NON-LINEARIZABLE REAL ALGEBRAIC ACTIONS
OF $O(2, \mathbb{R})$ ON $\mathbb{R}^4$

HIROYUKI MIKI

(Received March 18, 1994)

0. Introduction

In algebraic transformation groups, one of the important problems is the following.

Linearization problem ([6]). Let $G$ be a reductive complex algebraic group. Is any algebraic $G$ action on affine space $\mathbb{C}^n$ linearizable, i.e. isomorphic to some $G$ module as $G$ variety?

Some positive answers to this problem have been given (see [1] for a survey article) but in 1989, G.W. Schwarz [17] constructed counterexamples for many noncommutative groups with $O(2,\mathbb{C})$ being the most explicit case (in the case that the acting group is commutative, any counterexample have never found, and see [7], [9], [11], [12] for further recent results).

In this paper, we consider the analogous problem in the real algebraic category, which was posed in [15]. Then it would be appropriate to take a compact Lie group as acting group since there is a one-to-one correspondence between the family of compact Lie groups and that of reductive complex algebraic groups through the complexification (see [14] p.247).

Schwarz used the properties of complex algebraic geometry to find the counterexamples, so it is not clear whether his argument works in the real algebraic category because $\mathbb{R}$ is not algebraically closed. We use the methods of Masuda-Petrie [11] to obtain the following result.

Theorem. There is a continuous family of algebraically inequivalent, non-linearizable real algebraic $O(2,\mathbb{R})$ actions on $\mathbb{R}^4$.

Let $G$ be a compact real algebraic group and $G_c$ be the reductive complex algebraic group obtained from $G$ via the complexification. Let $\text{ACT}(G,\mathbb{R}^n)$ (resp. $\text{ACT}(G_c,\mathbb{C}^n)$) be the set of equivalence classes of real algebraic $G$ actions on $\mathbb{R}^n$ (resp. complex algebraic $G_c$ actions on $\mathbb{C}^n$), where the equivalence relation is defined by $G$ variety (resp. $G_c$ variety) isomorphism. Then there is a complexification map
$c_a : ACT(G,R^n) \rightarrow ACT(G_C,C^n)$.  

It is natural to ask that $c_a$ is injective, but it turns out that the examples in the theorem above give a negative answer to this question.

**Proposition.** The map $c_a$ is not injective.

This paper is organized as follows. We consider the relation between the linearization problem and algebraic $G$ vector bundles in section 1 and construct non-trivial real (affine) algebraic $O(2,R)$ vector bundles in section 2. In section 3 we consider the complexification of real algebraic $G$ vector bundles and that of algebraic actions. In section 4 we prove the theorem above using vector bundles constructed in section 2, and apply the complexifications to the examples in the theorem. We give an explicit description of a non-linearizable real algebraic $O(2,R)$ action in the appendix. Most of the results in this paper are from the author’s master thesis [13].

**ACKNOWLEDGEMENT.** I would like to thank Professor Mikiya Masuda heartily for not only his many helpful suggestions for writing this paper but his lead to my research. I would also like to thank Professor Sung Sook Kim for her many useful suggestions for writing this paper.

1. Algebraic $G$ vector bundles and non-linearizable actions

Let $K$ be the real numbers $R$ or the complex numbers $C$. We say that $X (\subset K^n)$ is an affine variety if $X$ is the set of the zeros of a map from $K^n$ to some $K^m$ whose coordinate functions are polynomials, and we say that $f : X \rightarrow Y$, where $X (\subset K^n)$ and $Y (\subset K^m)$ are affine varieties, is an algebraic map if $f$ extends to a map from $K^n$ to $K^m$ whose coordinate functions are polynomials. A group $G$ is an algebraic group if $G$ is an affine variety and the map $\phi : G \times G \rightarrow G$ defined by $(g_1,g_2) \mapsto g_1g_2^{-1}$ is algebraic, $X$ is an (affine) $G$ variety if $X$ is an affine variety and the action map $\phi : G \times X \rightarrow X$ is algebraic, and $f : X \rightarrow Y$ is an algebraic $G$ map (here $X$ and $Y$ are $G$ varieties) if $f$ is algebraic and $G$ equivariant. An algebraic $G$ map is an algebraic $G$ isomorphism if it is bijective and its inverse is also an algebraic $G$ map. Two $G$ varieties are isomorphic if there is an algebraic $G$ isomorphism between them.

