the mapping class group of a closed surface of genus g. With this proposition we obtain a representation of $\mathfrak{M}_g$ to $PU(V_g)=U(V_g)/U(1)$ as follows. Let F be a closed surface with parametrization of genus g and $f\colon F\to F$ a homeomorphism. We denote by C_f the mapping cylinder of f, that is, $F\times[0,1]$ with parametrization in $F\times\{1\}$ induced by f. For fixed N and q, we have a map $\mathfrak{M}_g\to PU(V_g)$, $f\mapsto Z_N(C_f;q)$. By Proposition 8.1 this map becomes a representation.

In the case that N is even and $q=\exp(\pi\sqrt{-1}/N)$, this representation coincides with a representation constructed by T. Gocho [8]. Let N and q as above in the following of this section. By a geometric method based on $U(1)$ gauge thory, Gocho constructed a representation ρ_g of $\mathfrak{M}_g$ to $PU(V_g)$ which factors $Sp(2g;\boldsymbol{Z})\ni f_*\colon H_1(F;\boldsymbol{Z})\to H_1(F;\boldsymbol{Z})$. The representation $\rho_g\colon Sp(2g;\boldsymbol{Z})\to PU(V_g)$ is given by the next formulas.

$$\rho_g\begin{pmatrix}0 & -1_g\\ 1_g & 0\end{pmatrix} e_h = \sqrt{N}^{-g}\sum_{h'} q^{2^t h\cdot h'} e_{h'}\,,$$

$$\rho_g\begin{pmatrix}X & 0\\ 0 & {}^tX^{-1}\end{pmatrix} e_h = e_{{}^tX^{-1}h}\,,$$

$$\rho_g\begin{pmatrix}1_g & Y\\ 0 & 1_g\end{pmatrix} e_h = q^{-{}^thYh} e_h\,.$$

Here $X\in GL(g;\boldsymbol{Z})$ and Y is a $g\times g$ symmetric integral matrix. Note that $\begin{pmatrix}0 & -1_g\\ 1_g & 0\end{pmatrix}$, $\begin{pmatrix}X & 0\\ 0 & {}^tX^{-1}\end{pmatrix}$, and $\begin{pmatrix}1_g & Y\\ 0 & 1_g\end{pmatrix}$ generate $Sp(2g;\boldsymbol{Z})$. We can check that this representation coincides with our representation by calculaing about generators of $\mathfrak{M}_g$. In [8], Gocho also defines a topological invariant of M by

$$W_N(M) = \sqrt{N}^{g-1}\langle\rho_g(f_*)e_0, e_0^*\rangle\in\boldsymbol{C}/U(1)\,,$$

where M is presented by a Heegaard splitting $M=H_g\cup_f(-H_g)$ with H_g a handlebody of genus g. Noting that $W_N(S^3)=\sqrt{N}^{-1}$, $Z_N(H_g;q)=\sqrt{N}^{g/2}\,e_0$, and $Z_N(-H_g;q)=\sqrt{N}^{g/2}\,e_0^*$, we immediately have the next proposition.

Proposition 8.2. *Let N be even. Then we have*

$$\frac{W_N(M)}{W_N(S^3)} = |Z_N(M;\exp\frac{\pi\sqrt{-1}}{N})|\,,$$

where $W_N(M)$ is Gocho's invariant defined above.

9. Invariants of Dijkgraaf and Witten for $G=\boldsymbol{Z}/N\boldsymbol{Z}$.

In this section we will show relations between our invariants and invariants of R. Dijkgraaf and E. Witten.

Let G be $\boldsymbol{Z}/N\boldsymbol{Z}$. We choose a class $q\in H^3(BG, U(1))$. Since $H^3(BG, U(1)) \cong \boldsymbol{Z}/N\boldsymbol{Z}$ (see for example [11, Lemma 9.2]) for a classifying space BG for G, we regard q as a (not necessarily primitive) N-th root of unity with an inclusion $\boldsymbol{Z}/N\boldsymbol{Z}\to U(1)$. Let M be a closed orientable 3-manifold. In [5], Dijkgraaf and Witten defined invariants as the sum over all possible G bundles over M:

$$D_N(M; q) = \sum_{\gamma\in \mathrm{Hom}(\pi_1(M), G)} \langle f_\gamma^* q, [M]\rangle \in \boldsymbol{C}\,,$$

where $f_\gamma\colon M\to BG$ is a classifying map corresponding to γ and $\langle f_\gamma^* q, [M]\rangle\in U(1)$. We regard $U(1)$ as the set of units in $\boldsymbol{C}$ and the sum is taken in $\boldsymbol{C}$.

Proposition 9.1. *Let N be a positive integer, K a divisor of N, and q an N^2-th (primitive) root of unity. Then the following formulas hold.*

$$\text{For } N \text{ odd}\quad D_N(M; q^{NK}) = Z_{N^2/K}(M; q^K) Z_K(M; q^{-N^2/K})\,.$$
$$\text{For } N \text{ even}\quad D_N(M; q^{NK}) = Z_{N^2/2K}(M; q^K) Z_{2K}(M; q^{-N^2/4K})\,.$$

Before we prove this proposition, we show some lemmas. Since $\mathrm{Hom}(\pi_1(M), G) = \mathrm{Hom}(H_1(M;\boldsymbol{Z}), \boldsymbol{Z}/N\boldsymbol{Z}) = H^1(M; \boldsymbol{Z}/N\boldsymbol{Z})$, we denote by $\bar\gamma$ the corresponding element to γ in $H^1(M; \boldsymbol{Z}/N\boldsymbol{Z})$.

