ON COHOMOLOGY GROUPS OF NEF LINE BUNDLES TENSORIZED WITH MULTIPLIER IDEAL SHEAVES ON COMPACT KÄHLER MANIFOLDS

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Introduction

Let $X$ be a compact Kähler manifold of dimension $n$ provided with a Kähler metric $\omega_X$ and let $E$ be a holomorphic line bundle on $X$. $E$ is said to be numerically effective, "nef" for short, if the real first Chern class $c_{R,1}(E)$ of $E$ is contained in the closure of the Kähler cone of $X$. If $X$ is projective algebraic, then $E$ is nef if and only if $C \cdot E = \int_C c_{R,1}(E) \geq 0$ for any irreducible reduced curve $C$ of $X$ (cf.[13], §2 and [1], §6).

If $E$ is nef, then for a fixed smooth metric $h_E$ of $E$ and a given sequence of positive numbers $\{\varepsilon_k\}_{k \geq 1}$ decreasing to zero, there exists a sequence of real-valued smooth functions $\{\varphi_k\}_{k \geq 1}$ such that every form $\Theta_E + dd^c\varphi_k + \varepsilon_k\omega_X$ yields a Kähler metric. Here $\Theta_E$ is the curvature form of $E$ relative to $h_E$ defined by $\Theta_E = dd^c(-\log h_E)$ with $d^c = \sqrt{-1}(\bar{\partial} - \partial)/2$. Normalizing $\varphi_k$ in such a way that $\sup_X \varphi_k = 0$, we can show that $\varphi_k$ converges to an integrable function $\varphi_\infty$ on $X$ so that $\Theta_E + dd^c\varphi_\infty$ is a positive current (cf. §2, Proposition 2.5). Such an integrable function $\varphi_\infty$ is said to be almost plurisubharmonic. In general $\varphi_\infty$ has singularities and $e^{-\varphi_\infty}$ is not integrable on $X$ (cf. [11], [18]), which implies that the nefness is strictly weaker than the semi-positivity of line bundle in the sense of Kodaira (cf. [4], Example 1.7). Hence we can define a coherent analytic sheaf of ideal $\mathcal{I}(\varphi_\infty)$ associated to $\varphi_\infty$ whose zero variety (possibly empty) is the set of points in a neighborhood of which $e^{-\varphi_\infty}$ is not integrable. The sheaf $\mathcal{I}(\varphi_\infty)$ is called the multiplier ideal sheaf associated to $\varphi_\infty$ and we obtain the canonical homomorphism $\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \otimes \Omega^n_X(E)) \rightarrow H^q(X, \Omega^n_X(E))$ induced by $\iota(\varphi_\infty) : \mathcal{I}(\varphi_\infty) \otimes \Omega^n_X(E) \rightarrow \Omega^n_X(E)$.

Though $\varphi_\infty$ can not be uniquely determined generally, the study of $H^q(X, \mathcal{I}(\varphi_\infty) \otimes \Omega^n_X(E))$ is deeply related to several interesting problems in analytic and algebraic geometry (cf. [2], [3], [11], [12], [18]). Nevertheless not much is known about the cohomology group except a vanishing theorem for multiplier ideal sheaves associated to nef and big line bundles by Nadel (cf. [11]). We study the cohomology group by establishing a certain harmonic representation theorem. In particular we...
can determine the structure of $\text{Image} \iota^q(\varphi_\infty)$. As a consequence we can get the following Lefschetz type theorem (cf. [5], Theorem 0.3).

**Theorem 1.** Let $X$ be a compact Kähler manifold of dimension $n$ provided with a Kähler metric $\omega_X$ and let $E$ be a nef line bundle on $X$ provided with a smooth hermitian metric $h_E$. Let $\varphi_\infty$ be an integrable function determined as above; i.e., $\Theta_E + dd^c \varphi_\infty$ is a positive current on $X$, and let $\mathcal{I}(\varphi_\infty)$ be the multiplier ideal sheaf associated to $\varphi_\infty$. Then the homomorphism

$$L^q : \Gamma(X, \mathcal{I}(\varphi_\infty) \otimes \Omega_X^{n-q}(E)) \longrightarrow \text{Image} \iota^q(\varphi_\infty) \subset H^q(X, \Omega_X^n(E))$$

is surjective and the Hodge star operator relative to $\omega_X$ yields a splitting homomorphism

$$\delta^q : \text{Image} \iota^q(\varphi_\infty) \longrightarrow \Gamma(X, \mathcal{I}(\varphi_\infty) \otimes \Omega_X^{n-q}(E))$$

with $L^q \circ \delta^q = \text{id}$ for any $q \geq 1$.

The theorem was formulated and proved by Enoki in the case where $E$ is semi-positive, in which case the zero variety defined by $\mathcal{I}(\varphi_\infty)$ is empty and $\iota^q(\varphi_\infty)$ is isomorphic. Furthermore we can obtain certain injectivity and vanishing theorems for the cohomology groups, which are weaker than the semi-positive line bundle case and are closely linked together to study a Kawamata-Viehweg type vanishing theorem on compact Kähler manifolds (cf. §4, Theorems 4.2 and 4.3). Actually the following vanishing theorem holds (cf. [5], [9], [10], [15], [17], [19]).

**Theorem 2.** Let the situation be the same as in Theorem 1. Then if $q > n - \kappa_*(E)$

$$\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \otimes \Omega_X^n(E)) \longrightarrow H^q(X, \Omega_X^n(E))$$

is the zero homomorphism. Especially if $\iota^q(\varphi_\infty)$ is surjective (resp. injective) and $q > n - \kappa_*(E)$, then

$$H^q(X, \Omega_X^n(E)) = 0 \quad (\text{resp. } H^q(X, \mathcal{I}(\varphi_\infty) \otimes \Omega_X^n(E)) = 0),$$

where $\kappa_*(E)$ is the numerical Kodaira dimension of $E$ defined by

$$\kappa_*(E) := \max\{l : \wedge c_{R,1}(E) \neq 0 \in H^{2l}(X, R)\}.$$

**Remark.** The above vanishing theorem is a variant of Kawamata-Viehweg's vanishing theorem for nef line bundles on projective algebraic manifolds (cf. [9],
We do not know whether Kawamata-Viehweg’s vanishing theorem still holds on any compact Kähler manifold even if $E$ is nef and good (cf. §3, Comment and §4, Remark 2).

1. **Harmonic representation theorem for cohomology groups of multiplier ideal sheaves**

1.1. Let $X$ be a complex manifold of dimension $n$ and let $T$ be a $d$-closed $(1,1)$ current on $X$. Setting $d\bar{\partial} = \sqrt{-1}(\partial - \partial)/2$ we suppose that $T$ is decomposed as follows:

$$T = \Theta + dd^c \varphi_\infty$$

for a $d$-closed smooth real $(1,1)$ form $\Theta$ and a locally integrable function $\varphi_\infty$ on $X$. In this article we represent the positivity of $T$ in the sense of current by the notation “$T \geq 0$” and the semi-positivity (resp. positivity) of $\Theta$ by the notation “$\Theta \geq 0$” (resp. “$\Theta > 0$”). A function $\varphi$ on $X$ is said to be almost plurisubharmonic if $\varphi$ is locally equal to the sum of a plurisubharmonic function and of a smooth function (cf. [1], §1). If $T \geq 0$ and $d\Theta = 0$, then locally there exist a plurisubharmonic function $\psi$ and a smooth function $h$ such that $T = dd^c \psi$, $\Theta = dd^c h$ and $h + \varphi_\infty$ is equal almost everywhere to $\psi$. Hence the function $\varphi_\infty$ is almost plurisubharmonic.

The representation $\varphi_\infty = \psi - h$ is not unique. However if $\varphi_\infty = \psi - h = \psi_\ast - h_\ast$ with $\Theta = dd^c h_\ast$, then $\psi - \psi_\ast$ is pluriharmonic. In particular $\psi$ is determined uniquely whenever $h$ is fixed. Therefore we can define the following:

**Definition.** The multiplier ideal sheaf $\mathcal{I}(\varphi_\infty) \subset \mathcal{O}_X$ associated to $\varphi_\infty$ satisfying with $T = \Theta + dd^c \varphi_\infty \geq 0$ is the sheaf of germs of holomorphic functions $f_x \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-\varphi_\infty}$ is integrable with respect to the Lebesgue measure in a local coordinates around $x$ for any point $x$ of $X$.

It is known that $\mathcal{I}(\varphi_\infty)$ is a coherent analytic ideal sheaf of $\mathcal{O}_X$ (cf. [11, 1.2] and [3, Lemma 4.4]). The zero variety $V(\mathcal{I}(\varphi_\infty))$ of $\mathcal{I}(\varphi_\infty)$ is the set of points in a neighborhood of which $e^{-\varphi_\infty}$ is not integrable.

