ON SPECIAL VALUES AT $s=0$ OF PARTIAL ZETA-FUNCTIONS FOR REAL QUADRATIC FIELDS

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1. Introduction

1.1 Let $F$ be a totally real algebraic number field with finite degree, $\alpha$ a fractional ideal of $F$, and $F_{ab}$ the maximal abelian extension of $F$. We define a map $\xi_{\alpha}$ from the quotient space $F/\alpha$ to the group $W(F_{ab})$ of roots of unity of $F_{ab}$ using the deep results of Coates-Sinnott $[C-S1]$, $[C-S2]$ and Deligne-Ribet $[D-R]$ on special values of partial zeta functions of $F$. Under the action of the Galois group $Gal(F_{ab}/F)$ over $F$ this map behaves formally in a manner similar to Shimura’s reciprocity law for elliptic curves with complex multiplication. This reciprocity law for the map $\xi_{\alpha}$ is also a direct consequence of those results of Coates-Sinnott and Deligne-Ribet. On the other hand we have studied in $[Ar1]$ a certain Dirichlet series and its relationship with parital zeta functions of real quadratic fields. In particular the special values at $s=0$ of partial zeta functions of real quadratic fields essentially coincide with the residues at the pole $s=0$ of our Dirichlet series. Using those residues, we give another expression for the map $\xi_{\alpha}$ in the case of $F$ a real quadratic field. We also show that the expression works in a reasonable manner under the action of the Galois group $Gal(F_{ab}/F)$.

1.2 We summarize our results. For an integral ideal $c$ of a totally real algebraic number field $F$, denote by $H_F(c)$ the narrow ray class group modulo $c$. For each integral ideal $b$ prime to $c$, we define the partial zeta-function $\zeta_c(b, s)$ to be the sum $\sum_{\alpha}(N\alpha)^{-s}, \alpha$ running over all integral ideals of the class of $b$ in $H_F(c)$. Let $\alpha$ be a fractional ideal of $F$. For each class $\bar{z}$ of the quotient space $F/\alpha$, we take a totally positive representative element $z \in F$ of the class $\bar{z}$, and write

$$z\alpha^{-1} = \overline{f}^{-1}b$$

with coprime integral ideals $\overline{f}, b$ of $F$. Thanks to some results of Coates-Sinnott ([C-S1], [C-S2], [Co]) and Deligne-Ribet ([D-R]), one can define a map $\xi_{\alpha} : F/\alpha \to W(F_{ab})$ as follows;

$$\xi_{\alpha}(\bar{z}) = \exp(2\pi i \zeta_{\overline{f}}(b, 0)),$$
where the value on the right hand side of the equality depends on the class \( \tilde{z} \) and not on a representative element \( z \) of \( \tilde{z} \). Denote by \( F^* \) the idele group of \( F \) and by \( F^*_A \) the subgroup of \( F^* \) consisting of ideles \( x \) whose archimedean components \( x_\infty \) are totally positive. Each element \( s \) of \( F^*_A \) induces a natural isomorphism \( s: F/a \cong F/sa \). We denote by \([s,F]\) the canonical Galois automorphism of the extension \( F_{ab}/F \) induced by \( s \in F^*_A \). The following theorem is a reformulation of a part of the results due to Coates-Sinnott and Deligne-Ribet ([C-S1], [C-S2], [D-R]).

**Theorem A** (Coates-Sinnott, Deligne-Ribet)

Let \( s \in F^*_A \) and set \( \sigma = [s,F] \). Then the following diagram is commutative.

\[
\begin{array}{ccc}
F/a & \xrightarrow{\xi_{s}a} & W(F_{ab}) \\
\downarrow{\cong} & & \downarrow{\sigma} \\
F/s^{-1}a & \xrightarrow{\xi_{s}^{-1}a} & W(F_{ab})
\end{array}
\]

Namely,

\[
\xi_{s}a(\bar{z})^\sigma = \xi_{s}^{-1}a(s^{-1}z),
\]

where \( s^{-1}z \) stands for the image of \( z \) by the isomorphism \( s^{-1}: F/a \cong F/s^{-1}a \).

In particular if we write, with \( \bar{z} \) being specialized at \( \bar{0} = 0 \mod a \),

\[
\xi(a) = \xi_{a}(\bar{0}),
\]

then, \( \xi(a) \) is a root of unity contained in the narrow Hilbert class field of \( F \). In this case the Galois action is described in the simple manner:

\[
\xi(a)^{[s,F]} = \xi(s^{-1}a) \quad \text{for any} \ s \in F^*_A.
\]

Theorem A will be interpreted as a formal analogy to Shimura's reciprocity law for elliptic curves with complex multiplication (see Theorem 5.4 of [Shm]).

For a real number \( x \), we denote by \( \langle x \rangle \) the real number satisfying \( x - \langle x \rangle \in \mathbb{Z} \) and \( 0 < \langle x \rangle \leq 1 \). Let \( F \) be a real quadratic field embedded in \( \mathbb{R} \). We set, for \( \alpha \in F - \mathbb{Q} \) and \((p, q) \in \mathbb{Q}^2\),

\[
\eta(\alpha, s, p, q) = \sum_{n=1}^{\infty} \frac{\exp(2\pi in(p\alpha + q))}{1 - \exp(2\pi i n \alpha)}
\]

and

\[
H(\alpha, s, (p, q)) = \eta(\alpha, s, \langle p \rangle, q) + e^{siz} \eta(\alpha, s, \langle -p \rangle, -q).
\]

