INJECTIVE PAIRS IN PERFECT RINGS

MITSUO HOSHINO and TAKESHI SUMIOKA

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Throughout this note, rings are associative rings with identity and modules are unitary modules. Sometimes, we use the notation $A X$ (resp. $X A$) to signify that the module $X$ considered is a left (resp. right) $A$-module. For each pair of subsets $X$ and $M$ of a ring $A$, we set $\ell_X(M) = \{a \in X | aM = 0\}$ and $r_M(X) = \{a \in M | Xa = 0\}$.

Following Baba and Oshiro [1], we call a pair $(eA, Af)$ of a right ideal $eA$ and a left ideal $Af$ in a ring $A$ an $i$-pair if (a) $e$ and $f$ are local idempotents; (b) $eA_A$ and $AAf$ have essential socles; and (c) $\text{soc}(eA_A) \cong fA/fJ$ and $\text{soc}(Af) \cong Ae/J e$, where $J$ is the Jacobson radical of $A$.

Generalizing a result of Fuller [3], Baba and Oshiro [1] showed that for a local idempotent $e$ in a semiprimary ring $A$, $eA_A$ is injective if and only if there exists a local idempotent $f$ in $A$ such that $(eA, Af)$ is an $i$-pair in $A$ and $r_{Af}(\ell_{eA}(M)) = M$ for every submodule $M$ of $Af/Af$, and that for an $i$-pair $(eA, Af)$ in a semiprimary ring $A$ the following are equivalent: (1) $eA_A$ is artinian; (2) $Af_Af$ is artinian; and (3) both $eA_A$ and $Af$ are injective.

Our aim is to extend the results mentioned above to perfect rings. Following Harada [4], we call a module $L_A M$-simple-injective if for any submodule $N$ of $M_A$ every $\theta : N_A \to L_A$ with $\text{Im} \theta$ simple can be extended to some $\phi : M_A \to L_A$. For a local idempotent $e$ in a left perfect ring $A$, we will show that $eA_A$ is $A$-simple-injective if and only if there exists a local idempotent $f$ in $A$ such that $(eA, Af)$ is an $i$-pair in $A$ and $r_{Af}(\ell_{eA}(M)) = M$ for every submodule $M$ of $Af_Af$, and that $eA_A$ is injective if it is $A$-simple-injective and has finite Loewy length. We will show also that for an $i$-pair $(eA, Af)$ in a left perfect ring $A$ the following are equivalent: (1) $eA_A$ is artinian; (2) $Af_Af$ is artinian; and (3) both $eA_A$ and $Af$ are injective.

1. Localization and injective objects

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ covariant functors, and $\varepsilon : 1_{\mathcal{A}} \to GF$ and $\delta : FG \to 1_{\mathcal{B}}$ homomorphisms of functors, where $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ and $1_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ are identity functors. We assume the conditions: (a) $\delta_F \circ F \varepsilon = \text{id}_F$; (b) $G \delta \circ \varepsilon = \text{id}_G$; (c) $F$ is exact; and (d) $\delta$ is an isomorphism.

Remark 1. (1) By the conditions (a) and (b), for each pair of $X \in \text{Ob}(\mathcal{A})$ and
$M \in \text{Ob}(B)$ we have a natural isomorphism

$$
\theta_{X,M} : \text{Hom}_B(FX, M) \rightarrow \text{Hom}_A(X, GM), \beta \mapsto G\beta \circ \epsilon_X
$$

with $\theta_{X,M}^{-1}(\alpha) = \delta_M \circ F\alpha$ for $\alpha \in \text{Hom}_A(X, GM)$. Namely, $G$ is a right adjoint of $F$. In particular, $G$ is left exact.

(2) By the conditions (a), (b) and (d), $G : B \rightarrow A$ is fully faithful.

(3) By the conditions (a) and (d), $F\epsilon : F \rightarrow FG\delta$ is an isomorphism with $F\epsilon^{-1} = \delta_F$.