Lemma 9.2.

$$\langle f_\gamma^* q, [M]\rangle = q^{\langle \bar\gamma \cup \delta^*(\bar\gamma), [M]\rangle}$$

where $\delta^*\colon H^1(M; \boldsymbol{Z}/N\boldsymbol{Z})\to H^2(M;\boldsymbol{Z})$ is the connecting homomorphism with respect to an exact sequence $0\to\boldsymbol{Z}\overset{N}{\to}\boldsymbol{Z}\to\boldsymbol{Z}/N\boldsymbol{Z}\to 0$ and $\cup\colon H^1(M;\boldsymbol{Z}/N\boldsymbol{Z})\times H^2(M;\boldsymbol{Z}) \to H^3(M;\boldsymbol{Z}/N\boldsymbol{Z})$.

Proof. Let $\gamma'\in\mathrm{Hom}(\pi_1(BG), G) = \mathrm{Hom}(G, G)$ be the identity map which is the monodromy representation of a classifying space $EG\to BG$. We denote by $\bar\gamma'$ a corresponding element to γ' in $H^1(BG, G)$. Some calculations show that $\bar\gamma'\cup\delta^*(\bar\gamma')$ is a generator of $H^3(BG, G)\cong\boldsymbol{Z}/N\boldsymbol{Z}\cong H^3(BG, U(1))$, where $\delta^*\colon H^1(BG;\boldsymbol{Z}/N\boldsymbol{Z})\to H^2(BG;\boldsymbol{Z})$ is the connecting homomorphism. Let q be $\exp(m\cdot 2\pi\sqrt{-1}/N)\in H^3(BG, U(1))\subset U(1)$. Then $f_\gamma^* q = \exp(m(\bar\gamma\cup\delta^*(\bar\gamma))\cdot 2\pi\sqrt{-1}/N)\in H^3(M, U(1)) = U(1)$ because $\bar\gamma = f_\gamma^*\bar\gamma'$. Hence we have the required formula. ∎

The following lemma is obtained in a similar way as a proof of Lemma 3.4.

Lemma 9.3. *Let $l\in\ker L_A\subset(\boldsymbol{Z}/N\boldsymbol{Z})^n$ be the corresponding element to $\bar\gamma$ under the isomorphism ι in Lemma 3.3. Then we have*

$$\langle \bar{\gamma} \cup \delta^*(\bar{\gamma}), [M] \rangle = \frac{1}{N} {}^t\tilde{l} A \tilde{l} \in \boldsymbol{Z}/N\boldsymbol{Z},$$

where $\tilde{l} \in \boldsymbol{Z}^n$ is a lift of l and A is the linking matrix of the framed link.

Proof of Proposition 9.1. By Lemmas 3.3, 9.2, and 9.3, we have

$$D_N(M; q^{NK}) = \sum_{l \in \ker L_A} q^{K\, {}^t\tilde{l} A \tilde{l}}$$

with $L_A: (\boldsymbol{Z}/N\boldsymbol{Z})^n \to (\boldsymbol{Z}/N\boldsymbol{Z})^n$, $l \mapsto Al$.

For N odd, we have

$$\begin{aligned}
&Z_{N^2/K}(M; q^K) Z_K(M; q^{-N^2/K}) \\
&= \left(\frac{\Gamma}{|\Gamma|}\right)^{-\sigma} |\Gamma|^{-n} \sum_{l_1 \in (\boldsymbol{Z}/N^2K^{-1}\boldsymbol{Z})^n} q^{K\, {}^t l_1 A l_1} \sum_{l_2 \in (\boldsymbol{Z}/K\boldsymbol{Z})^n} q^{-N^2K^{-1}\, {}^t l_2 A l_2} \\
&= \left(\frac{\Gamma}{|\Gamma|}\right)^{-\sigma} |\Gamma|^{-n} \sum_{l_2} \sum_{l_1'} q^{K\, {}^t l_1' A l_1' + 2N\, {}^t l_1' A l_2} \qquad (l_1 = l_1' + NK^{-1} l_2) \\
&= \left(\frac{\Gamma}{|\Gamma|}\right)^{-\sigma} |\Gamma|^{-n} \sum_{h} q^{K\, {}^t h A h} \sum_{l_2, l_3} q^{2N\, {}^t(K l_3 + l_2) A h} \qquad (l_1' = h + N l_3) \\
&= \left(\frac{\Gamma}{|\Gamma|}\right)^{-\sigma} |\Gamma|^{-n} N^n \sum_{h \in \ker L_A} q^{K\, {}^t h A h},
\end{aligned}$$

where $\Gamma = G_{N^2/K}(q^K) G_K(q^{-N^2/K})$. Similar calculations show $\Gamma = N$. Hence we obtain the required formula.

For N even the required formula is obtained in a similar way. ■

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