This type of infinite series has been intensively studied by Berndt [Be1], [Be2],
if \( \alpha \) is a complex number with positive imaginary part. In our case we have proved in [Ar1] that the infinite series \( \eta(\alpha, s, p, q) \) is absolutely convergent for \( \text{Re}(s)<0 \) and moreover that \( H(\alpha, s, (p, q)) \) can be analytically continued to a meromorphic function of \( s \) in the whole \( s \)-plane which has a possible simple pole at \( s=0 \). Let \( h_{-1}(\alpha, (p, q)) \) denote the residue at the pole \( s=0 \) of this function \( H(\alpha, s, (p, q)) \) (see §3 of this paper). We set

\[
h(\alpha, (p, q)) = \frac{1}{2} (h_{-1}(\alpha, (p, q)) - h_{-1}(\alpha', (p, q))) ,
\]

where \( \alpha' \) denotes the conjugate of \( \alpha \) in \( F \). This quantity \( h(\alpha, (p, q)) \) satisfies the transformation law under the action of \( SL_2(\mathbb{Z}) \):

\[(1.5) \quad h(V\alpha, (p, q)) = h(\alpha, (p, q)V) \quad \text{for any} \quad V \in SL_2(\mathbb{Z}) .
\]

We denote by \( F^\times \) the group of of invertible elements of \( F \). Let \( \alpha \) be a fractional ideal of \( F \) with an oriented basis \( \{\alpha_1, \alpha_2\} \) (i.e., \( \alpha = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2, \alpha_1\alpha_2 - \alpha_2^2 > 0 \)). Denote by \( q: F^\times \to GL_2(\mathbb{Q}) \) the injective homomorphism of \( F^\times \) into \( GL_2(\mathbb{Q}) \) defined via the basis \( \{\alpha_1, \alpha_2\} \) as follows;

\[(1.6) \quad \mu(\frac{\alpha_1}{\alpha_2}) = q(\mu)(\frac{\alpha_1}{\alpha_2}) \quad (\mu \in F^\times) .
\]

This homomorphism \( q \) is naturally extended to that of \( \mathfrak{F}_A^\times \) into the adele group \( G_A=GL_2(\mathbb{Q}_A) \). Denote by \( G_{A,+} \) the subgroup of \( G_A \) consisting of all elements \( x \in G_A \) whose archimedean components \( x_\infty \) have positive determinants. By the transformation law (1.5) of \( h(\alpha, (p, q)) \), one can define an action of any \( x \in G_{A,+} \) on the coefficient \( h(\alpha, (p, q)) \). This action will be denoted by \( h^x(\alpha, (p, q)) \) (for the precise definition see (3.12)). For an integral ideal \( \mathfrak{f} \) of \( F \), we denote by \( E_\infty(\mathfrak{f}) \) the group of totally positive units \( u \) of \( F \) with \( u-1 \in \mathfrak{f} \). Another expression for the map \( \xi_\alpha(z) \) is given by the following theorem.

**Theorem B** Let the notation be the same as above. Let \( \alpha \) be a fractional ideal of a real quadratic field \( F \) with the oriented basis \( \{\alpha_1, \alpha_2\} \). Choose a representative element \( z \in F, z \neq 0 \) of a class \( \bar{z} \in F/\alpha \) and determine the ideal \( \mathfrak{f} \) by (1.1). Denote by \( \eta \) the generator of the group \( E_\infty(\mathfrak{f}) \) with \( \eta > 1 \). Write \( z = p\alpha_1 + q\alpha_2 \) with \( (p, q) \in \mathbb{Q}^2 \) and set \( \alpha = \alpha_1/\alpha_2 \). Then,

\[(1.7) \quad \xi_\alpha(\bar{z}) = \exp(\log \eta \cdot h(\alpha, (p, q))) .
\]

Let \( s \in F_{A,+}^\times \). The Galois action on \( \xi_\alpha(\bar{z}) \) is given by the equality

\[(1.8) \quad \xi_\alpha(\bar{z})^{s, \mathfrak{f}} = \exp(\log \eta \cdot h^{s, \mathfrak{f}}(\alpha, (p, q))) .
\]

In Theorem 3.3 we obtain a stronger result than (1.7); namely, the special value \( \zeta(\mathfrak{f}, 0) \) is explicitly given by the value \( h(\alpha, (p, q)) \). We note that, as
is essentially known, the value $\xi(\alpha) = \xi_{\alpha}(0)$ is a twelfth root of unity in the narrow Hilbert class field of $F$ (see the end of §3).

2. Partial zeta-functions for totally real number fields

We recall a part of the results of [C-S1, 2], [Co], and [D-R] concerning special values at non-positive integers of partial zeta-functions for totally real algebraic number fields.

Let $\mu_m$ denote the group of $m$-th roots of unity. Let $L$ be an algebraic number field. If $K$ is a Galois extension of $L$, we write $\text{Gal}(K/L)$ for the Galois group of $K$ over $L$. For a positive integer $n$, we define $w_n(L)$ to be the largest integer $m$ such that the exponent of the group $\text{Gal}(L(\mu_m)/L)$ divides $n$ (see 2.2 of [Co]). In particular if $n=1$, $w_1(L)$ is nothing but the number of roots of unity of $L$. We denote by $W(L)$ the group of roots of unity of $L$.