(4) By the conditions (b) and (d), $\epsilon_G : G \rightarrow GFG$ is an isomorphism with $\epsilon_G^{-1} = G\delta$.

Though the following lemmas are well known and more or less obvious, we include proofs for completeness.

**Lemma 1.1.** Let $X \in \text{Ob}(A)$ be simple with $FX \neq 0$. Then $FX \in \text{Ob}(B)$ is simple.

Proof. Let $\beta : FX \rightarrow M$ be a nonzero morphism in $B$. We claim $\beta$ monic. Note that $\beta = \delta_M \circ F(G\beta \circ \epsilon_X)$. Thus $G\beta \circ \epsilon_X : X \rightarrow GM$ is nonzero and monic, so is $\beta = \delta_M \circ F(G\beta \circ \epsilon_X)$.

**Lemma 1.2.** Let $\mu : Y \rightarrow X$ be an essential monomorphism in $A$ with $\epsilon_Y$ monic. Then $F\mu : FY \rightarrow FX$ is an essential monomorphism in $B$.

Proof. Let $\beta : FX \rightarrow M$ be a morphism in $B$ with $\beta \circ F\mu$ monic. We claim $\beta$ monic. Since $(G\beta \circ \epsilon_X) \circ \mu = G\beta \circ GF\mu \circ \epsilon_Y = G(\beta \circ F\mu) \circ \epsilon_Y$ is monic, $G\beta \circ \epsilon_X$ is monic and so is $\beta = \delta_M \circ F(G\beta \circ \epsilon_X)$.

**Lemma 1.3.** Let $X \in \text{Ob}(A)$ be injective with $\epsilon_X$ monic. Then $\epsilon_X : X \rightarrow GFX$ is an isomorphism and $FX \in \text{Ob}(B)$ is injective.

Proof. Since $F\epsilon_X$ is an isomorphism, $F(\text{Cok} \epsilon_X) \cong \text{Cok} F\epsilon_X = 0$ and $\text{Hom}_A(\text{Cok} \epsilon_X, GFX) \cong \text{Hom}_B(F(\text{Cok} \epsilon_X), FX) = 0$. Thus, since $\epsilon_X : X \rightarrow GFX$ is a split monomorphism, $\text{Cok} \epsilon_X = 0$. Hence for each $M \in \text{Ob}(B)$ we have a natural isomorphism

$$
\eta_M : \text{Hom}_B(M, FX) \rightarrow \text{Hom}_A(GM, X), \beta \mapsto \epsilon_X^{-1} \circ G\beta.
$$

Let $\nu : N \rightarrow M$ be a monomorphism in $B$. Since $G\nu$ is monic, $\text{Hom}_A(G\nu, X)$ is epic and so is $\text{Hom}_B(\nu, FX) = \eta_N^{-1} \circ \text{Hom}_A(G\nu, X) \circ \eta_M$.

**Remark 2.** (1) An object $M \in \text{Ob}(B)$ is injective if and only if so is $GM \in$
Ob(A).

(2) The canonical monomorphism \( \text{Im} \varepsilon_X \to GFX \) is an essential monomorphism for every \( X \in \text{Ob}(A) \) with \( FX \neq 0 \).

(3) If \( \nu : N \to M \) is an essential monomorphism in \( B \), so is \( G\nu : GN \to GM \).

(4) For \( X \in \text{Ob}(A) \) with \( \varepsilon_X \) monic, a monomorphism \( \mu : Y \to X \) in \( A \) is an essential monomorphism if and only if so is \( F\mu : FY \to FX \).

2. Injective pairs

Throughout the rest of this note, \( A \) stands for a ring with Jacobson radical \( J \). For an \( \varepsilon \)-pair \((eA, Af)\) in \( A \), we denote by \( A_\varepsilon(eA, Af) \) the lattice of submodules \( X \) of \( eAeA \) with \( \ell_{eA}(r_{Af}(X)) = X \) and by \( A_r(eA, Af) \) the lattice of submodules \( M \) of \( Af_fAf \) with \( r_{Af}(\ell_{eA}(M)) = M \).