1.2.

**Definition.** A holomorphic line bundle $E$ on $X$ is said to be pseudo effective (resp. semi-positive, positive) if there exists a smooth hermitian metric $h_E$ and an almost plurisubharmonic function $\varphi_\infty$ (resp. a smooth hermitian metric $h_E$) such that $\Theta_E + dd^c \varphi_\infty \geq 0$ (resp. $\Theta_E \geq 0$, $\Theta_E > 0$) on $X$ for the curvature form $\Theta_E$ relative to $h_E$ defined by $\Theta_E = dd^c (-\log h_E)$. 
**Example.** Let \( D = \sum_{j=1}^{k} m_j D_j \) be an effective divisor on \( X \) with irreducible components \( D_j \) and non-negative integers \( m_j \), and let \([D_j]\) be the line bundle corresponding to each \( D_j \). Then one can verify that the line bundle \( F := \bigotimes_{j=1}^{k} [D_j] \otimes^{m_j} \) is pseudo effective by the Lelong-Poincaré formula. If \( D \) is a divisor with only normal crossings, then one can take a smooth hermitian metric \( h_F \) and an almost plurisubharmonic function \( \varphi_\infty \) such that \( \Theta_F + dd^c \varphi_\infty \geq 0 \) and \( \mathcal{I}(\varphi_\infty) = \mathcal{O}_X(F^*) \), where \( F^* \) is the dual line bundle of \( F \) (cf. [3], §5).

**1.3.** To study the cohomology groups of multiplier ideal sheaves of pseudo effective line bundles we need the following Dolbeault’s lemma which is formulated for our purpose (cf. [2, Proposition 4.1] and [3, (5.3) Corollary]).

**Theorem.** Let \( S \) be a Stein manifold of dimension \( n \) provided with a Kähler metric \( \omega_S \) defined by \( \omega_S := dd^c \Phi \) by a smooth strictly plurisubharmonic function \( \Phi \geq 0 \) on \( S \). Suppose \( E \) (resp. \( F \)) be a pseudo effective (resp. positive) line bundle provided with a smooth metric \( h_E \) and an almost plurisubharmonic function \( \varphi_\infty \) (resp. a smooth metric \( h_F \)) such that \( \Theta_E + dd^c \varphi_\infty \geq 0 \) (resp. \( \Theta_F + dd^c \Phi \geq 0 \)). Set \( (G, h_G) = (E \otimes F, h_E \otimes h_F) \). Then for any \( u \in L^{n,q}_{\text{loc}}(S, G) \), \( q \geq 1 \), with \( \bar{\partial} u = 0 \) and

\[
\int_S |u|^2_G e^{-\varphi_\infty} - 2\Phi d\nu_S < \infty
\]

there exists \( v \in L^{n,q-1}_{\text{loc}}(S, G) \) with \( \bar{\partial} v = u \) and

\[
q \int_S |v|^2_G e^{-\varphi_\infty} - 2\Phi d\nu_S \leq \int_S |u|^2_G e^{-\varphi_\infty} - 2\Phi d\nu_S.
\]

**1.4.** Let \( X \) be an \( n \) dimensional complex manifold provided with a hermitian metric \( \omega_X \). Let \( E \) be a pseudo effective line bundle provided with a smooth metric \( h_E \) and an almost plurisubharmonic function \( \varphi_\infty \) with \( \Theta + dd^c \varphi_\infty \geq 0 \) and let \( \mathcal{I}(\varphi_\infty) \) be the multiplier ideal sheaf associated to \( \varphi_\infty \). Let \( F \) be a holomorphic line bundle provided with a smooth metric \( h_F \) and set \( (G, h_G) = (E \otimes F, h_E \otimes h_F) \). We denote \( \| \|_\infty \) the \( L^2 \)-norm of \( G \)-valued forms relative to \( \omega_X \) and \( h_{GE^{-\varphi_\infty}} \), and denote \( \mathcal{F}_q \) the sheaf of germs of \( G \)-valued \((n, q)\) forms \( u \) with measurable coefficients such that both \( u \) and \( \bar{\partial} u \) are locally square integrable relative to \( \| \|_\infty \). By applying 1.3, Theorem to arbitrary small balls one can see that the complex of sheaves \( \{ \mathcal{F}^* \} \) provides a fine resolution of the sheaf \( \mathcal{I}(\varphi_\infty) \otimes \Omega_X^\bullet(G) \). Hence letting \( \Gamma(X, \mathcal{F}^q) \) be the space of global sections with values in \( \mathcal{F}^q \) and seting \( \mathcal{F}^{-1} = 0 \), we obtain the following:

\[
H^q(X, \mathcal{I}(\varphi_\infty) \otimes \Omega_X^\bullet(G)) \cong \frac{\{ u \in \Gamma(X, \mathcal{F}^q) : \bar{\partial} u = 0 \}}{\{ v \in \Gamma(X, \mathcal{F}^q) : v = \bar{\partial} w \text{ with } w \in \Gamma(X, \mathcal{F}^{q-1}) \}}
\]

for any \( q \geq 0 \).
1.5. Let $C^q(U,S)$ be the space of $q$-co-chains associated to the locally finite Stein open covering $U$ of $X$ with values in the sheaf $S := \mathcal{T}(\varphi_\infty) \boxtimes \Omega_X(G)$. Combining 1.3, Theorem with the above Dolbeault's theorem in 1.4 the Čech cohomology group $H^\bullet(U,S)$ defined by the complex $(C^\bullet(U,S),\delta)$ with the co-boundary operator $\delta$ is isomorphic to the Dolbeault cohomology group $H^\bullet(X,S)$ in view of Leray's theorem; i.e., the two complexes $(\Gamma(X,\mathcal{F}^\bullet),\partial)$ and $(C^\bullet(U,S),\delta)$ are quasi-isomorphic. In particular if $X$ is a compact complex manifold, then the Čech cohomology group $H^\bullet(U,S)$ has finite dimension and so it is a separated Fréchet topological vector space (cf. [7], Appendix B, 12. Theorem).

1.6. From now on we assume that $X$ is a compact complex manifold. Let $L^{p,q}(X,G)$ (resp. $L^{\infty}_{p,q}(X,G)$) be the $L^2$-space of $G$-valued square integrable $(p,q)$ forms provided with the inner product $(\ ,\ )$ (resp. $(\ ,\ )_\infty$) relative to $\omega_X$ and $h_G$ (resp. $\omega_X$ and $h_G e^{-\varphi_\infty}$). We denote $\partial : L^{p,q}(X,G) \to L^{p,q-1}(X,G)$ the adjoint operator of the closed densely defined operator $\bar{\partial} : L^{p,q}(X,G) \to L^{p,q+1}(X,G)$ relative to $(\ ,\ )$. Since $\varphi_\infty$ is bounded from above, $L^p_{p,q}(X,G)$ can be regarded as a subspace of $L^\infty_{p,q}(X,G)$. We denote the restriction of the operator $\bar{\partial} : L{n,q}(X,G) \to L{n,q+1}(X,G)$ onto $L^{n,q}_{\infty}(X,G)$ by $\bar{\partial}_{\infty}$ whose domain $\text{Dom}(\bar{\partial}_{\infty})$ coincides with $\Gamma(X,\mathcal{F}^q) \subseteq L^{n,q}_{\infty}(X,G)$. We claim the following.

**Lemma.** $\bar{\partial}_{\infty} : L{n,q}_{\infty}(X,G) \to L{n,q+1}_{\infty}(X,G)$ is a closed densely defined operator.

**Proof.** By Demailly's regularization result for almost plurisubharmonic functions on compact complex manifolds (cf. [1, Main Theorem 1.1]), there exists a sequence of smooth functions $\{\varphi_k\}$ on $X$ and an analytic subset $A$ of $X$ such that $\varphi_k$ decreases to $\varphi_\infty$ on $X$ as $k$ tends to infinity and $e^{-2\varphi_k}$ is locally integrable outside $A$. Set $(\ ,\ )_k := (\ ,\ e^{-\varphi_k})$ and let $L^{n,q}_k(X,G)$ be the $L^2$-space relative to the inner product $(\ ,\ )_k$ for any $k$. Let $C^0_{n,q}(X \setminus A,G)$ be the space of $G$-valued smooth $(n,q)$ forms with compact support in $X \setminus A$. Take a sequence $\{w_j\}$ in $C^0_{n,q}(X \setminus A,G)$ such that $w_j$ and $\bar{\partial}_{\infty} w_j$ converge strongly to $w$ and $\bar{\partial} w$ respectively. By the decreasing property of $\varphi_k$, $\bar{\partial} w = v$ in $L^{n,q+1}_{\infty}(X,G)$ for any $k$. For any $u \in C^0_{n,q+1}(X \setminus A,G)$, $(v,u) e^{-\varphi_\infty}$ and $(\bar{\partial} w, u) e^{-\varphi_\infty}$ are integrable on $X$ by Schwarz's inequality. Hence by Lebesgue's dominant convergence theorem we obtain:

$$
(v,u)_{\infty} = \lim_{k \to \infty} (v,u)_k \quad \text{and} \quad (\bar{\partial} w,u)_{\infty} = (\bar{\partial} w,u).$$

Since $C^0_{n,q}(X \setminus A,G)$ is dense in $L^{n,q}_{\infty}(X,G)$, $\bar{\partial}_{\infty}$ is densely defined and the above equality implies $\bar{\partial}_{\infty} w = v$ in $L^{n,q+1}_{\infty}(X,G)$; i.e., the closedness of $\bar{\partial}_{\infty}$.