Let $F$ be a totally real algebraic number field with finite degree throughout this paragraph. For an integral ideal $\mathfrak{f}$ of $F$, denote by $H_F(\mathfrak{f})$ the narrow ray class group modulo $\mathfrak{f}$. Namely, $H_F(\mathfrak{f})$ is the quotient group $I_\mathfrak{f}(\mathfrak{f})/P_\mathfrak{f}(\mathfrak{f})$, where $I_\mathfrak{f}(\mathfrak{f})$ is the group of fractional ideals of $F$ prime to $\mathfrak{f}$ and $P_\mathfrak{f}(\mathfrak{f})$ is the group of principal ideals of $F$ generated by totally positive elements $\theta$ of $F$ such that the numerators of $\theta - 1$ are divisible by $\mathfrak{f}$. We set, for each class $C$ of $I_\mathfrak{f}(\mathfrak{f})$,

$$\xi_\mathfrak{f}(C, s) = \sum_\alpha (Na)^{-s} \quad (\text{Re}(s)>1),$$

where $\alpha$ runs over all integral ideals of $C$ and $Na$ denotes the norm of $\alpha$. The partial zetafunction $\xi_\mathfrak{f}(C, s)$ is analytically continued to a meromorphic function in the whole $s$-plane which is holomorphic at non-positive integers. If $b$ is a representative ideal of $C$, we often write $\xi_\mathfrak{f}(b, s)$ in place of $\xi_\mathfrak{f}(C, s)$. Let $K=K_\mathfrak{f}(\mathfrak{f})$ be the maximal narrow ray class field of $F$ defined modulo $\mathfrak{f}$. We write $[C, K/F]$ for the Artin symbol of the class $C$ of $H_\mathfrak{f}(\mathfrak{f})$. By the class field theory there exists a canonical isomorphisms of $H_\mathfrak{f}(\mathfrak{f})$ to the Galois group $\text{Gal}(K/F)$ given by the correspondence: $C \rightarrow [C, K/F]$. If $b$ is a representative ideal of the class $C$, we write $[b, K/F]$ for $[C, K/F]$. The following theorem is due to Coates-Sinnott [C-S1, 2] in the case of real quadratic fields and to Deligne-Ribet [D-R] in general.

**Theorem 2.1.** (Coates-Sinnott, Deligne-Ribet) Let $\mathfrak{f}$ be an integral ideal of $F$ and $b, c$ integral ideals of $F$ which are prime to $\mathfrak{f}$. Set $K=K_\mathfrak{f}(\mathfrak{f})$. For each non-negative integer $n$,

(i) $w_{n+1}(K)\xi_\mathfrak{f}(b, -n)$ is an integer.

(ii) Moreover if $c$ is prime to $w_{n+1}(K)$, then the value

$$(Na)^{n+1}\xi_\mathfrak{f}(b, -n) - \xi_\mathfrak{f}(bc, -n)$$

is also an integer.
In the case of $n=0$, we reformulate the above theorem into a slightly different form suitable to our later situation. For that purpose we recall briefly the class field theory in the adelic language (see [C-F]).

Denote by $F^+_x$ the group of totally positive elements of $F$. Let $F^+_A$ denote the idele group of $F$, $F^+_x$ the archimedean part of $F^+_A$, and $F^+_x,+$ the connected component of the identity of $F^+_x$, respectively. We denote by $F^+_A,+$ the subgroup of $F^+_A$ consisting of elements $x \in F^+_A$ whose archimedean component $x_\infty$ are contained in $F^+_x,+$.

For each element $x$ of $F^+_A$ and for a finite prime $\mathfrak{p}$ of $F$, we denote by $x_\mathfrak{p}$ the $\mathfrak{p}$-component of $x$ and define a fractional ideal $il(x)$ of $F$ by putting $il(x)_{\mathfrak{p}} x_\mathfrak{p} \mathfrak{p}_\mathfrak{p}$ for all finite $\mathfrak{p}$, where $\mathfrak{p}_\mathfrak{p}$ is the maximal order of the completion $F_{\mathfrak{p}}$ of $F$ at $\mathfrak{p}$. Set

$$U = \{ x \in F^+_A | x_\mathfrak{p} \in \mathfrak{p}_\mathfrak{p} \} \quad \text{for all finite primes } \mathfrak{p} \text{ of } F \},$$

$\mathfrak{p}_\mathfrak{p}$ being the unit group of $\mathfrak{p}_\mathfrak{p}$. Set, for an integral ideal $\mathfrak{f}$,

$$W_+(\mathfrak{f}) = \{ x \in F^+_A | x_\infty \in F^+_x,+ \text{ and } x_{\mathfrak{p}} - 1 \in \mathfrak{p}_\mathfrak{p} \text{ for all } \mathfrak{p} \text{ dividing } \mathfrak{f} \},$$

$$U_+(\mathfrak{f}) = U \cap W_+(\mathfrak{f}).$$

By the class field theory there exists a canonical exact sequence

$$1 \longrightarrow \overline{F^+F^+_x, +} \longrightarrow F^+_x \longrightarrow Gal(F_{ab}/F) \longrightarrow 1,$$

where $\overline{F^+F^+_x, +}$ is the closure of $F^+F^+_x,+$ in $F^+_x$ and where we denote by $[s, F]$ the element of $Gal(F_{ab}/F)$ corresponding to an element $s$ of $F^+_A$. If we take an element $u$ of $W_+(\mathfrak{f})$, then the Galois automorphism $[u, F]$ coincides with the Artin symbol $[il(u)K_\mathfrak{f}(\mathfrak{f})/F]$ on the narrow ray class field $K_\mathfrak{f}(\mathfrak{f})$ over $F$.

Let $\alpha$ be a fractional ideal of $F$. To define the map $\xi_{\alpha}$ of the quotient space $F/\alpha$ to the group $W(F_{ab})$ by the equality (1.2), we have to prove that the right hand side of (1.2) depends only on the class $\bar{z} \in F/\alpha$ (not on the choice of a representative element $z$ of $\bar{z}$) and moreover that the image of $\xi_{\mathfrak{f}}$ is in $W(F_{ab})$. To see this we take another element $z_1$ of $F^+_x$ with the condition $z - z_1 \in \alpha$. Let $\mathfrak{f}, \mathfrak{b}$ be the same coprime integral ideals of $F$ as in (1.1). Then we have

$$z_1 \mathfrak{b}^{-1} \mathfrak{b}^{-1} \mathfrak{f},$$

with some integral ideal $\mathfrak{b}_1$ prime to $\mathfrak{f}$. We see easily that $\mathfrak{b}$ and $\mathfrak{b}_1$ are in the same class of $H_\mathfrak{f}(\mathfrak{f})$. Therefore,

$$\xi_{\mathfrak{f}}(\mathfrak{b}, 0) = \xi_{\mathfrak{f}}(\mathfrak{b}_1, 0)$$

By virtue of the assertion (i) of Theorem 2.1 the value

$$\exp(2\pi i \xi_{\mathfrak{f}}(\mathfrak{b}, 0))$$
is a root of unity of $K_F(f)$. Thus the map $\xi_\alpha$ given by (1.2) defines a map of $F/\alpha$ to $W(F_{st})$.

