EQUIVARIANT CRITICAL POINT THEORY AND IDEAL-VALUED COHOMOLOGICAL INDEX

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Introduction

We develop an equivariant critical point theory for differentiable $G$-functions on a Banach $G$-manifold with the aid of ideal-valued cohomological index theory, where $G$ is a compact Lie group. We obtain a lower bound for the number of critical orbits with values in a given interval $(a, b) = \{t \in \mathbb{R} \mid a < t \leq b\}$ and for the number of critical values in $(a, b)$. We also obtain cohomological information about the topology of the critical set $K$ of a $G$-function, which says a lot more about $K$ than that obtained by using the Lusternik-Schnirelmann category.

The Lusternik-Schnirelmann category is a numerical homotopical invariant which gives a lower bound for the number of critical points (see for example [16], [17]), and this category is successfully extended to the equivariant setting [2], [3], [5], [6], [7], [15]. Ideal-valued cohomological index theory also gives important information about the existence of critical points [8], [9], [10]. The index theory in these papers is a priori in the equivariant setting and contains the nonequivariant (absolute) setting as trivial case.

In their paper [6] M. Clapp and D. Puppe developed an equivariant critical point theory using an equivariant Lusternik-Schnirelmann category. In the present paper we will develop one using an ideal-valued cohomological index theory which contains the nonequivariant setting as nontrivial case. We will obtain a type of results corresponding to their Theorem 1.1 of [6] and further results which are derived only from our theory.

Throughout this paper $G$ always denotes a compact Lie group, and spaces considered are all paracompact Hausdorff. Let $M$ be a Banach $G$-manifold of class at least $C^1$, i.e., $M$ is a $C^1$ Banach manifold and $G$ acts differentiably by diffeomorphisms. Let $f : M \to \mathbb{R}$ be a $C^1$ $G$-function, i.e., $f$ is of class $C^1$ and satisfies $f(gx) = f(x)$ for all $x \in M$ and $g \in G$. Let $K = \{x \in M \mid df_x = 0\}$ the critical set of $f$, $M_c = f^{-1}(-\infty, c]$ and $K_c = K \cap f^{-1}(c)$ for any $c \in \mathbb{R}$.

If $x \in M$ is a critical point of $f$, then every point of $Gx = \{gx \mid g \in G\}$
is also a critical point, and $G_x$ is called a critical orbit of $f$. Note that $G_x$ is diffeomorphic to the homogeneous space $G/G_x$ where $G_x$ is the isotropy subgroup at $x$.

Consider the following deformation conditions ($D_0$)-($D_2$) for $f: M \to \mathbb{R}$ at $c \in \mathbb{R}$:

($D_0$) There is an $\varepsilon > 0$ such that $M_{c+\varepsilon}$ is $G$-deformable to $M_c$, i.e., there is a $G$-homotopy $\varphi_t: M_{c+\varepsilon} \to M_{c+\varepsilon}$ $(0 \leq t \leq 1)$ such that $\varphi_0 = \text{id}$ and $\varphi_1(M_{c+\varepsilon}) \subseteq M_c$.

($D_1$) $K_c$ is compact.

($D_2$) For every $\delta > 0$ and every $G$-invariant neighborhood $U$ of $K_c$ there is an $\varepsilon$ with $0 < \varepsilon < \delta$ such that $M_{c+\varepsilon} - U$ is $G$-deformable to $M_{c-\varepsilon}$ relative to $M_{c-\delta}$.

A $C^1$ Banach $G$-manifold $M$ admits a $G$-invariant Finsler structure $\| \|: TM \to \mathbb{R}$ (see Palais [16], Krawcewicz-Marzantowicz [14]). The Palais-Smale condition (or (PS) condition for abbreviation) for $f$ is:

(PS) Any sequence $\{x_n\}$ in $M$ such that $\{f(x_n)\}$ is bounded and $\{\|df_{x_n}\|\}$ converges to 0 contains a convergent subsequence.

As is well-known, ($D_1$) and ($D_2$) at any $c \in \mathbb{R}$ is a consequence of (PS) under suitable assumptions on differentiability and completeness. See for the proof Palais [16; Theorem 5.11], [17; Theorem 4.6] for the nonequivariant case, and Clapp-Puppe [6; Appendix A], Krawcewicz-Marzantowicz [14; Lemma 1.9] for the equivariant case. If $c$ is a regular value of $f$, ($D_0$) is also a consequence of (PS) (see [6; Appendix A]). Even if $c$ is not a regular value we can see that ($D_0$) follows from (PS) under the assumption that $c$ is an isolated critical value.