Hence the adjoint operator $\bar{\partial}_{\infty} := (\bar{\partial}_{\infty})^*$ of $\bar{\partial}_{\infty}$ can be defined and has the same property as $\bar{\partial}_{\infty}$ with $\bar{\partial}_{\infty} = (\bar{\partial}_{\infty})^{**}$. The domain of $\bar{\partial}_{\infty}$ is defined in the
following way.

\[ v \in \text{Dom} \left( \partial_{(\infty)} \right) \text{ if and only if there exists a positive constant } C \text{ such that} \]

\[ |(v, \partial_{(\infty)} w)_{\infty}| \leq C \|w\|_{\infty} \text{ for any } w \in \text{Dom}^{-1} \left( \partial_{(\infty)} \right). \]

For a given linear operator \( T \) acting on the Hilbert spaces \( L^{q}\cdot\cdot(X, G) \) and \( L^{q}\cdot\cdot(X, G) \), we denote \( N^{q}\cdot\cdot(T) \) (resp. \( R^{q}\cdot\cdot(T) \)) the null space of \( T \) (resp. the range of \( T \)). Setting \( L^{q}_{n-1}(X, G) = \{0\} \) and \( L^{q}_{n-1}(X, G) = \{0\} \) respectively, we define for any \( q \geq 0 \)

\[ H^{q,n}_{n}(X, G) := N^{n,q}(\partial) \cap N^{n,q}(\partial) \quad \text{and} \quad H^{q,n}_{\infty}(X, G) := N^{n,q}(\partial) \cap N^{n,q}(\partial). \]

\( H^{q,n}_{n}(X, G) \) is the \( E \)-valued \((n, q)\) harmonic space which is isomorphic to \( H^{q}(X, \Omega^{n}_{X}(G)) \). Usually the following weak decomposition of \( L^{q}_{\infty}(X, G) \) holds (cf. [8]) :

\[ L^{q}_{\infty}(X, G) = [R^{q,n}(\partial)] \bigoplus H^{q,n}_{\infty}(X, G) \bigoplus [R^{q,n}(\partial)] \text{ for any } q \geq 0, \]

where \([ \ ]\) means the closure of space in \( L^{q}_{\infty}(X, G) \). Since \( X \) is compact, for any \( q \geq 0 \) we note that

\[ R^{q,n}(\partial) = \partial \Gamma(X, \mathcal{F}^{q-1}) \text{ and } [R^{q,n}(\partial)] \subset N^{n,q}(\partial) = \Gamma(X, \mathcal{F}^{q}) \cap \text{Ker} \partial. \]

In view of the compactness of \( X \), it is natural to claim the following strong decomposition.

**Proposition.**

\[ L^{n}_{\infty}(X, G) = R^{n,q}(\partial) \bigoplus H^{n,q}_{\infty}(X, G) \bigoplus R^{n,q}(\partial) \text{ for any } q \geq 0. \]

**Proof.** Since the closedness of \( R^{n,q}(\partial) \) is equivalent to the one of \( R^{n,q-1}(\partial) \) (cf. [8, Theorem 1.1.1]), we have only to see that \([\partial \Gamma(X, \mathcal{F}^{q-1})] = \partial \Gamma(X, \mathcal{F}^{q-1}) \). Let \( v \in [\partial \Gamma(X, \mathcal{F}^{q-1})] \) and let \( \{\partial(\omega_{k})\}_{k \geq 1} \) be a sequence in \( \partial \Gamma(X, \mathcal{F}^{q-1}) \) such that \( \|v - \partial(\omega_{k})\|_{\infty} \to 0 \) as \( k \to \infty \). We must find \( w \in \Gamma(X, \mathcal{F}^{q-1}) \) with \( v = \partial(\omega_{k}) \). Let \( U \) be a finite Stein covering of \( X \) taken as in 1.5. Combining the \( L^{2} \)-estimate in 1.3, Theorem with the quasi-isomorphism theorem in 1.5, there exists a \( q \) cocycle \( \sigma(v) \in Z^{q}(U, S) \) and a sequence of \( q-1 \) cochains \( \{\tau(w_{k})\}_{k \geq 1} \subset C^{q-1}(U, S) \) such that \( \sigma(v) - \delta \tau(w_{k}) \) tends to zero with respect to the uniform convergence topology. From the separability of Fréchet topology induced on \( H^{q}(U, S) \), there is a \( q-1 \) cochain \( \tau(w) \in C^{q-1}(U, S) \) with \( \delta \tau(w) = \sigma(v) \) which implies the conclusion by the compactness of \( X \) and the quasi-isomorphism theorem (cf. [17, Proposition 4.6]).
1.7. We obtain the following theorem from the above observations:

**Theorem.** Let $X$ be a compact complex manifold of dimension $n$ provided with a hermitian metric $\omega_X$ and let $E$ be a pseudo effective line bundle on $X$ provided with a smooth hermitian metric $h_E$ and an almost plurisubharmonic function $\varphi_\infty$ with $\Theta_E + dd^c \varphi_\infty \geq 0$ on $X$ for $\Theta_E = dd^c(-\log h_E)$. Let $I(\varphi_\infty)$ be the multiplier ideal sheaf associated to $\varphi_\infty$. Then for any holomorphic line bundle $F$ provided with a smooth hermitian metric $h_F$ on $X$ and $q \geq 0$, the space

$$H^q_{\infty}(X, E \otimes F) := \{ u \in \text{Dom}(\bar{\partial}(\infty)) \cap \text{Dom}(\partial(\infty)) : \bar{\partial}(\infty) u = 0 \text{ and } \partial(\infty) u = 0 \}$$

defined in $L^q_{\infty}(X, E \otimes F)$ satisfies the following:

$$H^q(X, I(\varphi_\infty) \otimes \Omega^n_X(E \otimes F)) \cong H^q_{\infty}(X, E \otimes F)$$

and

$$\dim \mathbb{C} H^q_{\infty}(X, E \otimes F) < \infty.$$

Furthermore the following diagram is commutative:

$$\begin{array}{ccc}
H^q(X, I(\varphi_\infty) \otimes \Omega^n_X(E \otimes F)) & \xrightarrow{i^q(\varphi_\infty)} & H^q(X, \Omega^n_X(E \otimes F)) \\
\downarrow i^q_{\infty} & & \downarrow i^q \\
H^q_{\infty}(X, E \otimes F) & \xrightarrow{H^q_{\infty}} & H^q_{\infty}(X, E \otimes F)
\end{array}$$

where $i^q_{\infty}$ and $i^q$ (resp. $H^q_{\infty}$) are isomorphisms (resp. the orthogonal projection from $L^q_{\infty}(X, E \otimes F)$ to $H^q_{\infty}(X, E \otimes F)$).

2. A smoothing of almost plurisubharmonic functions associated to nef line bundles on compact Kähler manifolds

Let $X$ be a compact Kähler manifold of dimension $n$ provided with a Kähler metric $\omega_X$ and let $E$ be a holomorphic line bundle provided with a smooth hermitian metric $h_E$ on $X$.

**Definition 2.1.** $(E, h_E)$ is said to be nef if for any $\varepsilon > 0$ there exists a smooth function $\psi_\varepsilon$ on $X$ such that $\Theta_E + dd^c \psi_\varepsilon + \varepsilon \omega_X$ yields a Kähler metric for $\Theta_E := dd^c(-\log h_E)$.

The above definition depends on the choice of neither $h_E$ nor $\omega_X$ and is equivalent to that the real first Chern class $c_{R,1}(E)$ of $E$ is contained in the closure of
the Kähler cone of $X$ (cf. [13], §2). If $E$ has a smooth metric whose curvature is semi-positive, then $E$ is clearly nef. However the converse is not true in general even if $X$ is projective algebraic (cf. [4, Example 1.7]).

We begin with the following lemma suggested by [6], Lemma 2.1 and [18], Proposition 2.1 (compare [2, Lemma 6.6]).