Any element $x$ of $F_A^+$ acts naturally on a fractional ideal $\alpha$ of $F$. The ideal $\alpha x$ of $F$ is characterized by the property $\alpha x = i(\alpha)\alpha$. For each element $u$ of $F$, there exists an element $v$ of $F$ such that

$$v - x_p u \in x_p \alpha_p$$

for all prime ideals $p$ of $F$,

where $\alpha_p = \alpha_{\alpha_p}$ in $F_p$. Thus we obtain a natural isomorphism of $F/\alpha$ to $F/\alpha x$ by the correspondence $u \mod \alpha \mapsto v \mod x$. We denote this isomorphism by $x: F/\alpha \to F/\alpha x$ and write $xu \mod x$ for the image of $u \mod \alpha$.

A part of the theorem of Coates-Sinnott and Deligne-Ribet (Theorem 2.1) can be formulated in terms of the adele language as in Theorem A in the introduction. For the completeness we give its proof here.

Proof of Theorem A.

We take a representative element $z \in F_A^+$ of a class $\bar{z} \in F/\alpha$ and write $z \alpha^{-1} = \bar{f}^{-1} b$ with coprime integral ideals $\bar{f}, b$ of $F$ as in (1.1). Set $K = K_F(f)$. For $s \in F_A^{+,+}$, we decompose $s = au$ with $a \in F_A^+, u \in W_+(f)$. Moreover we may choose $u$ so that $i(\alpha)$ is an integral ideal prime to $\omega_l(K)$. Set, for simplicity, $c = i(\alpha)$. Since by definition

$$\xi_\alpha(\bar{z}) = \exp(2\pi i \xi_f(b, 0)) \in W(K),$$

we have, for $\sigma = [s, F]$,

$$\xi_\alpha(\bar{z})^\sigma = \xi_\alpha(\bar{z})^{|\alpha, F|}$$

$$= \exp(2\pi i \xi_f(b, 0))^{[c, F]}$$

$$= \exp(2\pi i c \omega_f(b, 0)).$$

Therefore Theorem 2.1 implies that

\begin{equation}
(2.1) \quad \xi_\alpha(\bar{z})^\sigma = \exp(2\pi i \xi_f(c, 0)).
\end{equation}

On the other hand since $u \in W_+(f)$ and $u_p \in \mathcal{O}_p$ for all prime ideals $p$ of $F$, we see immediately that

$$1 - u_p \in (\mathcal{f}^{-1} b_{-1}) \mathcal{O}_p$$

for all prime ideals $p$ of $F$.

Thus for every prime ideal $p$ of $F$,

$$u_p^{-1} z - z \in \mathcal{f}^{-1} u_p^{-1} \mathcal{O}_p,$$

which turns out that

$$u^{-1} z \equiv z \mod u^{-1} \alpha.$$

Hence,
(2.2) \[ s^{-1}z \equiv a^{-1}z \mod s^{-1}a, \]
where we see that
(2.3) \[ a^{-1}z \in F_+^x \quad \text{and} \quad a^{-1}z(s^{-1}a)^{-1} = f^{-1}bc. \]
Therefore,
\[ \xi s^{-1}a(s^{-1}z \mod s^{-1}a) = \exp(2\pi i \xi f(bc, 0)), \]
which together with (2.1) completes the proof of Theorem A. \[ \blacksquare \]

3. Special values at \( s=0 \) of partial zeta-functions for real quadratic fields

First we recall some results of [Ar1]. For a real number \( x \), denote by \( \{x\} \) (res. \( \langle x\rangle \)) the real number satisfying
\[ 0 \leq \{x\} < 1, \quad x - \{x\} \in \mathbb{Z} \quad \text{(resp.} \quad 0 < \langle x\rangle \leq 1, \quad x - \langle x\rangle \in \mathbb{Z}). \]
We note here that \( \{x\} + \langle -x\rangle = 1 \). In this paragraph let \( F \) be a real quadratic field embedded in \( \mathbb{R} \) and fix it once and for all. For each \( \alpha \) of \( F \), let \( \alpha' \) denote the conjugate of \( \alpha \) in \( F \). For \( \alpha \in F - \mathbb{Q} \) and \( (p, q) \in \mathbb{Q}^2 \), we define a Lambert series \( \eta(\alpha, s, p, q) \) by the equality (1.3) in the introduction. The infinite series \( \eta(\alpha, s, p, q) \) is absolutely convergent for \( \Re(s)<0 \) (see Lemma 1 of [Ar1]). We also define the function \( H(\alpha, s, (p, q)) \) of \( s \) by the equality (1.4) in the introduction. We note that \( H(\alpha, s, (p, q)) \) depends on \( (p, q) \mod \mathbb{Z}^2 \). As we have seen in [Ar1], this function \( H(\alpha, s, (p, q)) \) can be analytically continued to a meromorphic function of \( s \) in the whole \( s \)-plane and has a Laurent expansion at \( s=0 \) of the form:
\[ H(\alpha, s, (p, q)) = \frac{h_{-1}(\alpha, (p, q))}{s} + h_d(\alpha, (p, q)) + \cdots. \]
Moreover the first coefficient \( h_{-1}(\alpha, (p, q)) \) satisfies under the action of \( SL_2(\mathbb{Z}) \) the following transformation law.