By a $G$-pair $(X,A)$ we mean a $G$-space $X$ together with a $G$-invariant subspace $A$. A $G$-map $f: (X,A) \to (Y,B)$ means a $G$-map $f: X \to Y$, i.e., $f(gx) = gf(x)$ for $g \in G$ and $x \in X$, such that $f(A) \subseteq B$. Let $\mathcal{P}$ be the category of such $G$-pairs and $G$-maps. Let $h^*$ be a generalized $G$-cohomology theory on $\mathcal{P}$, i.e., $h^*$ is a contravariant functor into graded modules and $h^*$ is equipped with long exact sequences, excision and homotopy property. In this paper, moreover we require $h^*$ to be continuous and multiplicative with unit. See section 1 for the definition of the terms.

For $(X,A) \in \mathcal{P}$ the ideal-valued index of $A$ in $X$, denoted $\text{ind}(A,X)$, is defined to be the kernel of the homomorphism $i^*: h^*(X) \to h^*(A)$ where $i: A \to X$ is the inclusion and $h^*(X) = h^*(X,\emptyset)$. Then $\text{ind}(A,X)$ is an ideal of $h^*(X)$. 
We can now state our first theorem, which corresponds to Theorem 2.3 in section 2.

**Theorem 0.0.** Let $M$ be a $C^1$ Banach $G$-manifold with $h^*(M)$ Noetherian, and $f: M \to \mathbb{R}$ a $C^1$-function. For given $-\infty < a < b \leq \infty$, assume that $f$ satisfies $(D_0)$ at $a$ and $(D_1), (D_2)$ at every $c \in (a,b)$ ($c \neq \infty$). If $b = \infty$, assume in addition that $f(K)$ is bounded above. Then there are a finite number of critical values $c_1, \ldots, c_k \in (a,b]$ of $f$ such that

$$\text{ind}(M_a, M) \cdot \text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) \subseteq \text{ind}(M_b, M),$$

where $\cdot$ represents the products of ideals [1].

A ring $R$ is said to be nilpotent if $R^n = 0$ for some integer $n > 0$. The least such integer $n$ is called the index of nilpotency and written $\text{nil}(R)$. If no such integer $n$ exists we put $\text{nil}(R) = \infty$.

**Remark.** See Marzantowicz [15] for the relation between the index of nilpotency of $\tilde{h}^*(X)$ of a $G$-space $X$, the cup-length of $\Lambda^*(M)$ and the $G$-category of $X$.

If $-\infty < a < b \leq \infty$, we see $\text{ind}(M_b, M) \subseteq \text{ind}(M_a, M)$ in $h^*(M)$ since $M_a \subseteq M_b$. Define for any integer $s \geq 0$,

$$s\text{-nil}(M_a, M_b) = \text{nil}(\text{ind}^{\leq s}(M_a, M) / \text{ind}^{\leq s}(M_b, M)),$$

where

$$\text{ind}^{\leq s}(A, M) = \text{ind}(A, M) \cap h^{\leq s}(M), \ h^{\leq s}(M) = \bigoplus_{n \geq s} h^n(M).$$

Note that if $s \leq t$ then $t\text{-nil}(M_a, M_b) \leq s\text{-nil}(M_a, M_b)$, and if $b = \infty$ then $s\text{-nil}(M_a, M_b) = \text{nil}(\text{ind}^{\leq s}(M_a, M))$ since $M_b = M$ and $\text{ind}(M_b, M) = 0$.

Using a suitable $G$-cohomology theory $h^*$, we will derive the following theorem from Theorem 0.0, which summarizes Theorems 3.4, 3.5, 3.6 and 3.9 in section 3.

**Theorem 0.1.** Let $f: M \to \mathbb{R}$ be as in Theorem 0.0 except that $f(K)$ is bounded if $b = \infty$.

1. $f$ has at least $1\text{-nil}(M_a, M_b) - 1$ critical orbits in $M_{[a,b]} = f^{-1}(a,b]$.
2. If $h^{\leq s}(M) \subseteq \text{ind}(K_c, M)$ for all critical values $c \in (a,b]$, then $f$ has at least $s\text{-nil}(M_a, M_b) - 1$ critical values in $(a,b]$.
3. If $s\text{-nil}(M_a, M_b) - 1$ is greater than the number of critical values of $f$ in $(a,b]$, then there is a critical value $c \in (a,b]$ of $f$ such that $h^{\leq s}(K_c) \neq 0$. 