**Lemma 2.2.** Let $(X, \omega_X)$ be a compact Kähler manifold of dimension $n$ and let $\Theta$ be a $d$-closed smooth real $(1,1)$ form on $X$. Let $\mathcal{P}(\Theta)$ be the set of real-valued smooth functions $\psi$ so that $\Theta + dd^c\psi \geq 0$ and $\sup_X \psi = 0$. Then any sequence $\{\psi_k\}_{k \geq 1}, \psi_k \in \mathcal{P}(\Theta)$, contains a Cauchy subsequence in $L^1(X)$.

**Remark.** The existence of an $L^1$ Cauchy subsequence in $\{\psi_k\}_{k \geq 1}$, $\psi_k \in \mathcal{P}(\Theta)$, is not trivial because a local version of such a property is never true (cf. [18, p.238, Remark] and Remark 2 below).

**Proof.** Let $\{\psi_k\}_{k \geq 1}$ be a sequence belonging to $\mathcal{P}(\Theta)$. Setting $\tau_X = \omega_n/(n-1)!$ and $dv_X = \omega_X/n!$, there exists a positive constant $C(\Theta, \omega_X)$ not depending on $k$ such that

$$0 \leq \int_X e^{\psi_k}d\psi_k \wedge d^c\psi_k \wedge \tau_X = -\int_X e^{\psi_k}dd^c\psi_k \wedge \tau_X \quad \text{by Stokes' theorem}$$

$$= -\int_X e^{\psi_k}\{dd^c\psi_k + \Theta\} \wedge \tau_X + \int_X e^{\psi_k}\Theta \wedge \tau_X$$

$$\leq \int_X |\text{Trace}(\Theta, \omega_X)|dv_X \leq C(\Theta, \omega_X) < \infty.$$ 

Since $\{e^{\psi_k/2}\}$ and their first derivatives are bounded in $L^2(X)$ from the above inequality, $\{e^{\psi_k/2}\}$ has a Cauchy subsequence in $L^2(X)$ in view of Rellich’s lemma.

On the other hand there are three positive constants $C_j$ such that $C_1 \omega_X \leq C_2 \omega_X + \Theta \leq C_3 \omega_X$. Hence by [18], Proposition 2.1, there exist positive constants $\alpha$ with $0 < \alpha \ll 1$ and $C_*$ not depending on $\psi \in \mathcal{P}(\Theta)$ such that

$$\left(\int_X e^{-\alpha\psi}dv_X \right) \leq C_* < \infty$$

for any $\psi \in \mathcal{P}(\Theta)$. For any $\beta > 0$ by Schwarz’s inequality we obtain

$$\left(\int_X |e^{\beta(\psi_j - \psi_k)} - 1|dv_X \right)^2 \leq \left(\int_X |e^{\beta\psi_j} - e^{\beta\psi_k}|^2dv_X \right) \left(\int_X e^{-2\beta\psi_k}dv_X \right).$$

Taking $2\beta = \alpha$ the right hand side converges to zero from the above observation and (2.3). In particular we get

$$\int_X \max \{e^{\beta(\psi_j - \psi_k)} - 1\} - 1dv_X \to 0 \quad \text{as } j \text{ and } k \to \infty.$$
Here we may assume $\text{Vol}(X, \omega_X) = 1$ and use the following notation:

$$\log^+ t = \log \max\{t, 1\} \quad \text{and} \quad |\log t| = \log^+ t + \log^+ \left(\frac{1}{t}\right) \quad \text{for} \ t > 0.$$ 

By setting $\gamma = 1/\beta$ and the concavity of logarithmic functions we obtain:

$$\int_X |\psi_j - \psi_k| \, dv_X$$

$$= \gamma \int_X \log \left\{ e^{\beta(\psi_j - \psi_k)} \right\} \, dv_X$$

$$= \gamma \int_X \left\{ \log^+ e^{\beta(\psi_j - \psi_k)} + \log^+ e^{\beta(\psi_k - \psi_j)} \right\} \, dv_X$$

$$\leq \gamma \log \left\{ \left( \int_X \max\left\{ e^{\beta(\psi_j - \psi_k)}, 1 \right\} \, dv_X \right) \left( \int_X \max\left\{ e^{\beta(\psi_k - \psi_j)}, 1 \right\} \, dv_X \right) \right\}$$

Finally our assertion follows from the above inequality and (2.4). □

**Proposition 2.5.** Let $(E, h_E)$ be a nef line bundle on a compact Kähler manifold $(X, \omega_X)$. For a given sequence of positive numbers $\{\eta_k\}_{k \geq 1}$ decreasing to zero, let $\{\psi_k\}_{k \geq 1}$ be a sequence of smooth functions on $X$ such that

$$(2.5) \quad \Theta_E + dd^c \psi_k + \eta_k \omega_X > 0 \quad \text{on} \ X \quad \text{and} \ \sup_X \psi_k = 0,$$

where $\Theta_E = dd^c(-\log h_E)$.

Then there exist an almost plurisubharmonic function $\varphi_\infty$, a sequence of smooth functions $\{\varphi_k\}_{k \geq 1}$ on $X$, and a sequence of positive numbers $\{\epsilon_k\}_{k \geq 1}$ decreasing to zero such that

(i) $\Theta_E + dd^c \varphi_\infty \geq 0$; i.e., $E$ is pseudo effective on $X$

(ii) $\Theta_E + dd^c \varphi_k + \epsilon_k \omega_X > 0$ and $\varphi_\infty < \varphi_k \leq 1$ on $X$ for any $k \geq 1$

(iii) $\varphi_k$ converges to $\varphi_\infty$ in $L^1(X)$ and almost everywhere on $X$.

**Proof.** By applying Lemma 2.2 to $\Theta_E + \eta_k \omega_X$, if necessary, taking a subsequence, there exists a limit $\varphi_\infty \in L^1(X)$ such that $\{\psi_k\}_{k \geq 1}$ converges to $\varphi_\infty$ in $L^1(X)$. If necessary, taking a subsequence, we may assume that:

$$\|\psi_k - \varphi_\infty\|_{L^1(X)} < \frac{1}{2k}$$

$$(2) \quad \Theta_E + dd^c \varphi_\infty \geq 0.$$ 

(2) follows from the weak continuity of $\partial \bar{\partial}$ and (2.5) immediately. Locally $\omega_X$ can be written $\omega_X = dd^c \Phi$ by a smooth strictly plurisubharmonic function $\Phi$. By (2.5) (resp. (2)) $- \log h_E + \eta_k \Phi + \psi_k$ (resp. $- \log h_E + \varphi_\infty$) defines locally a smooth
plurisubharmonic function $\theta_k$ (resp. a plurisubharmonic function $\theta_\infty$). For every $k$ we put

$$\lambda_k := \max\{\psi_k, \varphi_\infty\}.$$  

Then $\lambda_k$ satisfies the following properties for any $k \geq 1$:

1. $\|\lambda_k - \varphi_\infty\|_{L^1(X)} < \frac{1}{2k}$
2. $\Theta_E + dd^c\lambda_k + \eta_k\omega_X \geq 0$.

(3) follows from (1) and (4) follows from the following local equality:

$$\lambda_k = \log h_E - \eta_k\Phi + \max\{\theta_k, \theta_\infty + \eta_k\Phi\}$$

because $\max\{\theta_k, \theta_\infty + \eta_k\Phi\}$ is plurisubharmonic. Since $\lambda_k$ is locally bounded, the Lelong number of $\lambda_k$ is zero at any point of $X$. Therefore by Demailly’s regularization result for almost plurisubharmonic functions (cf. [1], §3, the proof of Propositions 3.1 and 3.7), there exist a sequence of smooth functions $\{\varphi_k\}_{k \geq 1}$ and a sequence of positive numbers $\{\delta_k\}_{k \geq 1}$ decreasing to zero such that

1. $\varphi_\infty \leq \lambda_k < \varphi_k \leq 1$ on $X$
2. $\Theta_E + dd^c\varphi_k + (\eta_k + \delta_k)\omega_X \geq 0$ on $X$
3. $\|\varphi_k - \lambda_k\|_{L^1(X)} < \frac{1}{2k}$

for any $k \geq 1$. Setting $\varepsilon_k := \eta_k + 2\delta_k$ and if necessary, taking a subsequence, we obtain the desired sequence $\{\varphi_k\}_{k \geq 1}$. This completes the proof of Proposition 2.5.

