In this paper we assume that every ring $R$ is an associative ring with identity and $R$ is two-sided artinian. The author has defined almost projective modules and almost injective modules in [8], and by making use of the concept of almost projectives he has defined almost hereditary rings in [7], whose class contains that of hereditary rings and serial rings. Similarly to [7] we shall define an almost QF ring, which is a generalization of QF rings.

It is well known that an artinian ring $R$ is QF if and only if $R$ is self-injective. Following this fact, if $R$ is almost injective as a right $R$-module, we call $R$ a right almost QF ring. Analogously we call $R$ a right almost QF* ring if every injective is right almost projective. On the other hand, the author studied rings with $(\ast)$ (resp. $(\ast)^*$) (see §1 for definitions) in [4]. K. Oshiro called such a ring a right H- (resp. co-H) ring in [10]. In this note we shall show that a right almost QF (resp. almost QF*) ring coincides with a right co-H (resp. H-) ring. In the final section we shall give a characterization of serial rings in terms of almost projectives and almost injectives.

In the forthcoming paper [9] we shall study certain conditions under which right almost QF rings are QF or serial.

1. Almost QF rings

In this paper we always assume that $R$ is a two-sided artinian ring with identity and that every module is a unitary right $R$-module. We use the same notations in [7]. We have studied almost hereditary rings in [7], i.e. $J$, the Jacobson radical of $R$, is right almost projective. We shall study, in this paper, some kind of the dual concept to almost hereditary rings (see Theorem 1 below). We call $R$ a right almost QF ring if $R$ is right almost injective as a right $R$-module [8]. We can define similarly a left almost QF ring. It is clear that $R$ is right almost QF if and only if every finitely generated projective $R$-module is right almost injective. Hence the concept of almost QF rings is preserved under Morita equivalence and we may assume that $R$ is basic.

On the other hand, the author studied the two conditions in [3] and [4]. Let $M$ be an $R$-module. If $MSoc^1(R) \neq 0$ (resp. $M Soc^1(R) \neq 0$) then we call $M$ non-
small (resp. non-cosmall) [4], where \( \text{Soc}'(R) \) (resp. \( \text{Soc}''(R) \)) is the left (resp. right) socle of \( R \).

\( (*) \quad \) Every non-small module contains a non-zero injective module.

\( (*)' \quad \) Every non-cosmall module contains a projective direct summand.

K. Oshiro called rings with \( (*) \) (resp. \( (*)' \)) right H (resp. right co-H) rings in [10]. Relating with those two concepts we have

**Theorem 1.** Let \( R \) be an artinian ring. Then the following are equivalent:
1) \( R \) is right almost QF.
2) The Jacobson radical \( J \) of \( R \) is almost injective as a right \( R \)-module.
3) \( R \) is a right co-H ring.

**Proof.** 1) \( \leftrightarrow \) 2). This is clear from [8], Corollary 1* and Proposition 3.
1) \( \leftrightarrow \) 3). We know, from the above implication, [8], Proposition 3, [4], Theorem 3.6 and [11], Theorem 4.1, that the structure of right almost QF rings coincides with that of right co-H rings.

**Corollary.** \( R \) is right almost QF if and only if \( R \) is right QF-2, QF-3 and every submodule containing a projective submodule of \( eR \) is local for any primitive idempotent \( e \).

**Proof.** If \( R \) is right almost QF, \( R \) satisfies the conditions in the corollary by [8], Corollary 1* and Proposition 3. Conversely assume that \( R \) satisfies the conditions. Let \( A \) be a submodule of \( eR \) such that \( eR \supset A \supset fR \), where \( e, f \) are primitive idempotents. Then \( A \approx gR/B \) by assumption, where \( g \) is a primitive idempotent. Hence we obtain a natural epimorphism \( \theta: gR \to A \). Put \( K = \theta^{-1}(fR) \). Since \( fR \) is projective, \( K = B \oplus K' \). Further \( gR \) is uniform, and hence \( B = 0 \). Therefore \( A \) is projective, and \( R \) is right almost by Theorem 1 and [8], Proposition 3.

2. Almost QF* rings

In this section we shall study the dual concept to almost QF. If every indecomposable injective module is almost projective, we call \( R \) a right almost QF* ring. If every indecomposable injective module is local, we call \( R \) right QF-2*. If a projective cover of every (indecomposable) injective module is injective, we call \( R \) right QF-3*.

As the dual to Theorem 1, the following theorem is clear from [1], Theorem 2, [8], Theorem 1 and [10], Theorem 3.18.

**Theorem 2.** Let \( R \) be artinian. Then the following are equivalent:
1) \( R \) is right almost QF*.
2) \( R \) is a right H-ring.

K. Oshiro [11] showed that \( R \) is right almost QF* if and only if \( R \) is left al-
most QF.

The following is dual to Corollary to Theorem 1.