**Remark 3.** Let \((eA, Af)\) be an \( \varepsilon \)-pair in \( A \). Let \( X \) be a submodule of \( eAeA \). Then \( Xr_{Af}(X) = 0 \) implies \( X \subseteq \ell_{eA}(r_{Af}(X)) \) and thus \( r_{Af}(\ell_{eA}(r_{Af}(X))) \subseteq r_{Af}(X) \). Also, \( \ell_{eA}(r_{Af}(X))r_{Af}(X) = 0 \) implies \( r_{Af}(X) \subseteq r_{Af}(\ell_{eA}(r_{Af}(X))) \). Thus \( r_{Af}(X) \in A_\varepsilon(eA, Af) \). Similarly, \( \ell_{eA}(M) \in A_\varepsilon(eA, Af) \) for every submodule \( M \) of \( Af_fAf \). It follows that \( A_\varepsilon(eA, Af) \) is anti-isomorphic to \( A_r(eA, Af) \).

The following lemmas have been established in [5], [3], [1], [8], [6] and so on. However, for the benefit of the reader, we provide direct proofs.

**Lemma 2.1.** Let \( e, f \in A \) be idempotents and assume \( \ell_{eA}(Af) = 0 = r_{Af}(eA) \). Then the following hold.

1. For a two-sided ideal \( I \) of \( A \), \( ei = 0 \) if and only if \( If = 0 \).
2. \( \ell_{eA}(I) = \ell_{eA}(If) \) for every right ideal \( I \) of \( A \).
3. \( r_{Af}(I) = r_{Af}(ei) \) for every left ideal \( I \) of \( A \).

**Proof.**

1. Assume \( ei = 0 \). Then \( eAf = ei = 0 \) and \( If \subseteq r_{Af}(eA) = 0 \). By symmetry, \( If = 0 \) implies \( ei = 0 \).
2. Since \( If \subseteq I \), \( \ell_{eA}(I) \subseteq \ell_{eA}(If) \). For any \( x \in \ell_{eA}(If) \), since \( xI Af = xI f = 0 \), \( xI \subseteq \ell_{eA}(Af) = 0 \) and \( x \in \ell_{eA}(I) \). Thus \( \ell_{eA}(I) \subseteq \ell_{eA}(I) \).
3. Similar to (2).

**Lemma 2.2.** Let \((eA, Af)\) be an \( \varepsilon \)-pair in \( A \). Then the following hold.

1. \( \ell_{eA}(Af) = 0 = r_{Af}(eA) \).
2. \( eAf_fAf \) and \( eAeAf \) have simple essential socles and \( \text{soc}(eA_A)f = \text{soc}(eAf_fAf) = \text{soc}(eAeAf) = e(\text{soc}(A Af)) \).

**Proof.**

1. For any \( 0 \neq x \in eA \), since \( \text{soc}(eA_A) \subseteq xA \), \( 0 \neq \text{soc}(eA_A)f \subseteq xAf \) and \( x \notin \ell_{eA}(Af) \). Thus \( \ell_{eA}(Af) = 0 \). Similarly \( r_{Af}(eA) = 0 \).
(2) Since by Lemma 1.1 \( \text{soc}(e_A)_{fAf} \) and \( e_A e(\text{soc}(A_{Af})) \) are simple, and since by Lemma 1.2 \( \text{soc}(e_A)_{fAf} \subset eAf_{fAf} \) and \( e_A e(\text{soc}(A_{Af})) \subset e_A eAf \) are essential extensions, the assertion follows.

Lemma 2.3. Let \((eA, Af)\) be an i-pair in \(A\). Then for any \(n \geq 1\) \(eJ^n = 0\) if and only if \(J^n f = 0\), so that \(eA_A\) and \(A_Af\) have the same Loewy length.