3. On cohomology groups of nef line bundles tensorized with multiplier ideal sheaves on compact Kähler manifolds

Let $X$ be a connected compact Kähler manifold of dimension $n$ provided with a Kähler metric $\omega_X$. Let $E$ (resp. $F$) be a nef (resp. semi-positive) line bundle provided with a smooth metric $h_E$ (resp. $h_F$ with $\Theta_F = dd^c(-\log h_F) \geq 0$) on $X$. Let $\varphi_\infty$ be an almost plurisubharmonic function on $X$ with $\Theta_E + dd^c\varphi_\infty \geq 0$ determined in Proposition 2.5 and let $\mathcal{I}(\varphi_\infty)$ be the multiplier ideal sheaf associated to $\varphi_\infty$. For $\varphi_\infty$ we fix a sequence of smooth almost plurisubharmonic functions $\{\varphi_k\}_{k \geq 1}$ taken as in Proposition 2.5. We set:

$$G = E \boxtimes F, \quad h_G = h_E \boxtimes h_F, \quad \text{and} \quad h_{G,k} = h_G e^{-\varphi_k}$$
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for any $k$ with $0 \leq k \leq \infty$. Here if $k = 0$, then we set $\varphi_0 \equiv 0$ and do not specify it in the notations below.

$L_{k}^{p,q}(X, G)$ be the $L^2$-space of $G$-valued square integrable $(p, q)$ forms provided with the inner product $(\cdot, \cdot)_k$ relative to $\omega X$ and $h_{G,k}$, and let $\| \cdot \|_k$ denote the norm defined by the inner product. $L_{k}^{p,q}(X, G)$ can be regarded as a subspace of $L_{k}^{p,q}(X, G)$ for any $k$ with $0 \leq k < \infty$. Let $\bar{\partial}_{(k)}$ denote the adjoint operator of $\bar{\partial}$ in $L_{k}^{p,q}(X, G)$ (cf. 1.6). The space $N_{k}^{n,q}(\bar{\partial})$ of null solutions for $\bar{\partial}$ in $L_{k}^{n,q}(X, G)$ is decomposed strongly as follows:

\[
N_{k}^{n,q}(\bar{\partial}) = R_{k}^{n,q}(\bar{\partial}) \bigoplus H_{k}^{n,q}(X, G)
\]

where $H_{k}^{n,q}(X, G) := \{ u \in L_{k}^{n,q}(X, G) : \bar{\partial}u = \bar{\partial}_{(k)}u = 0 \}$ for any $q \geq 1$ and $0 \leq k \leq \infty$. We denote $H_{k}^{n,q}$ the orthogonal projection onto $H_{k}^{n,q}(X, G)$ for every $k$ with $0 \leq k \leq \infty$.

Setting $\mathcal{K}_{\infty}^{n,q}(X, G) := \text{Kernel} \{ H_{\infty}^{n,q} : H_{\infty}^{n,q}(X, G) \to H_{\infty}^{n,q}(X, G) \}$ (cf. 1.7, Theorem), we define a subspace $\mathcal{H}_{\infty}^{n,q}(X, G)$ of $H_{\infty}^{n,q}(X, G)$ by the following orthogonal decomposition relative to $(\cdot, \cdot)_{\infty}$:

\[
H_{\infty}^{n,q}(X, G) = \mathcal{H}_{\infty}^{n,q}(X, G) \bigoplus \mathcal{K}_{\infty}^{n,q}(X, G).
\]

Since $\mathcal{K}_{\infty}^{n,q}(X, G) = H_{\infty}^{n,q}(X, G) \cap R_{\infty}^{n,q}(\bar{\partial})$, the space $\mathcal{H}_{\infty}^{n,q}(X, G)$ is characterized as follows.

\[
(3.2) \quad u \in \mathcal{H}_{\infty}^{n,q}(X, G) \text{ if and only if } u \in N_{\infty}^{n,q}(\bar{\partial}_{\infty}) \text{ and } (u, \bar{\partial}w)_{\infty} = 0
\]

for any $w \in L_{\infty}^{n,q-1}(X, G)$ with $\bar{\partial}w \in L_{\infty}^{n,q}(X, G)$.

We define a homomorphism

\[
\mathcal{L}_{(\infty)}^{q} : \Gamma(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega_{X}^{n-q}(G)) \to \mathcal{H}_{\infty}^{n,q}(X, G)
\]

by the composition of the homomorphism

\[
\mathcal{L}^{q} : \Gamma(X, \mathcal{I}(\varphi_{\infty}) \bigotimes \Omega_{X}^{n-q}(G)) \to N_{(\infty)}^{n,q}(\bar{\partial}_{(\infty)})
\]

induced by the $q$-times left exterior product by $\omega X$ with the orthogonal projection from $N_{(\infty)}^{n,q}(\bar{\partial}_{(\infty)})$ to $\mathcal{H}_{\infty}^{n,q}(X, G)$.

The following lemma is very useful (cf. [3, (4.10)]).

**Lemma 3.3.** Let $W$ be a holomorphic line bundle on $X$ provided with a smooth hermitian metric $h_{W}$. Let $\Theta$ be a smooth real $(1, 1)$ differential form on $X$ and let $\{ \lambda_j \}$ be the eigen-values of $\Theta$ relative to $\omega X$ with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ (which are
continuous functions on $X$): i.e., $\Theta(x) = \sqrt{-1} \sum_{j=1}^{n} \lambda_j(x) d\bar{z}^j$ with $\omega_X(x) = \sqrt{-1} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j$, $x \in X$. Then if $v(x) = \sum v_{A_n,B_q} dz^A \wedge d\bar{z}^B \in C^{n,q}(X,W)$ with $q \geq 1$, the following holds

$$\langle e(\Theta)Av, v \rangle_W(x) = \sum_{|A_n| = n, |B_q| = q} \left( \sum_{j} \lambda_j(x) \right) |v_{A_n,B_q}|_W^2.$$  

In particular setting $\delta_q := \sum_{j=1}^{q} \lambda_j$ with $q \geq 1$ the following holds

(3.4) $\langle e(\Theta)Av, v \rangle_W \geq \delta_q(v,v)_W$ if $v \in C^{n,q}(X,W)$.

The nefness of $E$ enables us to show the following theorem.

**Theorem 3.5.** $\mathcal{L}^q(\infty)$ is surjective and the Hodge star operator $\ast$ relative to $\omega_X$ yields a splitting homomorphism

$$\delta^q_{(\infty)} : \mathcal{H}^{n,q}(X,G) \rightarrow \Gamma(X,\mathcal{I}(\varphi_{(\infty)}) \otimes \Omega^{n-q}_X(G))$$

with $\mathcal{L}^q(\infty) \circ \delta^q_{(\infty)} = \text{id}$. Furthermore $\mathcal{L}^q(\infty) = L^q$ on Image$\delta^q_{(\infty)}$ for any $q \geq 1$.

**Proof.** If $\mathcal{H}^{n,q}(X,G) = \{0\}$, then we have nothing to prove. Hence we assume $\mathcal{H}^{n,q}(X,G) \neq \{0\}$ and take $u \in \mathcal{H}^{n,q}(X,G)$ with $\|u\|_\infty = 1$. We claim that $u \in \Gamma(X,\mathcal{I}(\varphi_{(\infty)}) \otimes \Omega^{n-q}_X(G))$, which implies that $\mathcal{L}^q(\infty) = L^q$ is surjective by $L^q \circ \ast = c(n,q)id$ on the space of $(n,q)$ forms for the uniquely determined complex number $c(n,q) \neq 0$. We have only to define $\delta^q_{(\infty)} := c(n,q)^{-1} \ast$.

We note that $u$ has the following orthogonal decomposition by (3.1):

(3.6) $u = \bar{\partial}w_k + H^{n,q}_k(u)$, $\|\bar{\partial}w_k\|_k$ and $\|H^{n,q}_k(u)\|_k \leq 1$

for any $k$ with $0 \leq k < \infty$. Setting $u_k := H^{n,q}_k(u)$, we may assume $u_k \neq 0$ for any $k$. From $\|u_k\| \leq e\|u_k\|_k \leq e$, taking a subsequence, $\{u_k\}$ has a weak limit $u_\infty \in L^{n,q}(X,G)$ with $\bar{\partial}u_\infty = 0$. $\{\bar{\partial}w_k\}$ also has a weak limit $v_\infty$. Since $R^{n,q}(\bar{\partial})$ is closed, there exists $w_* \in L^{n,q-1}(X,G)$ with $w_\infty = \bar{\partial}w_*$. Therefore we obtain

(3.7) $u = \bar{\partial}w_* + u_\infty$ in $L^{n,q}(X,G)$.

We show that $u_\infty \in \Gamma(X,\mathcal{I}(\varphi_{(\infty)}) \otimes \Omega^{n-q}_X(G))$ and $u_\infty \in \mathcal{H}^{n,q}(X,G)$, which implies $\bar{\partial}w_* = 0$ by (3.2); i.e., $u_\infty = u$.