**Proposition 3.1.** Let \( \alpha \in F - \mathbb{Q} \) and \( (p, q) \in \mathbb{Q}^2 \). Then,
\[ h_{-1}(V\alpha, (p, q)) = h_{-1}(\alpha, (p, q)V) \quad \text{for any} \quad V = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}), \]
where we put \( V\alpha = \frac{a\alpha + b}{c\alpha + d} \).

**Proof.** For \( V = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \), set \( V^* = \left( \begin{array}{cc} a & -b \\ -c & d \end{array} \right) \) and \( (p^*, q^*) = (p, q)V \). If \( c>0 \) and \( c\alpha + d > 0 \), then the identity (3.1) is nothing but the first equality in
Proposition 4 of [Ar1]. Let $c<0$ and $ca+d>0$. In this case since $V^*(-\alpha) = -(V\alpha)$, we get, by Propositions 3, 4 of [Ar1],

\[
\begin{align*}
\delta(p, q) &= \begin{cases} 
1 & (p, q) \in \mathbb{Z}^2 \\
0 & \text{otherwise.}
\end{cases} \\
\end{align*}
\]

If $c=0$, $d=1$, then the assertion easily follows from the definition of $H(\alpha, s, (p, q))$. Finally let $ca+d<0$. Since $V\alpha=-(V)\alpha$, we have

\[
\begin{align*}
h_{-1}(V\alpha, (p, q)) &= h_{-1}(\alpha, (-p^*, q^*)) \\
&= h_{-1}(\alpha, (p^*, q^*))
\end{align*}
\]

With the help of Lemma 5 of [Ar1], the last term coincides with $h_{-1}(\alpha, (p^*, q^*))$. 

We set, for positive numbers $\omega, z$,

\[
G(z, \omega, t) = \frac{\exp(-zt)}{(1-\exp(-t))(1-\exp(-\omega t))} \quad (t \in \mathbb{C}),
\]

\[
\xi_2(s, \omega, z) = \sum_{m, n=0}^{\infty} (z+m+n\omega)^{-s} \quad (\text{Re}(s)>2).
\]

The Dirichlet series $\xi_2(s, \omega, z)$ is absolutely convergent for $\text{Re}(s)>2$. For a sufficiently small positive number $\varepsilon$, let $I_\varepsilon(\infty)$ be the integral path consisting of the oriented half line $(+\infty, \varepsilon)$, the counterclockwise circle of radius $\varepsilon$ around the origin, and the oriented half line $(\varepsilon, +\infty)$. Then as is well-known, the zeta-function $\xi_2(s, \omega, z)$ has the following expression by a contour integral:

\[
(3.2) \quad \xi_2(s, \omega, z) = \frac{1}{\Gamma(s)(e^{2\pi i s}-1)} \int_{I_\varepsilon(\infty)} t^{s-1} G(z, \omega, t) dt,
\]

where $\log t$ is understood to be real valued on the upper half line $(+\infty, \varepsilon)$. This expression (3.2) gives the analytic continuation of $\xi_2(s, \omega, z)$ to a meromorphic function over the whole $s$-plane which is holomorphic except at $s=1, 2$. We put, for $r \in \mathbb{R}$,

\[
\chi(r) = \begin{cases} 
1 & r \in \mathbb{Z} \\
0 & r \in \mathbb{R}-\mathbb{Z}.
\end{cases}
\]

For each $\alpha \in F-Q$ and a pair $(p, q) \in \mathbb{Q}^2$, we choose a totally positive unit $\eta$ of
F and an element \( V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of \( SL_2(\mathbb{Z}) \) which satisfy the following conditions

\[
(3.3) \quad c > 0, \quad (p, q) V \equiv (p, q) \mod \mathbb{Z}^2, \quad \eta(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

We have obtained in (3.2) of [Ar1] the following expression for \( h_{-1}(\alpha, (p, q)) \) using the data given in (3.3):

\[
(3.4) \quad h_{-1}(\alpha, (p, q)) = \frac{2\pi i}{\log \eta} \chi(p) \left( \frac{1}{2} - \langle q \rangle \right) - \frac{1}{\log \eta} L(\alpha, 0, (p, q), c, d),
\]

where \( L(\alpha, 0, (p, q), c, d) (s \in \mathbb{C}) \) is the special value at \( s = 0 \) of the function

\[
L(\alpha, s, (p, q), c, d) = -\sum_{j=1}^{c} \int_{t \in \mathbb{T}} t^{s-1} G \left( 1 - \left\{ \frac{jd + \rho}{c} \right\}, \eta, t \right) dt
\]

with \( \rho = \{q\}c - \{p\}d \). Since the above integral on the right hand side of the equality converges absolutely for any \( s \in \mathbb{C} \), this function \( L(\alpha, s, (p, q), c, d) \) of \( s \) is holomorphic in the whole complex plane.