Let $G$ denote an algebraic group over $K$ and let $B$, $F$, $S$ denote $G$ modules over $K$ whose representation maps $(G \times B \rightarrow B$ etc.) are algebraic.

**Definition 1.1.** Let $Vec(B,F;S)$ be the set of algebraic $G$ vector bundles $E$ over $B$ such that $E \oplus S$ is isomorphic to $F \oplus S$ as algebraic $G$ vector bundle, where $F = B \times F$ and $S = B \times S$ are product bundles over $B$. We define $VEC(B,F;S)$ to be the set of isomorphism classes of elements in $Vec(B,F;S)$ as algebraic $G$ vector
We recall some results about $Vec(B,F;S)$ from [11]. The following results are established in [11] when $K = C$. But the same argument works when $K = R$.

**Definition 1.2.** Let $\text{sur}(F \oplus S, S)$ be the set of algebraic $G$ vector bundle surjections $L: F \oplus S \to S$ which allow an algebraic $G$ splitting map from $S$ to $F \oplus S$, and let $\text{aut}(F \oplus S)$ be the group of algebraic $G$ vector bundle automorphisms $\tau$ of $F \oplus S$.

**Remark.** In the complex category, any algebraic $G$ vector bundle surjection from $F \oplus S$ to $S$ has a splitting (see [2]). But in the real category, this is not the case. For example, $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $(a,b) \mapsto (a, (a^2 + 1)b)$ has no splitting, where $\mathbb{R} \times \mathbb{R}$ is viewed as a trivial bundle with the projection on the first factor $\mathbb{R}$.

The group $\text{aut}(F \oplus S)$ acts on $\text{sur}(F \oplus S, S)$ by $L \mapsto L \circ \tau$ and $L \in \text{sur}(F \oplus S, S)$ defines an element $\ker L$ in $Vec(B,F;S)$.

**Theorem 1.3 ([11]).** The map sending $L \in \text{sur}(F \oplus S, S)$ to $\ker L \in Vec(B,F;S)$ induces a bijection

$$\text{sur}(F \oplus S, S)/\text{aut}(F \oplus S) \cong Vec(B,F;S).$$

Because of the solution of the Serre conjecture (see [16], [19]), any vector bundle $E \in Vec(B,F;S)$ is trivial if we forget the actions. So $E$ gives an algebraic $G$ action on some $K^n$. We consider the classification of (the total spaces of) elements in $Vec(B,F;S)$ as $G$ varieties.

**Definition 1.4.** Let $\text{VAR}(B,F;S)$ be the set of isomorphism classes of elements in $Vec(B,F;S)$ as $G$ varieties. Let $\text{Aut}(B)^G$ be the group of $G$ variety automorphisms of $B$.

The group $\text{Aut}(B)^G$ acts on $Vec(B,F;S)$ by taking pull back bundles and the trivial element in $Vec(B,F;S)$ is fixed under the action. One easily sees that the natural map from $Vec(B,F;S)$ to $\text{VAR}(B,F;S)$ factors through the map

$$Vec(B,F;S)/\text{Aut}(B)^G \to \text{VAR}(B,F;S).$$

This map is often (but not always) bijective ([11]). We recall a sufficient condition for the above map to be bijective.

**Definition 1.5.** Let $E_1, E_2 \in Vec(B,F;S)$ and let $f: E_1 \to E_2$ be a $G$ variety isomorphism. We say that $f$ maps $B$ as graph if the composition $pfs: B \to B$ is...
in \( \text{Aut}(B)^G \), where \( p: E_2 \to B \) is the projection and \( s: B \to E_1 \) is the zero-section.

**Theorem 1.6 ([11]).** Suppose that any \( G \) variety isomorphism between elements in \( \text{Vec}(B,F;S) \) maps \( B \) as graph. Then the natural map: \( \text{VEC}(B,F;S) \to \text{VAR}(B,F;S) \) induces a bijection

\[
\text{VEC}(B,F;S)/\text{Aut}(B)^G \cong \text{VAR}(B,F;S).
\]

In particular, if \( E \in \text{Vec}(B,F;S) \) is non-trivial, then the \( G \) action on \( E \) is non-linearizable.