By Calabi-Nakano-Vesentini's formula on compact Kähler manifolds (cf. [14, Proposition 1.2]), we obtain the following integral formula:

$$\|\bar{\partial}v\|^2_k + \|\bar{\partial}(k)v\|^2_k = \|\bar{\partial}v\|^2_k + (e(\Theta_G + dd^c\varphi_k)Av, v)_k$$
for any $G$-valued smooth $(n, q)$ form $v$ on $X$, $\Theta_G := \Theta_E + \Theta_F$ and $k \geq 1$. Since $q\|v\|_k^2 = (L \Lambda v, v)_k$, by Proposition 2.5, (ii) and the semi-positivity of $\Theta_F$ (cf. (3.4)), we obtain the following inequality:

$$
\varepsilon_k q\|u_k\|^2_k = \|\tilde{\partial} u_k\|^2_k + (e(\Theta_G + d \partial \varphi_k + \varepsilon_k \omega_X) \Lambda u_k, u_k)_k \\
\geq (e(\Theta_G + d \partial \varphi_k + \varepsilon_k \omega_X) \Lambda u_k, u_k)_k \geq 0.
$$

Therefore when $k$ tends to infinity, we obtain

$$
\|\tilde{\partial} u_k\|^2_k \leq \varepsilon_k q\|u_k\|^2_k \leq \varepsilon_k q \to 0.
$$

By $\tilde{\partial} = - * \tilde{\partial}*$ and $\|\tilde{\partial} * u_k\|^2 \leq \|\tilde{\partial} u_k\|^2$, $u_\infty$ satisfies $\tilde{\partial} * u_\infty = 0$ in the sense of distribution. Therefore $\star u_\infty \in \Gamma(X, \Omega^n_{X} (G))$. Setting $u_k = u_k e^{-\varphi_k/2}$ and, if necessary taking a subsequence, $u_k$ converges weakly to $u_\infty \in L^n_q(X, G)$ by $\|u_k\|_k \leq 1$.

Let $V$ be the analytic subset (might be empty) defined by $\mathcal{I}(\varphi_\infty)$. Since $e^{-\varphi_\infty}$ is locally integrable on $X \setminus V$, $e^{-\varphi_k}$ converges to $e^{-\varphi_\infty}$ in $L^1(K)$ for any compact subset $K$ in $X \setminus V$ by $\varphi_\infty < \varphi_k$ and Lebesgue's dominant convergence theorem. For every $E$-valued smooth $(n, q)$ form $v$ with compact support in $X \setminus V$, by setting $K := \text{Supp}(v)$ and denoting $|v|_G$ the pointwise length of $v$ relative to $\omega_X$ and $h_G$, we obtain from (3.6):

$$
\lim_{k \to \infty} \left| (u_k, \{e^{-\varphi_\infty/2} - e^{-\varphi_k/2}\} v) \right| \leq \lim_{k \to \infty} \sup_{K} |v|_G \|u_k\| \|e^{-\varphi_\infty/2} - e^{-\varphi_k/2}\| L^2(K) \\
\leq \varepsilon \sup_{K} |v|_G \lim_{k \to \infty} \sqrt{|e^{-\varphi_\infty} - e^{-\varphi_k}| L^1(K)} = 0.
$$

Here we have used: $(a - b)^2 < a^2 - b^2$ if $a > b > 0$. Hence we get:

$$
(u_\infty, v) = \lim_{k \to \infty} (u_k, v) = \lim_{k \to \infty} (u_k, v e^{-\varphi_\infty/2}) = (u_\infty e^{-\varphi_\infty/2}, v).
$$

This implies $u_\infty = u_\infty e^{-\varphi_\infty/2}$ on $X \setminus V$ as current and so $u_\infty \in L^n_q(X, G)$ because $u_\infty \in L^n_q(X, G)$. Therefore we get $\star u_\infty \in \Gamma(X, \mathcal{I}(\varphi_\infty) \boxtimes \Omega^n_{X} (G))$.

Furthermore if $w \in L^n_{q-1}(X, G)$ with $\tilde{\partial} w \in L^n_q(X, G)$, then $w \in L^n_k q_{-1}(X, G)$ with $\tilde{\partial} w \in L^n_q(X, G)$ for any $k$ with $1 \leq k < \infty$ because $\varphi_k$ is smooth. Therefore by $\tilde{\partial} u_k = 0$ and Lebesgue's dominant convergence theorem, we obtain:

$$
| (u_\infty, \tilde{\partial} w)_\infty | = \lim_{k \to \infty} \left| (u_k, \{e^{-\varphi_\infty/2} - e^{-\varphi_k/2}\} \tilde{\partial} w) \right| \\
\leq \lim_{k \to \infty} \sqrt{|\{e^{-\varphi_\infty} - e^{-\varphi_k}\}| \tilde{\partial} w|_G^2 \| L^1(X) = 0.
$$

Therefore $u_\infty \in \mathcal{H}^n_{\infty q}(X, G)$ by (3.2). This completes the proof of Theorem 3.5.
Proposition 3.8. Every $u \in \mathcal{H}^{n,q}_\infty(X,G)$ with $q \geq 1$ satisfies the following:

$$(3.9) \quad (e(\Theta_G + dd^c\varphi)Au, u)_\infty = 0$$

for any smooth real-valued function $\varphi$ on $X$.

Proof. By the equations $\bar{\partial}u = \bar{\partial}u = 0$, we get $\bar{\partial}\partial_G u = e(\Theta_G)Au$ and $\bar{\partial}e(\bar{\partial}\varphi)^*u = e(dd^c\varphi)Au$ by [14], Propositions 1.2 & 1.5. Since $\Theta_G$ and $dd^c\varphi$ are smooth on $X$, we obtain $\bar{\partial}\partial_G u$ and $\bar{\partial}e(\bar{\partial}\varphi)^*u \in L^{n,q}_\infty(X,G)$ by Lemma 3.3. The conclusion follows from (3.2). \(\square\)

In view of the $L^2$-estimate (3.9), we can show the following vanishing theorem for $\mathcal{H}^{n,q}_\infty(X,G)$.

Theorem 3.10. If $q > n - \max\{\kappa_*(E), \kappa_*(F)\}$, then $\mathcal{H}^{n,q}_\infty(X,G) = 0$, where $\kappa_*(E)$ is defined by $\kappa_*(E) := \max\{ l : \lambda c_{R,1}(E) \neq 0 \in H^{2l}(X,R) \}$ and so on.

Proof. By (3.9), if $u \in \mathcal{H}^{n,q}_\infty(X,G)$, then for any smooth real-valued function $\varphi$ on $X$ and $\varepsilon > 0$ we obtain

$$(3.11) \quad 0 < (e(\Theta_G + dd^c\varphi + \varepsilon\omega_X)Au, u)_\infty = q\varepsilon\|u\|_\infty$$

and particularly

$$(3.12) \quad (e(\Theta_F)Au, u)_\infty = 0.$$

If $q > n - \kappa_*(F)$, then the integrand of (3.12) is non-negative on $X$ and positive at least one point of $X$ by (3.4) (cf. [16], p. 277, Fact 2.7). Therefore $u$ should vanish on $X$ identically because $*u$ is holomorphic and $X$ is connected.

Assume $q > n - \kappa_*(E)$ and $u \neq 0 \in \mathcal{H}^{n,q}_\infty(X,G)$. For any $\varepsilon > 0$ we set:

$$p(\varepsilon) := \int_X (\Theta_G + \varepsilon\omega_X)^n / \int_X \omega_X^n.$$

Since $E$ is nef, for any $\varepsilon > 0$ there exists a smooth real-valued function $\varphi_\varepsilon$ on $X$ so that $\Theta_G + dd^c\varphi_\varepsilon + \varepsilon\omega_X$ is a Kähler metric. Furthermore by [21], there exists a smooth real-valued function $\psi_\varepsilon$ on $X$ such that $\gamma_\varepsilon := \Theta_G + dd^c(\varphi_\varepsilon + \psi_\varepsilon) + \varepsilon\omega_X$ is a Kähler metric on $X$ with

$$\gamma_\varepsilon^n = p(\varepsilon)\omega^n.$$

Let $\{\lambda_{\varepsilon,j}\}$ be the eigenvalues of $\gamma_\varepsilon$ relative to $\omega_X$ and let $\delta_{\varepsilon,\mu}$ be a continuous function defined as in Lemma 3.3 relative to $\{\lambda_{\varepsilon,j}\}$ for any $\varepsilon > 0$ and $1 \leq \mu \leq n$. 