**Proposition 1.** Let R be an artinian ring. Then R is right almost QF if and only if 1): R is right QF-2*, 2): R is right QF-3* and 3): if eR/A is injective for A≠0, then eR/B is uniform for any B⊂A. 1) together with 2) is equivalent to 4): every indecomposable injective is a factor module of some local projective and injective module, where e is a primitive idempotent.

**Proof.** If R is right almost QF, we obtain the conditions 1), 2) and 3) by [8], Corollary 1*. Conversely we assume 1), 2) and 3). Let eR/A be injective and B⊂A. Then eR/B is uniform by 3). Take a diagram

\[
\begin{array}{c}
0 \to eR/B \to E(eR/B) \\
\downarrow \quad \quad \quad \downarrow \\
eR/A
\end{array}
\]

where i is the inclusion and ν is the natural epimorphism. Since eR/A is injective, we have μ with \(μi=ν\). ν being an epimorphism, \(E(eR/B)=eR/B+μ^{-1}(0)\). Further \(E(eR/B)\) is local by 1) and 3), and hence \(E(eR/B)=eR/B\). Therefore eR/A is almost projective by [8], Theorem 1*. Hence R is right almost QF.

If R is hereditary and QF, then R is semisimple. Concerning with this fact, we have

**Proposition 2.** Let R be an artinian ring. Then the following are equivalent:

1) R is serial.
2) R is right almost QF and right co-serial. (cf. [11], Theorem 6.1.)
3) R is right almost QF and right almost hereditary.
4) R is right almost QF and right almost hereditary.
5) R is left almost QF and right almost hereditary.

**Proof.** 1) → 2), 3), 4) and 5). This is clear from [7] and [10].
2) → 1). Let f be a primitive idempotent and \(E=E(fR)\). Then we may assume \(E=e_1R⊕⋯⊕e_jR⊕g_1R/A_1⊕⋯⊕g_qR/A_q\), where the \(e_jR\) and the \(g_jR\) are injective and \(A_j≠0\) for all \(j\) by Proposition 1. Let \(θ\) be the natural epimorphism of \((Σ_{i=1}^j e_iR⊕Σ_{j=1}^q g_jR)\) onto E with \(θ^{-1}(0)=A_1⊕⋯⊕A_q\). Then since \(fR\) is projective, \(θ^{-1}(fR)=P⊕θ^{-1}(0); P奐fR\). Set \(E_1=Σ⊕e_1R\) and \(E_2=Σ⊕g_1R\) and \(π_i: E→E_i\) the projection. Since \(θ^{-1}(0)\) is essential in \(E_2\), \(P∩E_2=0\). Hence \(fR≅P≅π_1(P)⊂E_1\). Next we shall show that every submodule of \(E_1\) is standard. Take submodules \(K_i⊂L_i⊂e_iR\) for \(i=1, 2\) such that...
\( \mu: L_1/K_1 \cong L_2/K_2 \). We may assume \( |e_1 R/K_1| \leq |e_2 R/K_2| \). Since \( e_1 R \) is uniserial, we can suppose that \( e_1 R/K_1 \) and \( e_2 R/K_2 \) are contained in \( F = E(L_1/K_1) \), which is also uniserial. We can extend \( \mu \) to an automorphism \( \mu^* \) of \( F \). Since \( |e_1 R/K_1| \leq |e_2 R/K_2| \), \( \mu^*(e_1 R/K_1) \subseteq e_2 R/K_2 \). On the other hand, \( e_1 R \) being projective, \( \mu^* \) is liftable to a homomorphism of \( e_1 R \) into \( e_2 R \). Hence \( fR \) is a standard submodule of \( E_1 \) by [6], Lemma 5. Accordingly \( fR \) is some \( e_1 R \) (isomorphically) for \( fR \) is local. Therefore \( fR \) is uniserial for any \( f \), and hence \( R \) is serial by [2], Theorem 5.4.

3) \( \rightarrow \) 1). If \( R \) is hereditary, \( R \) is a serial ring in the first category by Corollary to Theorem 1 and [7], Corollary 3. Assume that \( R \) is of the form (9) in [7], Theorem 2. Now we follow [7] for the notations. Then \( h_\ast R \) is not injective provided \( R \) is not serial by [7], Corollary 3. However \( h_\ast R \) must be contained in an injective and projective \( fR \) by Corollary to Theorem 1, which is impossible from the construction of \( R \) in [7].

4) \( \rightarrow \) 1). Let \( R \) be right almost hereditary and right almost QF*. We use the same notations as in [7]. If \( R \) is not serial, \( T_1 \neq 0 \) in [7], the figure (9). Then \( E(h_\ast R/h_\ast I) \) is a factor module of an injective and projective \( fR \) by Proposition 1, which is impossible by the structure of \( R \). Hence \( R \) is serial.

4) \( \rightarrow \) 5). This is clear from the remark after Theorem 2 [11].

3. Serial rings

We have studied generalizations of QF rings in §§1 and 2. We shall consider, in this section, the remaining generalizations following previous sections.