2. **Non-trivial \( O(2,R) \) vector bundles**

In this section we show that \( \text{VEC}(B,F;S) \) can be non-trivial. Let \( O(2,R) \) be the real orthogonal group. We identify it with \( S^1 \times Z_2 \). Define a two dimensional real \( O(2,R) \) module \( W_n = \{(a,\bar{a}); a \in C \} \) \((n \in N)\) as follows (here \( \bar{a} \) denotes the complex conjugate of \( a \)). For \( g \in S^1 \) and \( 1 \neq j \in Z_2 \), the representation map is defined by

\[
g \mapsto \begin{pmatrix} g^n & 0 \\ 0 & \bar{g}^n \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

**Theorem 2.1.** There exists a bijection: \( \text{VEC}(W_1,W_m;R) \cong R^{m-1} \).

In order to prove this theorem, we use Theorem 1.3. We first calculate \( \text{sur}(W_m \oplus R,R) \) and \( \text{aut}(W_m \oplus R) \).

**Lemma 2.2.** (1) Any surjection \( L \in \text{sur}(W_m \oplus R,R) \) is of the following form on the fiber over \( (a,\bar{a}) \in W_1 \);

\[
L(a,\bar{a}) = (f \bar{a}^m f a^m h),
\]

where \( f, h \) are relatively prime polynomials of \( t = |a|^2 \) with real coefficients and \( h(0) \neq 0 \).

(2) Any automorphism \( \tau \in \text{aut}(W_m \oplus R) \) is of the following form on the fiber over \( (a,\bar{a}) \in W_1 \);

\[
\tau(a,\bar{a}) = \begin{pmatrix} u & a^{2m} & a^m s \\ \bar{a}^{2m} & u & \bar{a}^m s \\ a^{mr} & a^m r & w \end{pmatrix},
\]

where \( u, w, l, r, s \) are polynomials of \( t = |a|^2 \) and \( u, w \) are congruent to non-zero constants modulo \( t^m \).

**Proof.** (1) \( L \) is linear relative to each coordinate of \( W_m \) and \( R \), so one can write

\[
L(a,\bar{a}) = (L_1(a,\bar{a}), L_2(a,\bar{a}), L_3(a,\bar{a})),
\]
where \( L_i \) is a polynomial for \( i = 1, 2, 3 \). The \( S^1 \) equivariance of \( L \) means that
\[
L_1(ga, ga) = g^m L_1(a, \bar{a}), \quad L_2(ga, ga) = g^m L_2(a, \bar{a}), \quad L_3(ga, ga) = L_3(a, \bar{a}).
\]
An elementary computation shows that these imply
\[
L_1(a, \bar{a}) = f_1(t)a^m, \quad L_2(a, \bar{a}) = f_2(t)a^m, \quad L_3(a, \bar{a}) = h(t)
\]
for some polynomials \( f_1, f_2 \) and \( h \) with real coefficients. The \( \mathbb{Z}_2 \) equivariance shows that \( f_1 \) coincides with \( f_2 \), which we denote by \( f \). The property that \( f \) and \( h \) are relatively prime follows from the existence of a splitting of \( L \) and that \( h(0) \) is non-zero follows from the surjectivity of \( L \).

(2) Because of \( O(2, \mathbb{R}) \) equivariance, one can check that \( \tau \) is of the form in the statement. Since \( \tau \) is an automorphism,
\[
\det(\tau(a, \bar{a})) = (u - t^m)(uw - 2t^mrs + t^mhw)
\]
must be a unit polynomial, which is a non-zero constant. So each factor at the right hand side is also a non-zero constant. It follows that \( u \) and \( uw \) are congruent to non-zero constants modulo \( t^m \), hence so is \( w \).