Set \( U(\varepsilon) := \{ \delta_{\varepsilon,q} < 2q\varepsilon \} \) for any \( \varepsilon > 0 \). By applying \( \varphi_{\varepsilon} + \psi_{\varepsilon} \) to (3.11), and Lemma 3.3 we can show

\[
0 < \|u\|^2 \leq 2 \int_{U(\varepsilon)} |u|^2 e^{-\varphi_{\varepsilon}} dv_X.
\]

This implies \( U(\varepsilon) \neq \emptyset \) for any \( \varepsilon > 0 \). We claim that there exists a positive constant \( C_1 \) not depending on \( \varepsilon \) such that \( \int_{U(\varepsilon)} dv_X \geq C_1 > 0 \) for any \( \varepsilon > 0 \). If \( \int_{U(\varepsilon)} dv_X \) converges to zero, then \( \int_{U(\varepsilon)} |u|^2 e^{-\varphi_{\varepsilon}} dv_X \) also tends to zero because \( |u|^2 e^{-\varphi_{\varepsilon}} \) is integrable. However this contradicts to the above inequality.

Furthermore since \( \int_X e(\gamma_\varepsilon) \omega_{X}^{n-1} = \int_X e(\Theta_G + \varepsilon \omega_X) \omega_{X}^{n-1} \) is non-negative and bounded from above, there exists positive constant \( C_2 \) and \( C_3 \) not depending on \( \varepsilon \) such that \( 0 < \delta_{\varepsilon,n} \leq C_2 \) on an open subset \( Q(\varepsilon) \subseteq U(\varepsilon) \) with \( \int_{Q(\varepsilon)} dv_X \geq C_3 > 0 \).

Hence we obtain

\[
\prod_{j=1}^{n} \lambda_{\varepsilon,j} \leq (2q)^q C_2^{n-q} e^q \text{ on } Q(\varepsilon) \text{ for any } \varepsilon > 0.
\]

On the other hand since \( P(\varepsilon) = \prod_{j=1}^{n} \lambda_{\varepsilon,j} \) is a polynomial in \( \varepsilon \) of degree \( n \) and \( E \) is nef, letting \( P(\varepsilon) = \sum_{i=0}^{n} a_i \varepsilon^i \) we obtain : \( a_i > 0 \) if \( i \geq n - \kappa \) and \( a_i = 0 \) if \( i < n - \kappa \) by the definition of \( \kappa = \kappa_*(E) \) and (3.13). This implies that

\[
a_{n-n-\kappa} e^{n-\kappa} \leq \prod_{j=1}^{n} \lambda_{\varepsilon,j} \text{ on } X.
\]

By (3.14) and (3.15) we can get \( a_{n-\kappa} e^{n-\kappa} \leq (2q)^q C_2^{n-q} e^q \), which is a contradiction as \( \varepsilon \) tends to zero because \( q > n - \kappa \). The idea of this proof is due to Enoki [5].

This completes the proof of Theorem 3.10. □

Next we show the following injectivity theorem.

**Theorem 3.16.**

(i) If the \( j \)-times tensor product \( E \otimes^j \) of \( E \) admits a non-trivial holomorphic section \( \sigma \) with

\[
C(\sigma) := \text{ess. sup}_X |\sigma|^2_{E \otimes^j} e^{-j\varphi_{\varepsilon}} < \infty
\]

then the homomorphism

\[
\mathcal{H}^{n,q}_{\infty}(\sigma) : \mathcal{H}^{n,q}(X, E \otimes^i \bigotimes F) \rightarrow \mathcal{H}^{n,q}(X, E \otimes^{i+j} \bigotimes F)
\]

induced by the tensor product with \( \sigma \) is well defined and particularly injective for any \( q \geq 0 \), \( i \) and \( j \geq 1 \).
(ii) If the $k$-times tensor product $F^\otimes k$ of $F$ admits a non-trivial holomorphic section $\theta$, then

$$\mathcal{H}^{n,q}(\theta) : \mathcal{H}^{n,q}(X, E \bigotimes F^\otimes j) \rightarrow H^{n,q}_\infty(X, E \bigotimes F^\otimes (j+k))$$

induced by the tensor product with $\theta$ is well defined and particularly injective for any $q \geq 0$, $j$ and $k \geq 1$.

Proof of (i). For $u \in H^{n,q}(X, E^{\otimes i} \bigotimes F)$, setting $v = \sigma \bigotimes u$ we have only to show $(v, \bar{\partial}w) = 0$ for any $w \in L^{n,q}_{\overline{\partial}}(X, E^{\otimes (i+j)} \bigotimes F)$ with $\bar{\partial}w \in L^{n,q}_{\overline{\partial}}(X, E^{\otimes (i+j)} \bigotimes F)$. Since $\bar{\partial}v = \bar{\partial}v = 0$, and $\Theta_F$ is semi-positive, by Calabi-Nakano-Vesentini’s formula, Lemma 3.3 and Proposition 3.8, we can conclude:

$$\|\bar{\partial}(v)\|^2_k = (e((i+j)(\Theta_E + dd^c\varphi_k + \Theta_F)Au, v)_k$$

$$\leq \left(\frac{i+j}{i}\right)(e(i(\Theta_E + dd^c\varphi_k + \varphi_k\omega_X) + \Theta_F)Au, v)_k$$

$$\leq \varepsilon_k qC(\sigma)\left(\frac{i+j}{i}\right)\|u\|^2_\infty \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence by Lebesgue’s dominant convergence theorem we have

$$(v, \bar{\partial}w)_\infty = \lim_{k \rightarrow \infty} (v, \bar{\partial}w)_k = \lim_{k \rightarrow \infty} (\bar{\partial}(v), v)_k = 0.$$

Proof of (ii). Since the length of $\theta$ is bounded, the proof can be done similarly. This completes the proof of Theorem 3.16. □

REMARK. If the almost plurisubharmonic function $\varphi_\infty$ is determined independently of the choice of $\{\varepsilon_k\}$, then from the above proof it can be verified that $\mathcal{H}^{n,q}(\sigma) : H^{n,q}(X, E^{\otimes i} \bigotimes F) \rightarrow H^{n,q}_{\infty}(X, E^{\otimes (i+j)} \bigotimes F)$ is well defined.

Comment. In the situation of this section, setting $F = \text{the trivial line bundle}$, Enoki claims that $H^{n,q}(X, E) = 0$ if $q > n - \kappa_*(E)$, which implies that $H^q(X, \Omega^n_X(E)) = 0$ if $q > n - \kappa_*(E)$ (cf. [5, Theorem 0.1]). His idea of the proof consists of two parts; i.e., an $L^2$-estimate for the harmonic forms in $H^{n,q}(X, E)$ and the argument used to show Theorem 3.10. In fact he claims the following $L^2$-estimate (cf. [5, Proposition 3.1]):

Let $E$ be a holomorphic line bundle provided with a smooth hermitian metric $h_E$ on a compact Kähler manifold $X$ of dimension $n$ provided with a Kähler metric $\omega_X$. Then for any real-valued smooth function $\varphi$ on $X$ and $u \in H^{n,q}(X, E)$ with $q \geq 1$, setting $\eta := e^\varphi$ the following inequality holds

$$(\eta(e(\Theta_E + dd^c\varphi)Au, u) \leq 0.$$
Here we should note that any specific condition for the curvature of \((E, h_E)\) is not assumed to show the above inequality in his proof. However the sign of the left hand side can not be always determined in the following sense.

First for any \(E\)-valued smooth \((n, q)\) form \(v\) on \(X\) we can obtain the following integral formula (cf. [17, §1, Proposition 1.11]):

\[
\|\sqrt{\eta}(\bar{\partial} + e(\partial \varphi))v\|^2 + \|\sqrt{\eta}d_k v\|^2 = \|\sqrt{\eta}(\bar{\partial} - e(\partial \varphi)^*)v\|^2 + (\eta e(\Theta_E + d\bar{\partial} \varphi)\Lambda v, v).
\]

Hence if \(u \in H^{n,q}(X, E)\), by setting \(w = *u\) and using \(e(\partial \varphi)^* = *e(\partial \varphi)^*\) we can verify the following from the above formula:

\[
(\eta e(\Theta_E + d\bar{\partial} \varphi)\Lambda u, u) = -\|\sqrt{\eta}(\bar{\partial} - e(\partial \varphi)^*)u\|^2 + \|\sqrt{\eta}e(\partial \varphi)u\|^2 = -\|\sqrt{\eta}(\bar{\partial} + e(\partial \varphi))w\|^2 + \|\sqrt{\eta}e(\partial \varphi)^*w\|^2.
\]