Theorem 0.1.
(4) If \( 1\text{-nil}(M_a, M_x) = \infty \) for some \( a \in \mathbb{R} \), then there is an unbounded sequence of critical values of \( f \).

If in the above theorem \( f \) is bounded below and \( a < \inf f(M) \), then we will obtain a bit better results (see Theorem 3.7).

We will also obtain the following theorem more precisely than in Theorem 0.1 (3).

**Theorem 0.2.** Assume that \( f \) has \( k \) critical values \( c_1, \ldots, c_k \) in \( (a, b] \), and that there are \( x_0 \in \text{ind}(M_a, M) \) and \( x_1, \ldots, x_k \in h^*(M) \) such that \( x_0 x_1 \cdots x_k \notin \text{ind}(M_b, M) \). If each of \( x_1, \ldots, x_k \) is homogeneous, then

\[
h^{d_1}(K_{c_1}) \oplus \cdots \oplus h^{d_k}(K_{c_k}) \neq 0,
\]

where \( d_i = \deg x_i \).

This theorem corresponds to Theorem 3.11, and the following corollary corresponds to Corollary 3.13 in section 3.

**Corollary 0.3.** Assume that \( f \) is bounded (above and below) and has \( k \) critical values. Then \( h^m(K) \neq 0 \) for any integers \( m, l \geq 0 \) with \( kl \leq \cup_{m} (h^*(M)) \).

Here \( \cup_{m}(h^*(M)) \) is the \( \cup_{m}-\text{length} \) of \( h^*(M) \) defined to be the largest integer \( t \) such that \( (h_m(M))^t \neq 0 \) in \( h^*(M) \). Corollary 0.3 roughly says that the smaller the number of critical values is, the higher the dimension of the nonzero cohomology of \( K \) is.

**1. Ideal-valued cohomological index**

Let \( h^* \) be a generalized \( G \)-cohomology theory on \( \mathcal{P} \). \( h^* \) is said to be \textit{multiplicative} if it has products

\[
h^p(X, A) \times h^q(X, B) \rightarrow h^{p+q}(X, A \cup B)
\]

for any \( (X, A), (X, B) \in \mathcal{P} \) with \( \{A, B\} \) excisive and any \( p, q \in \mathbb{Z} \), which is natural, bilinear, associative, commutative (up to the sign \((-1)^{pq}) \). \( h^* \) is said to be \textit{continuous} if for any \( (X, A) \in \mathcal{P} \) with \( A \) closed,

\[
h^*(A) \cong \lim h^*(U)
\]

where the direct limit is taken over all \( G \)-invariant neighborhoods \( U \) of \( A \) in \( X \), and the isomorphism is induced by the inclusions.
EXAMPLE 1.1. Let $H^*$ be the Alexander-Spanier cohomology theory with coefficients in a field $F$. The following (1) and (2) are both generalized cohomology theories on $\mathcal{P}$ which are continuous and multiplicative with unit in $h^0(X)$.

(1) The Borel $G$-cohomology based on $H^*$,

$$h^*(X,A) := H^*(EG \times G X, EG \times gA; F),$$

where $EG$ is a universal $G$-space.

(2)

$$h^*(X,A) := H^*(X/G, A/G; F).$$

REMARK 1.2. The equivariant stable cohomotopy theory and the equivariant $K$-theory are also examples of a generalized $G$-cohomology theory. The former is employed in Bartsch-Clapp-Puppe [4].

In what follows we assume $h^*$ is a generalized $G$-cohomology theory on $\mathcal{P}$ which is continuous and multiplicative with unit. For $(X,A) \in \mathcal{P}$ the ideal-valued index $\text{ind}(A, X)$ is defined as in the Introduction. We summarize its properties in the following.

Proposition 1.3. Let $(X,A)$, $(X,A_1)$, $(X,A_2) \in \mathcal{P}$.