**NOTATION.** Let \( L_{f, h} \) denote \( L \) in Lemma 2.2 (1) and \( E(f, h) \) denote the kernel of \( L_{f, h} \). We abbreviate \( E(1, h) \) as \( E(h) \). Then the vector bundle \( E(h) \) (with the obvious projection on \( W_1 \)) is written as follows;
\[
E(h) = \{(a, \bar{a}, x, \bar{x}, z) \in W_1 \times W_m \times \mathbb{R}; \bar{a}^m x + a^m \bar{x} + h(t)z = 0\}.
\]
Note that if \( h \) is a non-zero constant, \( E(h) \) is isomorphic to \( W_m \) through the correspondence \((a, \bar{a}, x, \bar{x}, z) \mapsto (a, \bar{a}, x, \bar{x})\).

**Lemma 2.3.** There are three vector bundle isomorphisms.
(1) \( E(f, h) \cong E(f, h / h(0)) \).
(2) \( E(f, h) \cong E(h) \).
(3) \( E(h_1) \cong E(h_2) \) if and only if there is a non-zero constant \( c \) such that \( h_1 \equiv ch_2 \) modulo \( t^m \).

**Proof.**
(1) \((x, \bar{x}, z) \mapsto (x, \bar{x}, h(0)z)\) is the required isomorphism.
(2) By Theorem 1.3 and Lemma 2.2 (2), it suffices to show the existence of polynomials \( u, w, l, r, s \) such that
\[
(a^m \ a^m \ h) = (f a^m \ f a^m \ h) \left( \begin{array}{ccc} u & a^{2m}l & a^ms \\ a^{2m}l & u & a^ms \\ a^mr & a^ms & w \end{array} \right)
\]
and that the determinant of the above \( 3 \times 3 \) matrix is a non-zero constant. Choose polynomials \( \xi \) and \( \eta \) of \( t \) such that \( f \xi + h\eta = 1 \) (this is possible since \( f \) and \( h \) are
relatively prime by Lemma 2.2 (1)) and polynomials \( r' \) and \( r'' \) of \( t \) such that \( hr' = (1 - f) - t^m r'' \) (this is possible since \( h(0) \neq 0 \) by Lemma 2.2 (1)). Then one can check that

\[
u = 1 + t^m l, \quad w = 1 - 2t^m f l, \quad s = hl, \quad l = \xi r''/2, \quad r = r' + t^m \eta r''
\]
satisfies the required conditions.

(3) If \( E(h_1) \cong E(h_2) \) there is \( \tau \in aut(W_m \oplus R) \) such that \( L_{1,h_1} = L_{1,h_2} \circ \tau \), i.e.

\[
(a^m a^m h_1) = (a^m a^m h_2)
\]

where the determinant of the above \( 3 \times 3 \) matrix is a non-zero constant. Hence

\( h_1 = h_2w + 2t^m s \). Since \( w \) is a non-zero constant modulo \( t^m \) by Lemma 2.2 (2), the necessity is clear. Conversely if \( h_1 = ch_2 + t^m h_0 \) for some polynomial \( h_0 \) of \( t \), then \( \tau \in aut(W_m \oplus R) \) defined by

\[
\tau(a, \bar{a}) = \begin{pmatrix} 1 & 0 & \bar{a}^m h_0/2 \\ 0 & 1 & \bar{a}^m h_0/2 \\ 0 & 0 & c \end{pmatrix}
\]
is the isomorphism between \( E(h_1) \) and \( E(h_2) \).

Proof of Theorem 2.1. By Theorem 1.3 and Lemma 2.2 (1), any element in \( VEC(W_1,W_m \oplus R) \) is of the form \([E(f,h)]\), where \([ ]\) denotes the isomorphism class. Then Lemma 2.3 implies that the correspondence

\[
R^{m-1} \ni (a_1, \ldots, a_{m-1}) \mapsto [(E(h)]
\]

where \( h(t) = 1 + a_1 t + \cdots + a_{m-1} t^{m-1} \), gives the bijection.

3. Complexification

In this section, we assume that \( G \) is a real algebraic group and \( B, F, S \) are real \( G \) modules. We first define the complexification of real affine varieties and algebraic maps and prove some properties.

**Definition 3.1.** Let \( X(\subset R^n) \) be a real affine variety and let \( l(X) \) be the ideal of polynomial maps from \( R^n \) to \( R \) which vanish on \( X \). We define the complex affine variety \( X_C \) to be the common zeros of all the elements in \( l(X) \) regarded as maps from \( C^n \) to \( C \), and we call \( X_C \) the complexification of \( X \).

