1. Introduction

Let $G$ be a finite group and $k$ an algebraically closed field of characteristic $p$. An indecomposable $kG$-module with a vertex $Q$ is said to be a weight module if its Green correspondent with respect to $(G, Q, N_G(Q))$ is simple. Let $B$ be a block of $kG$. Alperin [1] conjectured that the number of the weight modules belonging to $B$ equals that of the simple modules in $B$. If this is the case and a defect group of $B$ a TI set, then it can be shown under some additional assumption that the socles of weight modules are simple, which in turn determine the isomorphism classes of the weight modules; this holds if $G$ is a simple group with a cyclic Sylow $p$-subgroup. This rather surprising property has been known to hold for finite groups of Lie type of characteristic $p$. However little is known about general properties of weight modules. In the final section we shall study solvable groups that have only simple weight modules.

Throughout this paper $G$ denotes a finite group and $k$ an algebraically closed field of prime characteristic $p$. For a $kG$-module $M$, $\text{hd}(M)$, $\text{soc}(M)$ and $P(M)$ denote the head, socle and projective cover of $M$ respectively. If $N$ is a $kG$-module, $N|M$ indicates that $N$ is isomorphic to a direct summand of $M$, and $(N, M)$ denotes the multiplicity of $N$ as a summand of $M$. We fix a block $B$ of $kG$ and let $D$ be its defect group. $\text{IRR}(B)$ denotes a full set of non-isomorphic simple modules in $B$, $l(B)$ its cardinality and $\text{WM}(B)$ a full set of non-isomorphic weight modules belonging to $B$. Let $f$ be the Green correspondence with respect $(G, D, H)$, where $H = N_G(D)$. If $\text{WM}(B|D)$ denotes the subset of $\text{WM}(B)$ consisting of the weight modules with vertices $D$ and $b$ the Brauer correspondent of $B$ in $kH$, then $f$ induces a bijection between $\text{WM}(B|D)$ and $\text{IRR}(b)$.

The author thanks the referee for improving the proof of Proposition 4 below.

2. Weight modules over blocks with TI defect groups

To begin with, we quote the following as a preliminary lemma.

Lemma 1 (Robinson [8]). Let $T$ be a subgroup of $G$. Let $M$ (resp. $N$) be
a simple \(kG\) (resp. \(kT\))-module. Then we have \((P(M), N^G) = (P(N), M|_H)\).

Throughout this section \(D\) is assumed to be a non-trivial TI subgroup of \(G\), i.e., \(D \cap D^x = 1\) if \(x \in G \setminus H\). Let \(\text{IRR}(B) = \{M_1, \ldots, M_r\}\), \(\text{IRR}(b) = \{W_1, \ldots, W_e\}\) and \(n_i = \dim_k W_i\). We set \(\text{WM}(B|D) = \{V_i = f^{-1}(W_i); 1 \leq i \leq e\}\). Note that \(\text{WM}(B) = \text{WM}(B|D)\). In fact, let \(V \in \text{WM}(B)\) and \(Q = \text{vx}(V)\). We may assume that \(D \supseteq Q\). If \(D > Q\), then \(N_D(Q) > Q\). On the other hand, since \(D\) is a TI set, it follows that \(H \supseteq N_G(Q)\) and hence \(N_P(Q)\) is normal in \(N_G(Q)\). So, \(N_G(Q)/Q\) fails to have a block of defect zero. This is a contradiction, since \(f(V)\) is simple and projective as an \(N_G(Q)/Q\)-module.

**Lemma 2.** \(M_i|_H = f(M_i) \oplus N_i\), where \(N_i|_D\) is projective and \(f(M_i)|_D\) has no projective summand.

**Proof.** If \(L\) is an indecomposable component of \(N_i\) with vertex \(P\), then \(P\) lies in \(\mathfrak{y}(D, H)\), where

\[
\mathfrak{y}(D, H) = \{Q; Q \subseteq D^x \cap H, x \in G \setminus H\}.
\]

By the Mackey decomposition theorem we have

\[
(L \otimes_P kH)|_D = \bigoplus_{y \in P^D \cap H} (L \otimes_P kH)|_{P^y \cap D} kD.
\]

There is \(x \in G \setminus H\) such that \(P \subseteq D^x \cap H\). Hence for any \(y \in H\), we have

\[
P^y \cap D \subseteq D^{xy} \cap D \cap H = 1, \text{ as } xy \in G \setminus H.
\]

Therefore \((L \otimes_P kH)|_D\) is projective. Since \(L|_D \otimes_P kH\), \(L|_D\) is also projective.