(1) Monotonicity: If there is a $G$-map $\varphi : A_1 \to A_2$ such that $i_2 \varphi$ is $G$-homotopic to $i_1$ where $i_1 : A_1 \to X$ and $i_2 : A_2 \to X$ are the inclusions, then

$$\text{ind}(A_2, X) \subseteq \text{ind}(A_1, X).$$

(2) Subadditivity: If $\{A_1, A_2\}$ is an excisive pair, then

$$\text{ind}(A_1, X) \cdot \text{ind}(A_2, X) \subseteq \text{ind}(A_1 \cup A_2, X).$$

(3) Continuity: If $A$ is closed in $X$ and $\text{ind}(A, X)$ is a finitely generated ideal of $h^*(X)$, then there is a $G$-invariant neighborhood $U$ of $A$ in $X$ such that

$$\text{ind}(A, X) = \text{ind}(U, X).$$

Proof. (1) Easy by the definition of the index.

(2) It suffices to show that if $x_n \in \text{ind}(A_n, X), n = 1, 2$, then $x_1 x_2 \in \text{ind}(A_1 \cup A_2, X)$. Consider the following commutative diagram.
where the homomorphisms are all induced from the inclusions. Note that the two sequences \( \{i^*_1, k^*_1\} \) and \( \{j^*_3, k^*_3\} \) are both exact. By the commutativity of the diagram we see \( h_n^* x_n = 0 \) in \( h^*(A_n) \) for \( n = 1, 2 \), and by the exactness we see that for \( n = 1, 2 \) there are \( y_n \in h^*(A_1 \cup A_2, A_n) \) such that \( j^*_n y_n = i^*_n x_n \). Hence

\[
i^*_3(x_1 x_2) = i^*_3 x_1 \cdot i^*_3 x_2 = j^*_3 y_1 \cdot j^*_3 y_2 = j^*_3(y_1 y_2) = 0.
\]

This implies \( x_1 x_2 \in \text{ind}(A_1 \cup A_2, X) \).

(3) Let \( x_1, \ldots, x_k \) be generators of \( \text{ind}(A, X) \). Since \( x_n \mid A = i^* x_n = 0 \) in \( h^*(A)(n = 1, 2, \ldots, k) \), by the continuity there is a \( G \)-invariant neighborhood \( U_n \) of \( A \) in \( X \) such that \( x_n \mid U = 0 \) in \( h^*(U_n) \). Then \( U = U_1 \cap \cdots \cap U_n \) is also a \( G \)-invariant neighborhood of \( A \), and \( x_n \mid U = 0 \), i.e., \( x_n \in \text{ind}(U, X) \). Hence \( \text{ind}(A, X) \subseteq \text{ind}(U, X) \). On the other hand we see \( \text{ind}(A, X) \supseteq \text{ind}(U, X) \) by the monotonicity of index.

**Remark 1.4.** In (3) of the above proposition \( \text{ind}(A, X) \) is finitely generated if \( h^*(X) \) is Noetherian. One can find in Fadell [8; §3] some sufficient conditions for \( h^*(X) \) to be Noetherian.

### 2. Indices of critical sets

**Lemma 2.1.** Let \( M \) be a \( C^1 \) Banach \( G \)-manifold and \( f : M \to \mathbb{R} \) a \( C^1 \) \( G \)-function. For given \( -\infty < a < b \leq \infty \), assume that \( f \) satisfies \( (D_0) \) at \( a \) and \( (D_2) \) at every \( c \in (a, b)(c \neq \infty) \). If \( f \) has no critical value in \( (a, b) \), then

\[
\text{ind}(M_a, M) = \text{ind}(M_b, M).
\]

Proof. By the conditions \( (D_0), (D_2) \) we can see that \( M_b \) is
By the monotonicity of index we see \( \text{ind}(M_a, M) \leq \text{ind}(M_b, M) \). Conversely, by the monotonicity again we see \( \text{ind}(M_a, M) \geq \text{ind}(M_b, M) \) since \( M_a \subseteq M_b \). Thus the lemma is proved.

Lemma 2.2. Let \( M \) be a \( C^1 \) Banach \( G \)-manifold with \( h^*(M) \) Noetherian. If a \( C^1 \) \( G \)-function \( f: M \to \mathbb{R} \) satisfies \((D_1)\) and \((D_2)\) at \( c \), then there is an \( \varepsilon > 0 \) such that

\[
\text{ind}(M_{c-\varepsilon}, M) \cdot \text{ind}(K_c, M) \leq \text{ind}(M_{c+\varepsilon}, M).
\]