Here are some elementary properties about the complexification.
Proposition 3.2. (1) Let \( I(X) \) be the ideal of polynomial maps from \( \mathbb{C}^n \) to \( \mathbb{C} \) which vanish on \( X \). Then \( I(X) = I(X) \otimes \mathbb{C} \).

(2) \((X \times Y)_c = X_c \times Y_c\).

(3) Any algebraic map \( f: X \to Y \) extends to a unique algebraic map \( f_c: X_c \to Y_c \).

Proof. (1) It is clear that \( I(X) \supset I(X) \otimes \mathbb{C} \) by definition. We prove the opposite inclusion. For \( f \in I(X_c) \), we express \( f = f_1 + if_2 \), where \( f_1 \) and \( f_2 \) are polynomials with real coefficients. Then \( f_1|_X + if_2|_X = f|_X = 0 \), so \( f_1 \) and \( f_2 \) are in \( I(X) \). This means that \( I(X) \subset I(X) \otimes \mathbb{C} \).

(2) The ideal \( I(X \times Y) \) is generated by the elements \( h_i h_s \), where \( f_i \in I(X) \) and \( h_s \in I(Y) \). This together with (1) shows that the ideal \( I((X \times Y)_c) \) is generated by the elements \( f_i h_s \), where \( f_i \in I(X) \) and \( h_s \in I(Y) \). This implies (2).

(3) Suppose \( X \subset \mathbb{R}^n \) and \( Y \subset \mathbb{R}^m \) and let \( F: \mathbb{R}^n \to \mathbb{R}^m \) be an extension of \( f \). We regard \( F \) as a map from \( \mathbb{C}^n \) to \( \mathbb{C}^m \). One easily checks that \( F \) maps \( X \) to \( Y \). Therefore \( F|_{X_c}: X_c \to Y_c \) is an extension of \( f \). Now we prove the uniqueness. Suppose that two maps \( f_1, f_2: X_c \to Y_c \) are extensions of \( f \). Let \( F_j: \mathbb{C}^n \to \mathbb{C}^m \) be an extension of \( f_j \) \( (j = 1, 2) \). Then \( F_1 - F_2 \) is algebraic and vanishes on \( X \). Therefore \( F_1 - F_2 \) vanishes on \( X_c \) by (1). Hence \( f_1 - f_2 = (F_1 - F_2)|_{X_c} = 0 \), i.e. \( f_1 = f_2 \). \( \square \)

We call \( f_c \) the complexification of \( f \). By Proposition 3.2, we obtain the following.

Corollary 3.3. (1) The complexification of a real algebraic group is a complex algebraic group.

(2) If \( G \) is a real algebraic group and \( X \) is a real \( G \) variety, \( X_c \) is a complex \( G_c \) variety.

(3) If \( X \) and \( Y \) are real \( G \) varieties and \( f: X \to Y \) is \( G \) equivariant, then \( f_c: X_c \to Y_c \) is \( G_c \) equivariant.

(4) If \( f: X \to Y \) and \( h: Y \to Z \) are algebraic \( G \) maps between real \( G \) varieties, then \( (f \circ h)_c = f_c \circ h_c \).

Now we define a complexification of elements in \( VEC(B,F;S) \) and an involution on \( VEC(B_c,F_c;S_c) \). Note that the usual complexification of vector bundles means to complexify only fibers, but our definition means to complexify also base space. Let \( L \) be an element in \( \text{sur}(F \oplus S, S) \). The map \( L_c: (F \oplus S)_c \to S_c \) is \( G_c \) equivariant and has a splitting because if \( P \) is an algebraic \( G \) splitting of \( L \) then \( P_c \) is an algebraic \( G_c \) splitting of \( L_c \). Hence \( L_c \) is in \( \text{sur}(F \oplus S)_c, S_c \). Let \( L' \) be another element of \( \text{sur}(F \oplus S, S) \). If \( L' = L \circ \tau \) for some \( \tau \in \text{aut}(F \oplus S) \), then \( L'_c = L_c \circ \tau_c \) and \( \tau_c \in \text{aut}(F \oplus S)_c \). Therefore the following definition makes sense, i.e. it does not depend on the choice of \( L \).