We next show that \(f(M_i)|_D\) is projective-free. Actually, this is a general fact. Note that \(f(M_i)\) belongs to \(b\) and \(b\) has the normal defect group \(D\). So, it suffices to show that if \(L\) is a non-projective indecomposable \(b\)-module, then \(L|_D\) is projective-free. But since \(L\) is \(D\)-projective, this is a routine work, using Mackey decomposition.

**Lemma 3.** \(\text{Hom}_{kG}(M_i, V_j) \simeq \text{Hom}_{kH}(f(M_i), W_j)\) for all \(i, j\).

**Proof.** There is an isomorphism

\[
\text{Hom}_{kG}(M_i, V_j) / \text{Tr}^G_{\mathfrak{x}}(M_i, V_j) \simeq \text{Hom}_{kH}(f(M_i), W_j) / \text{Tr}^H_{\mathfrak{x}}(f(M_i), W_j),
\]

where \(\mathfrak{x} = \mathfrak{x}(G, H) = \{Q; Q \subseteq D^x \cap D, x \in G \setminus H\}\). However, since \(D\) is a TI set, we have \(\mathfrak{x} = \{1\}\). And if \(M\) and \(V\) are non-projective indecomposable and if one of them is simple, then \(\text{Tr}^G_{\mathfrak{x}}(M, V) = 0\), whence the result follows.
Proposition 4. Let \( \varepsilon \) be the block idempotent of \( B \). Then we have

\[
(k_D)^G \varepsilon \cong \bigoplus_{i=1}^e n_i V_i \oplus \bigoplus_{i=1}^r a_i P(M_i), \text{ with } a_i = (kD, M_{i\mid D}).
\]

Proof. Let

\[
k[H/D] = \sum_{i=1}^m n_i W_i
\]

be an indecomposable decomposition. Note that no \( W_j \) belongs to \( b \) if \( j \geq e+1 \). Since \( D \) is a TI set, we have

\[
W_j^G = f^{-1}(W_j) \oplus \text{(projectives)}.
\]

Moreover we know by Green's theorem that \( V_j^G = f^{-1}(W_j) \) does not belong to \( B \) if \( j \geq e+1 \). Thus

\[
(k_D)^G = (k_D^H)^G = k[H/D]^G = \bigoplus_{i=1}^e n_i V_i \oplus \bigoplus_{j=e+1}^m n_j V_j \oplus \text{(projectives),}
\]

whence we have

\[
(k_D)^G \varepsilon = \bigoplus_{i=1}^e n_i V_i \oplus \bigoplus_{i=1}^r a_i P(M_i), \text{ with } a_i \geq 0
\]

and by Lemma 1, \( a_i = (kD, M_{i\mid D}) \) for \( i = 1, 2, \ldots, r \).

Theorem 5. Assume that \( D \) is a TI set and that \( \text{hd}(f(M_i)) \) is simple for all \( i \). Then we have the following:

1. \( l(B) \geq l(b) \);
2. the equality sign in the above holds if and only if \( \text{soc}(V_i) \) is simple for all \( i \) \( (1 \leq i \leq e) \), in which case we have that

\[
\text{soc}(V_i) \cong \text{soc}(V_j) \text{ if and only if } V_i \cong V_j.
\]

Proof. From the assumption we may set \( \text{hd}(f(M_i)) = W_{i\tau(i)} \) \( (1 \leq i \leq r, 1 \leq \tau(i) \leq e) \). By lemma 3 we find easily that

(i) \( M_{i\mid \text{soc}(V_{i\tau(i)})} \) with multiplicity one.
(ii) \( M_{i\mid \text{soc}(V_j)} \), then \( j = \tau(i) \).

Now, the second assertion yields that the map

\[
\tau: \{1, 2, \ldots, r\} \rightarrow \{1, 2, \ldots, e\}
\]
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is a surjection. In fact, for an arbitrary \( V_j \), take \( M_j \) such that \( M_j \mid \text{soc}(V_j) \). Then \( j=\tau(i) \). Thus \( \tau \) is surjective. In particular, we have that \( r \geq e \).

To show the second part of the theorem, suppose that \( \text{soc}(V_j) \) is simple for all \( j \). Then \( M_j \mid \text{soc}(V_{\tau(j)}) \) and hence \( \tau \) is a bijection. Therefore we have \( r = e \). If, conversely, \( r = e \), then \( \tau \) is a bijection. This implies by (ii) above that \( \text{soc}(V_j) \) must be simple and \( V_j \cong V_j \) if and only if \( \text{soc}(V_j) \cong \text{soc}(V_j) \).

Remark 1. If the Alperin conjecture is true, we always have \( l(B) = l(b) \) when \( D \) is a TI set.