In particular, if \( M_{c-\varepsilon} = \emptyset \) then

\[
\text{ind}(K_c, M) = \text{ind}(M_{c+\varepsilon}, M),
\]

and if \( K_c = \emptyset \) then

\[
\text{ind}(M_{c-\varepsilon}, M) = \text{ind}(M_{c+\varepsilon}, M).
\]

Proof. By the assumptions, \( K_c \) is compact and \( h^*(M) \) is Noetherian. So by the continuity of index there is a \( G \)-invariant neighborhood \( U \) of \( K_c \) such that \( \text{ind}(K_c, M) = \text{ind}(U, M) \). There is also a \( G \)-invariant neighborhood \( V \) of \( K_c \) such that \( K_c \subseteq V \subseteq \bar{V} \subseteq \bar{U} \). By the monotonicity we see \( \text{ind}(K_c, M) = \text{ind}(V, M) \). Take an \( \varepsilon > 0 \) satisfying \((D_2)\) for this \( V \). Then we have

\[
\text{ind}(M_{c+\varepsilon}, M) = \text{ind}((M_{c+\varepsilon} - V) \cup U, M)
\]

\[
\geq \text{ind}(M_{c+\varepsilon} - V, M) \cdot \text{ind}(U, M) \quad \text{by subadditivity}
\]

\[
= \text{ind}(M_{c+\varepsilon} - V, M) \cdot \text{ind}(K_c, M)
\]

\[
\geq \text{ind}(M_{c-\varepsilon}, M) \cdot \text{ind}(K_c, M) \quad \text{by \((D_2)\) and monotonicity.}
\]

Thus the first half of the lemma is proved. If \( A = \emptyset \) then \( \text{ind}(A, M) = h^*(M) \). This fact and the monotonicity implies the second half.

We will obtain the following theorem:

Theorem 2.3. Let \( M \) be a \( C^1 \) Banach \( G \)-manifold with \( h^*(M) \) Noetherian. For given \( -\infty < a < b \leq \infty \), assume that \( C^1 \) \( G \)-function \( f: M \to \mathbb{R} \) satisfies \((D_0)\) at \( a \) and \((D_1),(D_2)\) at every \( c \in (a, b) \) \((c \neq \infty)\). If \( b = \infty \), assume in addition that \( f(K) \) is bounded above. Then there are a finite number of critical values \( c_1, \ldots, c_k \in (a, b) \) of \( f \) such that

\[
\text{ind}(M_{a, M}) \cdot \text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) \leq \text{ind}(M_{b, M}).
\]
Proof. First assume \( b < \infty \). Let \( \varepsilon(a) \) be such an \( \varepsilon > 0 \) as in \((D_0)\) at \( a \). For any \( c \in (a, b] \) let \( \varepsilon(c) \) be such an \( \varepsilon > 0 \) as in Lemma 2.2, i.e.,

\[
\text{ind}(M_{c - \varepsilon(c)}, M) \cdot \text{ind}(K_{c}, M) \subseteq \text{ind}(M_{c + \varepsilon(c)}, M).
\]

Let \( V_c \) denote the open interval \((c - \varepsilon(c), c + \varepsilon(c))\) for any \( c \in [a, b] \). Then \( \{V_c|c\in[a,b]\}\) is an open covering of \([a,b]\). Since \([a,b]\) is compact, there are a finite number of \( d_1, \ldots, d_m \in [a, b] \) such that

\[
[a,b] \subseteq V_{d_1} \cup \cdots \cup V_{d_m}.
\]

By the monotonicity and Lemma 2.2 we have

\[
\text{ind}(M_b, M) \supseteq \text{ind}(M_{b + \varepsilon(b)}, M) \supseteq \text{ind}(K_b, M) \cdot \text{ind}(M_{b - \varepsilon(b)}, M).
\]

\( b - \varepsilon(b) \) is contained in \( V_d \) for some \( d \in \{d_1, \ldots, d_m\} \). Since \( b - \varepsilon(b) < d + \varepsilon(d) \) we have

\[
\text{ind}(M_{b - \varepsilon(b)}, M) \supseteq \text{ind}(M_{d + \varepsilon(d)}, M) \supseteq \text{ind}(K_d, M) \cdot \text{ind}(M_{d - \varepsilon(d)}, M) \text{ by Lemma 2.2}.
\]

By the above we have

\[
\text{ind}(M_b, M) \supseteq \text{ind}(K_b, M) \cdot \text{ind}(K_d, M) \cdot \text{ind}(M_{d - \varepsilon(d)}, M)
\]

Repeating this we have

\[
(2.4) \quad \text{ind}(M_b, M) \supseteq \text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) \cdot \text{ind}(M_{a}, M)
\]

for some \( c_1, \ldots, c_k \in (a, b] \). If \( c \) is not a critical value then \( K_c = \emptyset \) and \( \text{ind}(K_c, M) = h^*(M) \in 1 \). So we may assume that \( c_1, \ldots, c_k \) in \((2.4)\) are all critical values. Thus the theorem is proved for the case \( b < \infty \).