Definition 3.4. Let \( [E] \in VEC(B,F;S) \) and let \( L \in \text{sur}(F \oplus S, S) \) represent \( E \), i.e.
Then we define the complexification of $[E]$ by $[\ker L] \in VEC(B_C, F_C; S_C)$.

Let $X( \subset R^n)$ be a real $G$ variety. For $x \in X_C (\subset C^n)$, the complex conjugation $\bar{x}$ is also in $X_C$ since $f(\bar{x}) = 0$ for any $f \in I(X)$. Hence $X_C$ has an involution defined by $x \mapsto \bar{x}$. Similarly, $G_C$ has an involution. Since the action map: $G \times X \to X$ is real algebraic, we have $g \cdot \bar{x} = \bar{g} \cdot \bar{x}$ for any $g \in G_C$ and $x \in X_C$.

**Definition 3.5.** For $L \in \text{sur}(F \otimes S)_{C}, S_C)$, we define $\bar{L} : (F \otimes S)_C \to S_C$ by

$$\bar{L}(b, f, s) = \overline{L(b, \overline{f}, \overline{s})}.$$ 

One can check that $\bar{L}$ is in $\text{sur}(F \otimes S)_{C}, S_C)$. So the correspondence $L \mapsto \bar{L}$ induces an involution on $VEC(B_C, F_C; S_C)$. Since $\overline{L_C} = L_C$ for $L \in \text{sur}(F \otimes S, S)$, the complexification in Definition 3.4 induces a map

$$c_b : VEC(B, F; S) \to VEC(B_C, F_C; S_C)^{\mathbb{Z}_2}.$$

We ask

**Complexification problem (vector bundle case).** Is the above map $c_b$ bijective?

We turn to the complexification of actions. Let $ACT(G, R^n)$ (resp. $ACT(G_C, C^n)$) be the set of the equivalence classes of real algebraic $G$ actions on $R^n$ (resp. complex algebraic $G_C$ actions on $C^n$), where the equivalence relation is defined by $G$ variety (resp. $G_C$ variety) isomorphism. By the complexification of real $G$ varieties, we obtain a map

$$c_a : ACT(G, R^n) \to ACT(G_C, C^n).$$

**Complexification problem (action case).** Is the above map injective?

We deal with these problems in the next section.

4. Non-linearizable actions and the complexification problems

We first classify the elements in $Vec(W_1, W_m; R)$ as $O(2, R)$ varieties, i.e. we calculate $VAR(W_1, W_m; R)$. We show that the assumption of Theorem 1.6 is satisfied.

**Lemma 4.1.** Any $O(2, R)$ variety isomorphism between elements in $Vec(W_1, W_m; R)$ maps $W_1$ as graph.

Proof. Let $E_1, E_2$ be elements in $Vec(W_1, W_m; R)$ and $f : E_1 \to E_2$ be an $O(2, R)$ variety isomorphism. We show that $p \circ f \circ s$ is in $Aut(W_1)^{O(2, R)}$, where $p : E_2 \to W_1$ is the projection and $s : W_1 \to E_1$ is the zero-section. Take the complexification


Let $f_c: (E_1)_c \rightarrow (E_2)_c$, which is an $O(2, \mathbb{C})$ variety isomorphism. According to [11], $f_c$ maps $(W_1)_c$ as graph, in fact, $p_C f_c c^*: (W''_1)_c \rightarrow (W''_1)_c$ is a non-zero scalar multiplication. We recall the proof. The map $f_C c^*$ is $O(2, \mathbb{C})$ equivariant, so it is of the form

$$(W_1)_c \ni (a,b) \mapsto (af_0, bf_0, a^m h_0, b^m h_0, k_0),$$

where $f_0, h_0$ and $k_0$ are polynomials of $t = ab$. If $f_0$ is not a non-zero constant, $f_0$ has some zero $t_0$. Let $\zeta$ be a primitive $m$-th root of 1. Then $f_C c^*$ maps $(t_0, 1)$ and $(\zeta t_0, \zeta^{-1})$ to the same element $(0, a^m h_0(t_0), b^m h_0(t_0), k_0(t_0))$, which contradicts to the injectivity of $f_C c^*$. Hence $f_0$ must be a non-zero constant. Finally, since $p_C f_c c^*$ is the complexification of $p C c^*$, it preserves $W$. This proves $p f s \in Aut(W_1)^{O(2, \mathbb{R})}$.