**Theorem 3.** Let \( R \) be artinian. Then the following are equivalent:

1) Every almost projective is almost injective.
2) Every almost injective is almost projective.
3) \( R \) is serial.

Proof. 3) \( \rightarrow \) 1) and 2). This is clear from [7], Figure (2) and [8], Corollary 1 and Corollary 1*.

1) or 2) \( \rightarrow \) 3). Let \( \{g,R\} \) be the set of indecomposable, projective and injective modules, which is not empty by 1) or 2) and [8], and rename \( \{g,R\} = \{e_1 R, \ldots, e_q R, f_1 R, \ldots, f_q R\} \), where the \( e_i R \) are uniserial and the \( f_j R \) are not. Assume \( q > 0 \), i.e. \( R \) is not serial. We shall find the set of non-projective, non-injective, non-uniserial and indecomposable almost projectives. Since \( f_j R \) is not uniserial, there exists an integer \( k_j \) such that \( f_j R/Soc_k (f_j R) \) is injective for all \( 0 \leq r < k_j \) and \( f_j R/Soc_{k_j} (f_j R) \) is not injective. Then \( f_j R/Soc_{k_j} (f_j R) \) is non-projective, non-injective, non-uniserial and almost projective module by [8], Corollary 1. Further \( f_j R/Soc_{k_j} (f_j R) \not\cong f_i R/Soc_{k_i} (f_i R) \) for \( i \neq j \). Therefore since \( e_j R \) is uniserial, we obtain just \( q \) non-projective, non-injective, non-uniserial almost projective modules.
Almost QF Rings

\{ f_j R / \text{Soc}_k (f_j R) \}_{j \leq q}

by [8], Theorem 1. On the other hand, if \( f_i J \) is projective, \( f_i J \approx g R \) for some primitive idempotent \( g \) (note that \( f_i R \) is uniform). Continuing this argument, we may suppose

\[ f_i R \supseteq f_i J \supseteq \cdots \supseteq f_i J^{i-1} (i \leq q), f_i J^{i-1} \approx f_i h R \text{ and } f_i r_i J \text{ is not projective, where} \]

the \( f_i s \) are primitive idempotents.

Hence since \( f_i R \) is not uniserial, non-projective, non-injective, non-uniserial almost injective is of \( f_i J^s (= f_{sr_i} J) \) for \( s \leq q \) by [8], Theorem 1'. Further \( f_i J^s \approx f_{sr_i} J^t \) for \( s \neq t \), since \( f_i R \) and \( f_i J \) are indecomposable and injective. Hence we obtain just \( q \) non-projective, non-injective, non-uniserial and almost injective modules

\[ \{ f_j J^t \}_{j \leq q}. \]

From 1) we shall show that \( f_{jr} J \) is local. By 1) \( f_j R / \text{Soc}_k (f_j R) \) is almost injective. Hence \( \{ f_j R / \text{Soc}_k (f_j R) \}_{j \leq q} = \{ f_j J^t \}_{j \leq q} \) up to isomorphism. Therefore the \( f_j J^t = f_{sr} J \) are local. On the other hand, since every indecomposable projective \( g R \) is almost injective by assumption, \( g R \approx e_i J' \) for some \( t \) or \( g R \approx f_j J^{r_t} \) for some \( t \leq r_i - 1 \) by [8], Corollary 1'. Hence \( g J \) is local from the above. Therefore \( R \) is right serial by [5], Proposition 1, and \( R \) is serial by [10], Theorem 6.1.

Assume 2). Then \( \{ f_j R / \text{Soc}_k (f_j R) \}_{j \leq q} = \{ f_j J^t \}_{j \leq q} \) as above. Hence the \( f_j R / \text{Soc}_k (f_j R) \) are uniform and \( R \) is co-serial by [8], Corollary 1 and [5], Proposition 1'. Therefore \( R \) is serial by Proposition 2.

**Proposition 3.** Let \( R \) be artinian. Then the following are equivalent:

1) Every almost projective is injective.
2) Every almost injective is projective.
3) \( R \) is semi-simple.

Proof. Assume 1). Then \( R \) is QF. Since \( e R \text{Soc}(e R) \) is almost projective by [8], Theorem 1, \( e R \text{Soc}(e R) \) is injective by 1). Hence \( e R \text{Soc}(e R) \) is projective. Therefore \( \text{Soc}(e R) = e R \). Remaining parts are also clear.

The final type is

**Proposition 4.** Let \( R \) be as above. Then the following are equivalent:

1) Every almost projective is projective.
2) Every almost injective is injective.
3) \( R \) is a direct sum of semi-simple rings and rings whose every projective is never injective.

Proof. 1) \( \rightarrow \) 3). Let \( R \) be basic and \( e R \) injective. Then \( e R \text{Soc}(e R) \) is almost projective by [8], Theorem 1, and hence \( e R \text{Soc}(e R) \) is projective by 1). Therefore \( e R \) is simple and \( R = e R \oplus (1 - e) R \) as rings. Thus we obtain 3).
The remaining implications are also clear.

References


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