**Proposition 3.2.** Let \( \alpha \in F - \mathbb{Q} \) and \( (p, q) \in \mathbb{Q}^2 \). Choose a totally positive unit \( \eta \) of \( F \) and \( V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of \( SL_2(\mathbb{Z}) \) as in (3.3). Then,

\[
h_{-1}(\alpha, (p, q)) \chi(p) \chi(q) = \frac{2\pi i}{\log \eta} \sum_{\eta \mod c} \zeta(0, \eta, x_k + y_k \eta),
\]

\[
h_{-1}(\alpha', (p, q)) \chi(p) \chi(q) = -\frac{2\pi i}{\log \eta} \sum_{\eta \mod c} \zeta(0, \eta', x_k + y_k \eta'),
\]

where we put, for each integer \( k \),

\[
x_k = 1 - \left\{ \frac{(k+p)d}{c} - q \right\} \quad \text{and} \quad y_k = \left\{ \frac{k+p}{c} \right\}.
\]

**Proof.** We know by Lemma 5 of [Ar1] that

\[
h_{-1}(\alpha, (-p, -q)) = h_{-1}(\alpha, (p, q)).
\]

It follows from the identities (3.2) and (3.4) that

\[
h_{-1}(\alpha, (-p, -q)) \chi(p) \chi(q) = \frac{2\pi i}{\log \eta} \chi(p) \left( \frac{1}{2} - \langle q \rangle \right) + \frac{2\pi i}{\log \eta} \sum_{j=1}^{c} \zeta(0, \eta, 1 - \left\{ \frac{jd + \rho^*}{c} \right\} + \left\{ \frac{j-(p)}{c} \right\} \eta).
\]
where \( \rho^* = \{-q\} c - \{-p\} d \). A slight modification of the summation in (3.6) yields

\[
(3.7) \quad \sum_{j=1}^n \zeta_2(0, \eta, 1 - \left\{ \frac{j}{c} \right\} \eta - \left\{ \frac{j - \{-p\}}{c} \right\} \eta) = \zeta_2(0, \eta, \eta) - \sum_{k \mod e} \zeta_2(0, \eta, \eta + y_k \eta)
\]

\[= \chi(p)(\zeta_2(0, \eta, 1 - \{-q\} + \eta) - \zeta_2(0, \eta, 1 - \{-q\})) \cdot \]

An easy computation with the use of the identity (3.2) shows that

\[\zeta_2(0, \eta, \eta + y \eta) = \frac{1}{2} B_2(x) \eta^{-1} + \frac{1}{2} B_2(y) \eta + B_1(x) B_1(y) \]

(see (1.10) of [Sht2]),

where \( x, y > 0 \) and \( B_k(x) \) is the \( k \)-th Bernoulli polynomial. Thus the right hand side of the equality (3.7) coincides with

\[\chi(p) \left( \frac{\langle q \rangle}{2} \right) \cdot \]

Therefore the identity (3.6) with the help of (3.7) turns out the first identity in Proposition 3.2. Another identity is similarly verified.

Let \( a = (a_1, a_2) \) be a pair of positive numbers and \( x = (x_1, x_2) \) a pair of non-negative numbers with \( x > (0, 0) \). Shintani [Sht2] defined the following zeta-function \( \zeta(s, a, x) \):

\[\zeta(s, a, x) = \sum_{m,n=0}^{\infty} \prod_{j=1}^2 (x_1 + m + (x_2 + n)a_j)^{-s} \cdot \]

which is absolutely convergent for \( \text{Re}(s) > 1 \). It has been proved that the zeta-function \( \zeta(s, a, x) \) is continued analytically to a meromorphic function of \( s \) in the whole complex plane which is holomorphic at \( s = 0 \) and moreover that

\[\zeta(0, a, x) = \frac{1}{2} (\zeta_2(0, a_1, x_1 + x_2a_1) + \zeta_2(0, a_2, x_1 + x_2a_2)) \]

(see [Sht1], (1.11) of [Sht2] and [Eg]).

Let \( \mathfrak{f} \) be an integral ideal of \( F \) and \( E_+(\mathfrak{f}) \) the group of totally positive unit \( u \) of \( F \) with \( u - 1 \in \mathfrak{f} \). We denote by \( \eta \) the generator of the group \( E_+(\mathfrak{f}) \) with \( \eta > 1 \). For each class \( C \) of \( H_2(\mathfrak{f}) \), take an integral ideal \( b \) of \( C \) and a basis \( \{\beta_1, \beta_2\} \) of the ideal \( \mathfrak{f}b^{-1} \) with the conditions \( \beta_1 \beta_2' - \beta_1' \beta_2 > 0 \), \( \beta_2 \beta_2' > 0 \). We represent the unit \( \eta \) via the basis \( \{\beta_1, \beta_2\} \) to get an element \( V \) of \( SL_2(\mathbb{Z}) \) such that

\[\eta(\beta_1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\beta_1) \cdot V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \]

A pair \( (p, q) \) of \( \mathbb{Q}^* \) is uniquely determined by the relation
\[
(3.9) \quad p\beta_1 + q\beta_2 = 1.
\]

Since \(\eta \in E_+(*1)\), we necessarily have

\[
(p, q)V \equiv (p, q) \mod Z^2.
\]

Set \(\beta = \beta_1/\beta_2\). Then, \(\beta, \eta, V\) and \((p, q)\) satisfy the conditions in (3.3) with \(\alpha\) being replaced by \(\beta\). We have proved in 4 of [Ar1] that the partial zeta-function \(\zeta_f(b, s)\) has the decomposition

\[
\zeta_f(b, s) = N(b^t)^{-s} \sum_{k \mod c} \sum_{m, n = 0}^{s} \xi(\beta, \eta, (x_k + y_k\eta + m + n\eta)^{-s})
\]

where \(x_k, y_k\) are given by (3.5) (see also p.409, §2 of [Sht1] and [Ar2]). Therefore it is immediate to see from (3.8) that the special value \(\zeta_f(b, 0)\) at \(s = 0\) is given by the identity

\[
(3.10) \quad \zeta_f(b, 0) = \frac{1}{2} \sum_{k \mod c} ((\xi_2(0, \eta, x_k + y_k\eta) + \xi_2(0, \eta', x_k + y_k\eta')).
\]

The following theorem is immediate from Proposition 3.2 and (3.10).