We can check $Aut(W_1)^{O(2, \mathbb{R})} = R^*$ using the $O(2, \mathbb{R})$ equivariance. Suppose that $E(h_1)$ is isomorphic to $E(h_2)$ as $O(2, \mathbb{R})$ varieties. Then $E(h_1)$ is isomorphic to $c^* E(h_2)$ as $O(2, \mathbb{R})$ vector bundles for some $c \in Aut(W_1)^{O(2, \mathbb{R})} = R^*$ by Theorem 1.6 and Lemma 4.1. The fiber of $c^* E(h_2)$ over $(a, \bar{a})$ is the set of points satisfying the equation; $c^*(\tilde{a}^m x + a^m \bar{x}) + h_2(c^2 t)z = 0$. Then

$$c^* E(h_2) = \{(a, \bar{a}, x, \bar{x}, z); c^*(\tilde{a}^m x + a^m \bar{x}) + h_2(c^2 t)z = 0\} \cong \{(a, \bar{a}, x, \bar{x}, z); a^m x + a^m \bar{x} + h_2(c^2 t)z = 0\}$$

by Lemma 2.3 (1). Hence $h_1(t)$ is congruent to $h_2(c^2 t)$ modulo $t^m$ by Lemma 2.3 (3) and we obtain the following bijection.

**Theorem 4.2.** $VAR(W_1, W^R_m; \mathbb{R}) \cong R^{m-1}/R^*$, where the $R^*$ action on $R^{m-1}$ is defined as follows. For $c \in R^*$ and $(a_1, \ldots, a_{m-1}) \in R^{m-1}$,

$$(a_1, \ldots, a_{m-1}) \mapsto (c^2 a_1, c^4 a_2, \ldots, c^{2(m-1)} a_{m-1}).$$

Proof of the Theorem (in introduction). By Theorem 1.6, it suffices to show that the set $VAR(W_1, W^R_m; \mathbb{R})$ can be continuous density, but Theorem 4.2 says that the case $m \geq 3$ satisfies this condition.

Next we apply the complexification defined in section 3 to the $O(2, \mathbb{R})$ case. We recall Schwarz's [17] and Masuda-Petrie's [11] results in the complex category. Here $O(2, \mathbb{C}) = C^* \times Z_2$ and its action on $(W_m)_c = \{(a, b) \in C^2 \}$ is defined as follows. For $g \in C^*$, $1 \neq J = Z_2$ and $(a, b) \in (W_m)_c$,

$$(a, b) \mapsto (g^m a, g^{-m} b) \quad (a, b) \mapsto (b, a).$$

**Theorem 4.3** ([11],[17]). $VEC(W_1)_c(W_m)_c; C) \cong C^{m-1}$, where the correspon-
Theorem 4.4 \([\text{11}]\). \(\text{VAR}(W_1)_c(W_m)_c;C) \cong C^{m-1}/C^*\), where the \(C^*\) action on \(C^{m-1}\) is defined similarly to Theorem 4.2.

We study the involution on \(\text{VEC}(W_1)_c(W_m)_c;C)\). Any element of \(\text{VEC}(W_1)_c(W_m)_c;C)\) is represented by \(L \in \text{sur}(W_m \oplus R)_c,C)\) of the form;
\[
L(a,b,x,y,z) = b^m x + a^m y + f(t)z,
\]
where \(t=ab\) and \(f\) is a polynomial with real coefficients. Then
\[
\tilde{L}(a,b,x,y,z) = \bar{L}(\bar{a},\bar{b},\bar{x},\bar{y},\bar{z}) = b^m x + a^m y + \bar{f}(t)z,
\]
where \(\bar{f}\) is a polynomial whose coefficients are complex conjugate of those of \(f\). So the involution on \(\text{VEC}((W_1)_c(W_m)_c;C)\) coincides with the complex conjugate on \(C^{m-1}\) through the bijection in Theorem 4.3. This together with Theorem 2.1 shows that the complexification map
\[
c_b : \text{VEC}(W_1,W_m;R) \to \text{VEC}((W_1)_c(W_m)_c;C)
\]
is bijective.