3. Weight modules for the symmetric group \( S_p \)

In this section we assume that \( G=S_p \) is the symmetric group on \( p \) letters. If \( D \) is a Sylow \( p \)-subgroup of \( G \), then \( D \) has order \( p \) and \( C_G(D) = D, H/D \cong (Z/(p))^* \), the group of units of \( Z/(p) \). In particular, it follows that \( b=kH \) is the block of \( kH \). Let us write

\[
\text{IRR}(b) = \{ W_0, \cdots, W_{p-2} \}, \text{ where } \dim_k W_i = 1 \ (0 \leq i \leq p-2).
\]

If \( B \) denotes the principal block of \( G \), then \( B \) is a unique block of \( kG \) of positive defect and \( l(B) = p-1 \). The decomposition matrix of \( B \) is known. It can be displayed as follows, see James [5].

\[
\begin{array}{cccc}
\varphi_0 & \varphi_1 & \varphi_2 & \varphi_{p-2} \\
\chi_0 = (p) = 1_G & 1 & & \\
\chi_1 = (p-1,1) & 1 & 1 & 0 \\
\chi_2 = (p-2,1^2) & 1 & 1 & \\
\vdots & & & \\
\chi_{p-2} = (2,1^{p-2}) & 0 & 1 & 1 \\
\chi_{p-1} = (1^p) & 1 & & \\
\end{array}
\]

Since \( \chi_i = \varphi_{i-1} + \varphi_i \) and \( \deg \chi_i = p-2 C_i \), we find via induction that \( \deg \varphi_i = p-2 C_i \) \( (0 \leq i \leq p-2) \). So we can label the simple modules is \( B \) such that

\[
\text{IRR}(B) = \{ M_0, \cdots, M_{p-2} \}, \text{ with } m_i = \dim_k M_i = p-2 C_i.
\]

Here we note the following facts on binomial coefficients \( C_i \).

Lemma 6.

\[
(1) \quad m_i = p-2 C_i = \begin{cases} 
& i+1 \text{ mod } p, \quad \text{if } i \text{ is even;} \\
& p-i-1 \text{ mod } p, \quad \text{if } i \text{ is odd.}
\end{cases}
\]
Suppose that \( n \geq 4 \). If \( 2 \leq i \leq n-2 \), then \( nC_i \geq n+2 \).

Now, since \( H/D \) is abelian, every principal indecomposable module over \( kH \) has dimension \( p \), and thus every non-projective indecomposable module has dimension smaller than \( p \). In particular, it follows that \( \dim_k f(M_i) < p \). By Lemma 2, we can write

\[
M_{iD} = f(M_i|D) \oplus a_k kD.
\]

For \( i = 0, 1, p - 3 \) or \( p - 2 \), we have that \( m_i < p \) and so \( a_i = 0 \). This is true for all \( i \), provided \( p \leq 5 \). Suppose \( p > 5 \). If \( 2 \leq i \leq p - 4 \), then \( m_i \geq p \) by Lemma 6(2) and hence \( a_i > 0 \). This, together with Lemma 6(1) yields that \( \dim_k f(M_i) = i + 1 \) or \( p - i - 1 \) according as whether \( i \) is even or odd \( (2 \leq i \leq p - 4) \). Thus we have:

\[
a_i = \begin{cases} 
(m_i - i - 1)/p, & \text{if } i \text{ is even;} \\
(m_i - (p - i - 1))/p, & \text{if } i \text{ is odd.}
\end{cases}
\]

Now we have the following result by Lemma 1 and Proposition 4.

**Proposition 7.** Let \( \text{WM}(B) = \{ V_0, \ldots, V_{p-2} \} \), where \( V_i = f^{-1}(W_i) \), and let \( \{ U_j; 1 \leq j \leq q \} \) be the set of simple \( kG \)-modules belonging to the blocks of defect zero. Then we have

\[
(k_B)^G \cong \bigoplus_{i=0}^{p-2} V_i \oplus \bigoplus_{i=2}^{p-4} a_i P(M_i) \oplus \bigoplus_{i=1}^q \left( \dim_k U_i / p \right) U_i.
\]

### 4. Socles of weight modules

In view of Theorem 5, it seems to be natural to consider the following situation:

(1) Every weight module belonging to \( B \) has a simple socle, and for \( U, V \in \text{WM}(B) \), we have

\[
\text{soc}(U) \cong \text{soc}(V) \quad \text{if and only if } U \cong V.
\]

We first remark that

**Proposition 8.** If \( G \) is a simple group with a cyclic Sylow \( p \)-subgroup, the condition (1) holds for every block \( B \).