**Theorem 3.3.** Let \(b, t\) be coprime integral ideals of \(F\). Choose a basis \(\{\beta_1, \beta_2\}\) of the ideal \(b^{-1}\) with \(\beta_1, \beta_2 > 0\) and \(\beta_2 > 0\). Let \(\eta\) denote the generator of the group \(E_+(*)\) with \(\eta > 1\). Let \((p, q) \in Q^2\) be the same as in (3.9). Set \(\beta = \beta_1/\beta_2\). Then,

\[
\zeta_f(b, 0) = \frac{\log \eta}{4\pi i} (h^{-1}(\beta, (p, q)) - h^{-1}(\beta', (p, q))).
\]

Now we describe the map \(\xi: \mathbb{F}_a \rightarrow W(F_{ad})\) in terms of the coefficient \(h^{-1}(\alpha, (p, q))\). We set, for \(\alpha \in \mathbb{F} - \mathbb{Q}\) and \((p, q) \in Q^2\),

\[
\xi(\alpha, (p, q)) = \frac{1}{2} (h^{-1}(\alpha, (p, q)) - h^{-1}(\alpha', (p, q))).
\]

We denote by \(G\) the group \(GL_2\) defined over \(\mathbb{Q}\). Let \(G_A = GL_{2,A}\) be the adelized group of \(G\). For each \(x \in G_A\), denote by \(x_a\) the archimedean component of \(x\). Set

\[
G_{a, +} = GL_{2, +}(R) = \{x \in GL_2(R) | \det x > 0\},
\]

\[
G_{Q, +} = GL_{2, +}(Q) = \{x \in GL_2(Q) | \det x > 0\},
\]

and

\[
U = \prod_p GL_2(Z_p) \times G_{a, +},
\]
where $\mathbb{Z}_p$ is the ring of $p$-adic integers. We have the decomposition

$$G_{A,+} = G_{Q,+} U = U G_{Q,+}.$$  

Let $L$ be a $\mathbb{Z}$-lattice in $Q^2$. Set $L_p = L \otimes \mathbb{Z}_p$. For an element $x$ of $G_A$, we define $L x$ to be the $\mathbb{Z}$-lattice characterized by $(L x)_p = L_p x_p$ in $Q_p^2 = L \otimes Q_p$. Moreover any element $x$ of $G_A$ has a natural action on the quotient space $Q^2/L$ by the right multiplication and defines an isomorphism of $Q^2/L$ to $Q^2/L x$. We denote by $r x$ the image of an element $r \in Q^2/L$ by this isomorphism. For any $x \in G_{A,+}$, we write

$$x = u g \quad \text{with} \quad u \in U, g \in G_{Q,+}.$$  

We define the action of $x$ on $\mathfrak{h}(\alpha, (p, q))$ to be

$$\mathfrak{h}'(\alpha, (p, q)) = \mathfrak{h}(g \alpha, (p, q) u),$$  

where we note that the element $(p, q) u$ is uniquely determined as an element of $Q^2/\mathbb{Z}^2$. Since $G_{Q,+} \cap U = SL_2(\mathbb{Z})$, the right hand side of the equality (3.12) is independent of the decomposition $x = u g (u \in U, g \in G_{Q,+})$ according to (3.1).

Let $\alpha$ be a fractional ideal of $F$ with an oriented basis $\{a_1, a_2\}$ (namely, $\alpha = \mathbb{Z} a_1 + \mathbb{Z} a_2$, $a_1 a_2 - a_2 a_1 > 0$). Choose a representative element $z \equiv 0$ of the class $z \in F/\alpha$ and write

$$z \alpha^{-1} = f^{-1} b$$  

with coprime integral ideals $f$, $b$ of $F$. A pair $(p, q)$ of rational numbers is uniquely determined by

$$z = p a_1 + q a_2.$$  

Let $q: F^x \rightarrow GL_2(\mathbb{Q})$ be the homomorphism given by (1.6) in the introduction which is defined via the basis $\{\alpha_1, \alpha_2\}$ of $\alpha$. We also use the same symbol $q$ for the natural extension of $q$ to the homomorphism of $F^x_\alpha$ to $G_A$. Obviously, $q(F^x_\alpha) \subset G_{A,+}$.

A description of the map $\xi_\alpha : F/\alpha \rightarrow W(F_\alpha)$ in this case is formulated in Theorem $B$ in the introduction. Now under the above preparations we can give its proof.

**Proof of Theorem $B$.** Let the notation be the same as in the assertion of Theorem $B$. The expression on the right hand side of (1.7) is independent of the choice of an oriented basis $\{\alpha_1, \alpha_2\}$ of $\alpha$ in virtue of Proposition 3.1. Therefore we may assume that

$$\alpha_1 \alpha_2' - \alpha_1' \alpha_2 > 0, \quad \alpha_2 \alpha_2' > 0,$$  

if necessary, by change of a basis $\{\alpha_1, \alpha_2\}$ of $\alpha$. We choose an element $z_1$ of $F^x_\alpha$ such that $z - z_1 \in \alpha$ and set $z_1 = p_1 a_1 + q_1 a_2$ with a pair of rational numbers
We can write
\[ x_1 \alpha^{-1} = \mathfrak{f}^{-1} \mathfrak{b}_1 \]
with an integral ideal \( \mathfrak{b}_1 \) of \( \mathcal{F} \) prime to the same \( \mathfrak{f} \). Then,
\[ \mathfrak{f} \mathfrak{b}_1^{-1} = \pi \mathfrak{a} = \mathcal{Z}(\alpha_1/x_1) + \mathcal{Z}(\alpha_2/x_1), \]
\[ p_1(\alpha_1/x_1) + q_1(\alpha_2/x_1) = 1. \]
Noticing that \( x_1 \) is also a representative element of the class \( \bar{z} \), we get, by the definition (1.2) of the map \( \xi_\alpha \),
\[ \xi_\alpha(\bar{z}) = \exp(2\pi i \zeta_1(\mathfrak{b}_1, 0)) . \]
By virtue of Theorem 3.3 the special value \( \zeta_1(\mathfrak{b}_1, 0) \) has the expression
\[ \zeta_1(\mathfrak{b}_1, 0) = \frac{\log \eta}{2\pi i} b(\alpha, (p_1, q_1)) , \]
where we put \( \alpha = \alpha_1/\alpha_2 \). Since \((p_1, q_1) \equiv (p, q) \) mod \( \mathcal{Z}^2 \), we immediately have the identity (1.7).