Now assume \( b = \infty \). Take an \( r > 0 \) such that \( \sup f(K) < r < \infty \). By the above we see that there are a finite number of critical values \( c_1, \ldots, c_k \in (a, r] \) such that

\[
\text{ind}(M_{a}, M) \cdot \text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) \subseteq \text{ind}(M_{r}, M).
\]

Since there is no critical value in \([r, \infty)\) we can see by \((D_2)\) that \( M_b = M \) is \( G \)-deformable to \( M_r \). Thus \( \text{ind}(M_r, M) = \text{ind}(M_b, M) = 0 \). Thus the theorem is also proved for the case \( b = \infty \). \( \square \)

If \( f \) is bounded below and \( a < \inf f(M) \), then \( M_a = \emptyset \) and \( \text{ind}(M_a, M) = h^*(M) \in 1 \). Thus we obtain the following corollary from Theorem 2.3.
Corollary 2.4. If \( f \) is bounded below and \( a < \inf f(M) \) in Theorem 2.3, then there are a finite number of critical values \( c_1, \ldots, c_k \leq b \) of \( f \) such that

\[
\text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) \subseteq \text{ind}(M_b, M).
\]

In particular, if \( b = \infty \) then

\[
\text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) = 0.
\]

3. The number of critical orbits and values

In this section we will derive some results from Theorem 2.3. Before doing that we need a lemma.

Lemma 3.1. Let \( \mathfrak{U} \supseteq \mathfrak{B} \) be two ideals of a ring \( R \). If \( \mathfrak{U} \cdot R^k \subseteq \mathfrak{B} \) for some \( k \geq 0 \), then \( \text{nil}(\mathfrak{U}/\mathfrak{B}) \leq k + 1 \).

Proof. Assume to the contrary that \( k + 1 \leq \text{nil}(\mathfrak{U}/\mathfrak{B}) \). Then there were \( k + 1 \) elements \( x_0, x_1, \ldots, x_k \in \mathfrak{U} \) such that \([x_0] \cdot [x_1] \cdots [x_k] \neq 0 \) in \( \mathfrak{U}/\mathfrak{B} \), i.e., \( x_0 x_1 \cdots x_k \notin \mathfrak{B} \). This contradicts the assumption \( \mathfrak{U} \cdot R^k \subseteq \mathfrak{B} \).

For a function \( f: M \to \mathbb{R} \) and a subset \( S \subseteq \mathbb{R} \) define \( M_s := f^{-1}(S) \) and \( K_s := K \cap M_s \). In the theorems below we will assume (3.2) and (3.3).

Assumption 3.2. A generalized \( G \)-cohomology theory \( h^* \) is continuous and multiplicative with unit and satisfies \( h^{>1}(G/H) = 0 \) for all closed subgroups \( H \) of \( G \).

The \( G \)-cohomology theory of Example 1.1 (2) satisfies Assumption 3.2. Note that if \( K \) is a disjoint union of a finite number of orbits \( G/H_1, \ldots, G/H_m \) in \( M \) then

\[
\text{ind}(K, M) = \bigcap_{i=1}^m \text{ind}(G/H_i, M) \supseteq h^{>1}(M)
\]

under Assumption 3.2.

Assumption 3.3. \( M \) is a \( C^1 \) Banach \( G \)-manifold with \( h^*(M) \) Noetherian. For given \( -\infty < a < b \leq \infty \), a \( C^1 \) \( G \)-function \( f: M \to \mathbb{R} \) satisfies (\( D_0 \)) at \( a \) and (\( D_1 \), (\( D_2 \)) at every \( c \in (a,b) \) (\( c \neq \infty \)).

Theorem 3.4. \( f \) has at least \( 1 - \text{nil}(M_a, M_b) - 1 \) critical orbits in \( M_{[a,b]} \). In particular, if \( 1 - \text{nil}(M_a, M_b) = \infty \) then \( f \) has infinitely many critical
orbits in $M_{(a,b)}$.