In fact we know that a Sylow \( p \)-subgroup is a TI set (Blau[2]) and that \( l(B) = l(b) \), hence the result follows from Theorem 5.

On the other hand, we have the following, as is shown on pp.370–371 in
Alperin [1].

**Proposition 9** (Alperin). Let $G$ be a finite group of Lie type of characteristic $p$. Then the condition ($\#$) holds for every block $B$.

Before proceeding let us recall that a simple module is a weight module if and only if it has trivial source (Okuyama [7]).

Now, for the rest of this paper we assume that $G$ is solvable. In this case the Alperin conjecture has been proved by Okuyama.

**Definition.** A solvable group $G$ is said to be $p'$-supersolvable if all of its chief composition factors of order prime to $p$ are cyclic.

**Proposition 10.** If $G$ is $p'$-supersolvable, every simple module has trivial source. Hence $WM(B) = IRR(B)$ for every block $B$.

Proof. Let $G$ be a counter-example of minimum order and let $V$ be a simple $kG$-module with source not isomorphic to $k$. Let $K$ be a maximal abelian normal $p'$-subgroup of $G$ and $W$ a simple summand of $V|_K$. By Fong's reduction and the minimality of $G$, $W$ must be $G$-invariant. So $W$ is faithful as $K$-module and hence $K$ must be central. If $O_p(G/K) = 1$, $G/K$ has a cyclic normal $p'$-subgroup, say $M/K$. Then $M$ is abelian, contradicting the choice of $K$. Thus $O_p(G/K) > 1$, which implies that $O_p(G) > 1$, since $K$ is central. This is a contradiction.

The second statement is clear since the number of weight modules belonging to $B$ equals $l(B)$.

Now we give a definition:

**Definition.** A finite group is said to be a CR1-group if all of its characteristic abelian subgroup are cyclic.

We say that the group $G$ involves a group $T$ provided there are subgroups $L > M$ of $G$ such that $L/M \simeq T$. For a prime number $q$, let us denote a Sylow $q$-subgroup of $G$ by $G_q$.

**Theorem 11.** Let $G$ be a solvable group and suppose that $G_q$ involves no non-abelian CR1-group for each prime $q$ different from $p$. Then the conclusion of Proposition 10 holds.

Proof. We shall show that every simple $kG$-module has trivial source by the induction on the order of $G$. We may assume $O_p(G) = 1$. Let $K$ be the Fitting subgroup of $G$, so we have that $C_G(K) \subseteq K$. If $K$ is cyclic, $Aut(K)$ is abelian and
so is $G/K$. Thus $G$ is supersolvable and the result follows from Proposition 10. If $K$ is non-cyclic, our assumption implies that $G$ has a non-cyclic abelian normal $q$-subgroup, say $L$, for some prime $q$. Let $V$ be a simple $kG$-module and $W$ a simple summand of $V_K$. If the inertial group of $W$ is proper, the result follows by induction. If $W$ is $G$-invariant, $N=\ker(W)$ is a non-trivial normal subgroup of $G$. Then we get the result by applying the inductive hypothesis to $G/N$.

**Remark 2.** The CR1-$q$-groups are classified (Gorenstein [3], Chap. 5). In particular, a non-abelian CR1-$q$-group contains $D_3$ or $Q_3$ if $q=2$, while it contains $M(q)$ if $q$ is odd, where

$$M(q) = \langle x, y, z; x^q = y^q = z^q = 1, [x, z] = [y, z] = 1, [x, y] = z \rangle,$$

which has order $q^3$ and exponent $q$.

**Remark 3.** One may show that the following $q$-group $Q$ involves no non-abelian CR1-$q$-group:

$$Q = \langle x, y; x^{aq} = y^{bq} = 1, x^q = x^{1 + a^{-1}} \rangle,$$

where $a \geq 2$, $b \geq 1$, and $a \geq 3$ if $q = 2$.

In fact every proper subgroup of $Q$ is abelian (cf. Huppert [4] III, Aufgaben 22). So it suffices to show that $Q$ has no factor group isomorphic to $D_3$, $Q_3$ or $M(q)$, which will be easily done.

**Remark 4.** Let $G = \langle \sigma \rangle$ be the semidirect product, where $\sigma$ is an automorphism of the quaternion group $Q_2$ of order 3. Then $kG$ has a simple module whose source is not trivial, where $k$ is of characteristic 3. On the other hand, if $G$ is the symmetric group $S_4$, every simple $kG$-module has trivial source, $k$ being the same as above. In both groups the Sylow 2-subgroups are CR1-groups.

References

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