Next let \( s \in F_{\mathcal{A},+} \) and write
\[ g(s)^{-1} = ug \] with \( u \in U, g \in G_{\mathcal{A},+} \).

We set
\[ \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = g \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} . \]

Obviously,
\[ \beta_1 \beta_2 - \beta_1 \beta_2 > 0 . \]

Then we see easily that
\[ s^{-1} \alpha = \mathcal{Z}^2 g(s)^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \mathcal{Z}^2 \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \]
\[ = \mathcal{Z} \beta_1 + \mathcal{Z} \beta_2 \]
and moreover that
\[ s^{-1} \mathfrak{x} \equiv (p, q) u \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \text{ mod } s^{-1} \alpha , \]
where \((p, q) u\) stands for an element of \( \mathcal{Q}'/\mathcal{Z}^2 \) and where \( s^{-1} \mathfrak{x} \) is not determined as an element of \( \mathcal{F} \) but uniquely determined modulo \( s^{-1} \alpha \). Choose a representative element \( \theta(\theta \equiv 0) \) of the class \( s^{-1} \mathfrak{x} = s^{-1} \mathfrak{x} \) mod \( s^{-1} \alpha \). We see from (2.2), (2.3) in the proof of Theorem A that
\[ \theta(s^{-1} \alpha)^{-1} = \mathfrak{f}^{-1} \mathfrak{b}_2 \]
with some integral ideal \( \mathfrak{b}_2 \) of \( \mathcal{F} \) prime to \( \mathfrak{f} \). Set \( \beta = \beta_1/\beta_2 \). Thus we have,
by the expression (1.7) and the definition (3.12),

\[ \xi_{1-a}(s^{-1}z) = \exp(\log \eta \cdot h(\beta, (p, q)u)) = \exp(\log \eta \cdot h(g\alpha, (p, q)u)) = \exp(\log \eta \cdot h^{(s)}(\alpha, (p, q))). \]

Finally thanks to Theorem A in the introduction we obtain the identity (1.8).

We continue the assumption that \( F \) is a real quadratic field. For \( F - Q \), we define \( \xi(s, \alpha) \) to be the Dirichlet series

\[ \sum_{n=1}^{\infty} \frac{\cot \pi n\alpha}{n} \cdot \gamma \]

We have proved in [Ar2] that \( \xi(s, \alpha) \) is absolutely convergent for \( \text{Re}(s)>1 \) and that it can be continued analytically to a meromorphic function in the whole \( s \)-plane. Moreover, \( \xi(s, \alpha) \) has a simple pole at \( s=1 \). We denote by \( \xi_{1-1}(\alpha) \) the residue of \( \xi(s, \alpha) \) at the simple pole \( s=1 \). The function \( H(\alpha, s, (0, 0)) \) given by (1.4) has the following obvious connection with \( \xi(s, \alpha) \):

\[ H(\alpha, s, (0, 0)) = \frac{1+e^{\pi it}}{2}(\xi(1-s, \alpha)-\xi(1-s)), \]

where \( \xi(s) \) is the Riemann zeta function. Thus we have

\[ h_{1-1}(\alpha, (0, 0)) = -i\xi_{1-1}(\alpha)+1. \]

Since \( \xi_{1-1}(\alpha') = -\xi_{1-1}(\alpha) \) (see Proposition 2.10 of [Ar2]), it follows that

\[ \xi(\alpha, (0, 0)) = -i\xi_{1-1}(\alpha). \]

Let \( \varepsilon \) be the totally positive fundamental unit of \( F \) with \( \varepsilon>1 \). Choose a basis \( \{\alpha_1, \alpha_2\} \) of a fractional ideal \( a \) of \( F \) such that

\[ \alpha_1\alpha_2 - \alpha_1\alpha_2 > 0, \quad \alpha_2\alpha_2 > 0. \]

We represent \( \varepsilon \) by the basis \( \{\alpha_1, \alpha_2\} \) to get a matrix \( V \) of \( SL_2(Z) \):

\[ \varepsilon(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}) = V(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}), \quad V = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}). \]

We get, by Theorem B,

\[ \xi_a(0 \mod a) = \exp(\log \varepsilon \cdot h((\alpha, (0, 0))) = \exp(-i \log \varepsilon \cdot c_{-1}(\alpha)), \]

where we put \( \alpha = \alpha_1/\alpha_2 \). Taking the facts \( V \varepsilon = \alpha, c>0, c \alpha + d > 0 \) into account, we have, with the help of Proposition 2.9, (i) of [Ar2],
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\[ c_{-1}(\alpha) = -\frac{2\pi}{\log \epsilon} \left( \frac{a+d}{12\epsilon} - s(d, c) - \frac{1}{4} \right), \]

where \( s(d, c) \) is the Dedekind sum (for the Dedekind sum we refer the reader to [R-G]). Hence,

\[ \xi_\alpha(0 \mod \alpha) = \exp \left( 2\pi i \left( \frac{a+d}{12\epsilon} - s(d, c) - \frac{1}{4} \right) \right). \]

It is known that the value \((a+d)/c - 12s(d, c)\) is a rational integer (see Ch. 4 of [R-G] and Remark 3.2 of [Ar2]). Therefore the value \( \xi_\alpha(0 \mod \alpha) \) is a twelfth root of unity.

References


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