Proof. It suffices to consider only the case where the number of critical values in $(a,b)$ is finite. Let $c_1, \ldots, c_k \in (a, b]$ be such critical values. It also suffices to consider the case where $K_{c_i}$ is a finite union of orbits for all $1 \leq i \leq k$. In this case we see $h^{\geq 1}(M) \subseteq \text{ind}(K_{c_i}, M)$. Thus by Theorem 2.3 we have

$$\text{ind}(M_a, M) \cdot (h^{\geq 1}(M))^k \subseteq \text{ind}(M_b, M).$$

By Lemma 3.1 we see $1\text{-nil}(M_a, M_b) \leq k + 1$. This implies that the number of critical orbits in $M_{(a,b]}$ is at least $1\text{-nil}(M_a, M_b) - 1$.

A similar proof to above also shows the following.

**Theorem 3.5.** If $h^{\geq s}(M) \subseteq \text{ind}(K_{c_i}, M)$ for all critical values $c \in (a, b]$ and for some integer $s \geq 0$, then $f$ has at least $s\text{-nil}(M_a, M_b) - 1$ critical values in $(a, b]$.

The contraposition of this theorem is:

**Theorem 3.6.** If $s\text{-nil}(M_a, M_b) - 1$ is greater than the number of critical values of $f$ in $(a, b]$, then there is a critical value $c \in (a, b]$ of $f$ such that

$$h^{\geq s}(M) \nsubseteq \text{ind}(K_{c_i}, M)$$

and hence $h^{\geq s}(K_c) \neq 0$.

If $f$ is bounded below and $a < \inf f(M)$, then we may use Corollary 2.4 instead of Theorem 2.3 in the proofs of Theorems 3.4, 3.5, 3.6, and obtain

**Theorem 3.7.** Assume that $f$ is bounded below and $a < \inf f(M)$. Then

1. $f$ has at least $1\text{-nil}(0, M_b)$ critical orbits in $M_b$,
2. if $h^{\geq s}(M) \subseteq \text{ind}(K_{c_i}, M)$ for all critical values $c \leq b$ of $f$, then $f$ has at least $s\text{-nil}(0, M_b)$ critical values in $(-\infty, b]$,
3. if $s\text{-nil}(0, M_b)$ is greater than the number of critical values of $f$ in $(-\infty, b]$, then there is a critical value $c \leq b$ of $f$ such that $h^{\geq s}(K_c) \neq 0$.

Note that $s\text{-nil}(0, M_b) = \text{nil}(h^{\geq s}(M)/\text{ind}^{\geq s}(M_b, M))$.

**Lemma 3.8.** If $A$ is a $G$-invariant compact subspace of a $G$-space $X$ with $h^*(X)$ Noetherian, then
\[(h^{\geq 1}(X))^k \leq \text{ind}(A, X)\]

for some integer \(k > 0\).

**Proof.** Since \(A\) is compact, there are a finite number of orbits in \(A\), say \(G/H_i (1 \leq i \leq k)\), and \(G\)-invariant open neighborhoods \(U_i\) of \(G/H_i\) such that \(A\) is covered by \(U_i(1 \leq i \leq k)\) and \(\text{ind}(G/H_i, X) = \text{ind}(U_i, X)\). This fact shows

\[\text{ind}(G/H_1, X) \cdots \text{ind}(G/H_k, X) \leq \text{ind}(A, X)\]

by the monotonicity and subadditivity of index. Then Assumption 3.2 implies the lemma. \(\square\)

**Theorem 3.9.** If \(1\)-nil\((M_a, M_b) = \infty\) and \(b = \infty\), then \(f(K)\) is not bounded, i.e., there is an unbounded sequence of critical values of \(f\).

**Proof.** If \(f(K)\) were bounded, then by Theorem 2.3 there were a finite number of critical values \(c_1, \ldots, c_k > a\) such that

\[(3.10) \quad \text{ind}(M_a, M) \cdot \text{ind}(K_{c_1} M) \cdots \text{ind}(K_{c_k} M) = 0.\]

Since \(\text{nil}(\text{ind}^{-1}(M_a, M)) = 1\)-nil\((M_a, M) = \infty\), for every \(n > 0\) there are \(x_1, \ldots, x_n \in \text{ind}^{-1}(M_a, M)\) with \(x_1 \cdots x \neq 0\). Since \(K_{c_i}(1 \leq i \leq k)\) is compact, Lemma 3.8 shows that for a sufficiently large \(n\) there is an \(m < n\) such that

\(x_1 \cdots x_m \in \text{ind}(K_{c_1} M) \cdots \text{ind}(K_{c_k} M)\).