Now we turn to the case of actions. Remember that we have the complexification map
\[
c_a : \text{ACT}(O(2,R),R^4) \to \text{ACT}(O(2,C),C^4).
\]
The sets \(\text{VAR}(W_1,W_m;R)\) and \(\text{VAR}((W_1)_c(W_m)_c;C)\) are subsets of \(\text{ACT}(O(2,R),R^4)\) and \(\text{ACT}(O(2,C),C^4)\) respectively and \(c_a\) maps \(\text{VAR}(W_1,W_m;R)\) into \(\text{VAR}((W_1)_c(W_m)_c;C)\). Through the bijections in Theorems 4.2 and 4.4, one can see that the map \(c_a\) restricted to \(\text{VAR}(W_1,W_m;R)\) is nothing but the map from \(R^{m-1}/R^*\) to \(C^{m-1}/C^*\) induced from the natural inclusion \(R^{m-1} \subset C^{m-1}\). An elementary observation shows that the map from \(R^{m-1}/R^*\) to \(C^{m-1}/C^*\) is not injective, in fact, the inverse image of an element in \(C^{m-1}/C^*\) consists of one or two elements. This gives a negative answer to the complexification problem in the action case. However \(c_a^{-1}([0]) = [0]\), where \([0]\) denotes the element in \(R^{m-1}/R^*\) or \(C^{m-1}/C^*\) represented by 0. Since \([0]\) corresponds to a linear action, we pose

\[\text{Weak complexification problem.} \text{ If the complexification of a real algebraic action on } R^n \text{ is linearizable, then is the action itself linearizable?}\]

\[\text{Appendix}\]

We give an explicit description of a non-linearizable real algebraic \(O(2,R)\) action on \(R^4\) obtained from Theorem 4.2. For example, we take \(E(1-t^2) \in \text{Vec}(W_1,W_4;R)\).
The following (nonequivariant) algebraic vector bundle automorphism of $W_4 \oplus \mathbb{R}$ gives a trivialization of $E(1-t^2) \cong W_4 \subset W_4 \oplus \mathbb{R}$.

\[
\tau(a, \alpha) = \begin{pmatrix} 1 + it & 0 & -a^4/2 \\ 0 & 1 - it & -a^{-4}/2 \\ a^{-4} & a^4 & 1 - t^2 \end{pmatrix}.
\]

We define $\sigma : \mathbb{R}^4 \to W_4$ by $(a, b, x, y) \mapsto (a + ib, a - ib, x + iy, x - iy)$. Then it suffices to calculate the correspondence of the composition map in the following:

\[
\begin{array}{c}
\begin{array}{c}
\sigma \\
\tau^{-1} \text{ action} \\
R^4 \to W_4 \to E(1-t^2) \to E(1-t^2) \to W_4 \to \mathbb{R}^4.
\end{array}
\end{array}
\]

It turns out that the actions on $\mathbb{R}^4$ of $g = \cos \theta + i \sin \theta \in S^1$ and $1 \neq J \in Z_2 \subset O(2, \mathbb{R})$ are as follows.

\[
\begin{pmatrix} (a) \\ (b) \\ (x) \\ (y) \end{pmatrix} \mapsto \begin{pmatrix} (\cos \theta & -\sin \theta) \\ (\sin \theta & \cos \theta) \\ (a) \\ (b) \end{pmatrix} \begin{pmatrix} (\cos 4 \theta & -\sin 4 \theta) \\ (\sin 4 \theta & \cos 4 \theta) \\ (x) \\ (y) \end{pmatrix}
\]

\[
\begin{pmatrix} (a) \\ (b) \\ (x) \\ (y) \end{pmatrix} \mapsto \begin{pmatrix} (a) \\ (b) \end{pmatrix} \begin{pmatrix} (-f_2 t + 2a^4 - 2t^2 + 1 & f_1 t + t^3 - 2t^3 + 2t) \\ (-f_1 t + t^3 - 2t^3 + 2t & -f_2 t - 2a^4 + 2t^2 - 1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]

where $t = a^2 + b^2$, and $f_1, f_2$ are polynomials of $a, b$ with the real coefficients such that $(a + ib)^8 = f_1 + if_2$.

References