Here we note that \(\bar{\partial} w\) is primitive ; i.e., \(\Lambda \bar{\partial} w = 0\) by \(\bar{\partial} u = 0\) and \(\bar{\partial} = -\sqrt{-1}[\bar{\partial}, \Lambda]\). For any \(E\)-valued smooth \((n - q, 1)\) form \(\alpha\), let \(\alpha = \alpha_1 + \alpha_2\) be the primitive decomposition of the form ; i.e., \(\Lambda \alpha_1 = 0\) and \(\alpha_2 = 1/(q+1)\Lambda \alpha\) (cf.[20, Chap.V, Theorem 1.8]). Here the coefficient \(1/(q+1)\) of \(\alpha_2\) is crucial. Since \(e(\partial \varphi)^* = \sqrt{-1}[e(\partial \varphi), \Lambda]\), by applying the decomposition to \(\alpha := e(\partial \varphi)\) and the above equality it can be verified that

\[
(\eta e(\Theta_E + d\bar{\partial} \varphi)\Lambda u, u) = -\|\sqrt{\eta}(\bar{\partial} w + \alpha_1)\|^2 + q\|\sqrt{\eta}\alpha_2\|^2
\]

and

\[
\alpha_2 = 0 \quad \text{if and only if} \quad e(\partial \varphi)u = 0.
\]

Therefore if \(u \in H^{n,q}(X, E)\) satisfies the equality

\[
(\eta e(\Theta_E + d\bar{\partial} \varphi)\Lambda u, u) = -\|\sqrt{\eta}(\bar{\partial} w + \alpha_1)\|^2 \leq 0
\]

for any real-valued smooth function \(\varphi\) on \(X\) as he claims (see the last line of his proof of Proposition 3.1 in [5]), then by the above observations an \(E^*\) (the dual of \(E\))-valued harmonic \((0, n - q)\) form \(*\bar{h}u\) satisfies the \(\bar{\partial}\)-Neumann condition on every open ball with smooth boundary contained in any local coordinate neighborhood of \(X\). Hence such a form should vanish on it in view of the solvability for \(\bar{\partial}\) on open balls and its boundary condition (cf.[17, §4. Theorem 4.3, (iv)]), and so identically on \(X\) by a unique continuation property for harmonic forms, which implies \(H^q(X, \Omega_X^p(E)) = 0\). However \(H^q(X, \Omega_X^p(E))\) does not vanish without any specific condition in general.

4. On cohomology groups of nef line bundles on compact Kähler manifolds

First we state the following Lefschetz type theorem (cf. [5, Theorem 0.3]).
**Theorem 4.1.** Let $X$ be a connected compact Kähler manifold of dimension $n$ provided with a Kähler metric $\omega_X$. Let $E$ (resp. $F$) be a nef (resp. semi-positive) line bundle provided with a smooth metric $h_E$ (resp. $h_F$) with $\Theta_F = dd^c(-\log h_F) \geq 0$ on $X$. Let $\varphi_\infty$ be an almost plurisubharmonic function with $\Theta_E + dd^c \varphi_\infty \geq 0$ determined in Proposition 2.5 and let $\mathcal{I}(\varphi_\infty)$ be the multiplier ideal sheaf associated to $\varphi_\infty$. Then for any $q \geq 1$ the homomorphism

$$L^q : \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(E \bigotimes F)) \longrightarrow \text{Image}(q)(\varphi_\infty) \subset H^q(X, \Omega_X^n(E \bigotimes F))$$

is surjective and the Hodge star operator relative to $\omega_X$ yields a splitting homomorphism

$$\delta^q : \text{Image}(q)(\varphi_\infty) \longrightarrow \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(E \bigotimes F))$$

with $L^q \circ \delta^q = \text{id}$, where $\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E \bigotimes F)) \longrightarrow H^q(X, \Omega_X^n(E \bigotimes F))$ is the canonical homomorphism induced by $\iota : \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E \bigotimes F) \hookrightarrow \Omega_X^n(E \bigotimes F)$.

**Proof.** The conclusion follows from Theorem 3.5 because the image of $\iota^q(\varphi_\infty)$ can be identified with $\mathcal{H}^q_\infty(X, E \bigotimes F)$ by the commutative diagram in 1.7, Theorem.

We denote $V(\varphi_\infty)$ the compact analytic subset of $X$ defined by the multiplier ideal sheaf $\mathcal{I}(\varphi_\infty)$ and define $d(\varphi_\infty) := \max \{\dim_{\mathbb{C}} V(\varphi_\infty) \alpha : V(\varphi_\infty) \alpha \text{ is any irreducible component of } V(\varphi_\infty)\}$ (we set $d(\varphi_\infty) = -1$ if $V(\varphi_\infty) = \emptyset$; i.e., $\mathcal{I}(\varphi_\infty) \cong \mathcal{O}_X$). It is clear that $d(j \varphi_\infty) \leq d(k \varphi_\infty)$ if $1 \leq j < k$, and $\iota^q(\varphi_\infty)$ is bijective (resp. surjective) if $q > d(\varphi_\infty) + 1$ (resp. $q > d(\varphi_\infty)$). If the Lelong number of $\varphi_\infty$ is less than one everywhere on $X$, then $d(\varphi_\infty) = -1$ (cf. [3, (5.6) Lemma]). Under the hypothesis of Theorem 4.1, by Theorem 3.10 we can obtain the following vanishing theorem immediately (cf. [5], [9], [15], [19]).

**Theorem 4.2.** Suppose $q > n - \max \{\kappa_*(E), \kappa_*(F)\}$. Then

$$\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E \bigotimes F)) \longrightarrow H^q(X, \Omega_X^n(E \bigotimes F))$$

is the zero homomorphism. Especially the following assertions hold:

(i) If $\iota^q(\varphi_\infty)$ is surjective (resp. injective) and $q > n - \max \{\kappa_*(E), \kappa_*(F)\}$, then

$$H^q(X, \Omega_X^n(E \bigotimes F)) = 0 \quad (\text{resp. } H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E \bigotimes F)) = 0)$$

(ii) If $q > \max \{n - \max \{\kappa_*(E), \kappa_*(F)\}, d(\varphi_\infty)\}$, then

$$H^q(X, \Omega_X^n(E \bigotimes F)) = 0$$

where $\kappa_*(E)$ (resp. $\kappa_*(F)$) is the numerical Kodaira dimension of $E$ (resp. $F$).
REMARK 1. The homomorphism $\iota^q(\varphi_\infty)$ is not always injective (cf. [4, Example 1.7]).

At last we can get the following theorem from Theorem 3.16 (cf. [5, Theorem 0.2] and [10, Theorem 2.2]).

Theorem 4.3. Under the hypothesis of Theorem 4.1 the following assertions hold:

(i) Suppose a non-trivial holomorphic section $\sigma$ of $E^{\otimes j}$ satisfies $\text{ess. sup}_X |\sigma|^2_{E^{\otimes j}} \times e^{-j\varphi_\infty} < \infty$ and $q > d((i+j)\varphi_\infty) + 1$. Then the homomorphism

$$H^{n,q}(\sigma) : H^q(X, \Omega^i_X(E^{\otimes j} \otimes F)) \longrightarrow H^q(X, \Omega^i_X(E^{\otimes(i+j)} \otimes F))$$

induced by the tensor product with $\sigma$ is injective for any $i$ and $j \geq 1$.

(ii) Suppose $\theta$ is a non-trivial holomorphic section of $F^{\otimes j}$ and $q > d(\varphi_\infty) + 1$. Then the homomorphism

$$H^{n,q}(\theta) : H^q(X, \Omega^i_X(E \otimes F^{\otimes i})) \longrightarrow H^q(X, \Omega^i_X(E \otimes F^{\otimes(i+j)}))$$

induced by the tensor product with $\theta$ is injective for any $i$ and $j \geq 1$.

REMARK 2. Theorems 4.2 and 4.3 yield us an indication about Kawamata-Viehweg type vanishing theorem for nef line bundles on compact Kähler manifolds; i.e., $H^q(X, \Omega^n_X(L)) = 0$ if a holomorphic line bundle $L$ on a compact Kähler manifold $X$ with $\dim_C X = n$ is nef and good; i.e., $\kappa(L) = \kappa_*(L)$ and $q > n - \kappa_*(L)$, where $\kappa(L)$ is the Kodaira dimension of $L$. In this situation by replacing $X$ by a bimeromorphic Kähler model of $X$ there exist a surjective morphism $\pi : X \to Y$ with connected fibres from $X$ to a projective algebraic manifold $Y$ with $\dim_C Y = \kappa_*(L)$ and a nef-big $Q$-divisor $B$ on $Y$ such that (i) $L = \pi^*B$, (ii) $kB = A + D$ with a very ample divisor $A$ and an effective divisor $D$ on $Y$ for $k \gg 0$ (cf. [13, §2, Proposition 2.14]). This implies that $L^{\otimes k}$ is written by the tensor product of a semi-positive line bundle $\pi^*[A]$ and a pseudo effective one $\pi^*[D]$, and admits a non-trivial section $\theta$ which vanishes along $\pi^*D$ (cf. Theorem 4.3 and [17, §6]).
References


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