Then (3.10) implies \(x_1 \cdots x_m \cdots x_n = 0\). This is a contradiction. So \(f(K)\) is not bounded. \(\square\)

**Theorem 3.11.** Assume that \(f\) has \(k\) critical values \(c_1, \ldots, c_k\) in \((a, b]\), and that there are \(x_0 \in \text{ind}(M_a, M)\) and \(x_1, \ldots, x_k \in h^*(M)\) such that \(x_0 x_1 \cdots x_k \notin \text{ind}(M_b, M)\). If each of \(x_1, \ldots, x_k\) is homogeneous, then

\[(3.12) \quad h^{d_1}(K_{c_1}) \bigoplus \cdots \bigoplus h^{d_k}(K_{c_k}) \neq 0,\]

where \(d_i = \deg x_i\).

**Proof.** If the left hand side of (3.12) were zero, then \(x_i \in \text{ind}(K_{c_i}, M)\) for all \(1 \leq i \leq k\). This implies

\(x_0 x_1 \cdots x_k \in \text{ind}(M_a, M) \cdot \text{ind}(K_{c_1} M) \cdots \text{ind}(K_{c_k} M),\)
and by Theorem 2.3 we see $x_0 x_1 \cdots x_k \in \text{ind}(M_b, M)$. This contradicts the assumption of the theorem.

**Corollary 3.13.** Assume that $f$ is bounded (above and below) and has $k$ critical values. Then $h^m(K) \neq 0$ for any integers $m, l \geq 0$ with $kl \leq \cup_m (h^*(M))$.

Proof. If $\cup_m (h^*(M)) < k$, then the corollary is trivial since $l = 0$ can only be taken. So assume $k \leq \cup_m (h^*(M)) = t$. Then there are $y_1, \cdots, y_t \in h^m(M)$ for $i = 1, \cdots, t$ such that $y_1, \cdots, y_t \neq 0$. If we take $a$ and $b$ such that $-\infty < a < \inf f(M) \leq \sup f(M) < b < \infty$, then $\text{ind}(M_a, M) = h^*(M)$ and $\text{ind}(M_b, M) = 0$. Thus we can take $x_0, x_1, \cdots, x_k$ in Theorem 3.11 so as

$$x_0 = 1, x_i = y_{i-1} y_{i-1} y_{i-1} y_{i-1} y_{i-1} y_{i-1} y_{i-1} y_{i-1} (1 \leq i \leq k).$$

Since $\deg x_i = ml$ for all $i$ with $1 \leq i \leq k$, Theorem 3.11 shows $h^m(K) \neq 0$.

Finally, we give an application of Corollary 3.13. Let $K$ be the reals $R$, the complexes $C$, or the quaternions $H$, and according to that $G$ be the group $Z_2, S^1$ or $S^3$ of $g \in K$ with $|g| = 1$. Then $G$ acts on $K^n$ by coordinate-wise multiplication, and the unit sphere $S(K^n)$ of $K^n$ is a $G$-invariant submanifold with the orbit space $S(K^n)/G = KP^{n-1}$, the projective space. Let $h^*(X) = H^*(X/G; F)$ where $H^*$ is the Alexander-Spanier cohomology and $F = Z_2, Q$ or $Q$ according to $K = R, C$ or $H$. Then

$$h^*(S(K^n)) \cong F[x]/(x^n), \quad d = \deg x = 1, 2 \text{ or } 4,$$  

and we see $\cup_m (h^*(S(K))) = n - 1$. Thus Corollary 3.13 shows that if a $C^1 G$-function $f : S(K^n) \rightarrow R$ has $k$ critical values, then $h^{dl}(K) \neq 0$ for any integer $l$ with $0 \leq kl \leq n - 1$. This says a lot more about the cohomology of $K$ than in Clapp-Puppe [5; §2].

For many spaces other than $S(K^n)$ we already know the $\cup_1$-length or a lower bound of that. See for example Fadell-Husseini[10; Theorem 3.16], Hiller [11], Jaworowski [12; §5] and Komiya [13; Remark 5.10]. So we can apply Corollary 3.13 to functions on such spaces.

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**References**


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