Proof. By Lemmas 2.2(1) and 2.1(1).

Lemma 2.4. Let \((eA, Af)\) be an i-pair in \(A\). Let \(N, M\) be submodules of \(Af_{fAf}\) with \(N \subset M\) and \(M/N\) simple. Assume \(N \in A_r(eA, Af)\). Then the following hold.

1. \(e_A eA(N)/e_A eA(M)\) is simple.
2. \(M \in A_r(eA, Af)\).

Proof. (1) Let \(a \in M\) with \(a \notin N\). Then \(M = N + aAf\). Also, since \(M \neq N = r_Af(e_A eA(N)), e_A eA(M) \subset e_A eA(N)\) with \(e_A eA(N)/e_A eA(M) \neq 0\). Since \(0 \neq e_A eA(N)M = e_A eA(N)aAf\) and \(e_A eA(N)aAf = \text{soc}(eA_{fAf})\), thus by Lemma 2.2(2) \(e_A eA(N) = \text{soc}(e_A eA)\) and, since \(e_A eA(M) = 0\), \(e_A eA(N)/e_A eA(M) \cong \text{soc}(e_A eA)\).

(2) Since \(e_A eA(M) \subset e_A eA(N) \subset e_A eA\) with \(e_A eA(M) \in A_r(eA, Af)\) and \(e_A eA(N)/e_A eA(M)\) simple, we can apply the part (1) to conclude that \(r_Af(e_A eA(M))/r_Af(e_A eA(N))\) is simple. Thus \(r_Af(e_A eA(N)) = N \subset M \subset r_Af(e_A eA(M))\) with both \(r_Af(e_A eA(M))/r_Af(e_A eA(N))\) and \(M/N\) simple, so that \(M = r_Af(e_A eA(M))\).

Lemma 2.5. Let \((eA, Af)\) be an i-pair in \(A\). Then \(M \in A_r(eA, Af)\) for every submodule \(M\) of \(Af_{fAf}\) of finite composition length.

Proof. Lemma 2.4(2) together with Lemma 2.2(1) enables us to make use of induction on the composition length.

Lemma 2.6. Let \((eA, Af)\) be an i-pair in \(A\). Then \(e_A eA\) and \(Af_{fAf}\) have the same composition length.

Proof. By symmetry, we may assume \(Af_{fAf}\) has finite composition length. Let \(0 = M_0 \subset M_1 \subset \cdots \subset M_n = Af\) be a composition series of \(Af_{fAf}\). Put \(X_i = e_A eA(M_i)\) for \(0 \leq i \leq n\). Since by Lemma 2.5 \(M_i \in A_r(eA, Af)\) for all \(0 \leq i \leq n\), by Lemmas 2.4(1) and 2.2(1) we have a composition series \(0 = X_0 \subset \cdots \subset X_1 \subset X_n = eA\) of \(e_A eA\).
Lemma 2.7. Let \((eA, Af)\) be an i-pair in \(A\). Then the following are equivalent.

1. \(eA_A\) is \(A\)-simple-injective.
2. \(\ell_{eA}(M) = \ell_{eA}(N)\) implies \(N = M\) for submodules \(N, M\) of \(Af_{Af}\) with \(N \subseteq M\).
3. \(M \in A_r(eA, Af)\) for every submodule \(M\) of \(Af_{Af}\).

Proof. (1) \(\Rightarrow\) (2). Let \(N, M\) be submodules of \(Af_{Af}\) with \(N \subseteq M\) and \(M/N \neq 0\). Since \((MA/NA)f \cong M/N \neq 0\), there exist submodules \(K, I\) of \(MA_A\) such that \(NA \subseteq K \subseteq I\) and \(I/K \cong fAf/fJ\). Let \(\mu : I_A \to A_A\) denote the inclusion. Since we have \(\theta : I_A \to eA_A\) with \(\text{Im} \theta = \text{soc}(eA_A)\) and \(\text{Ker} \theta = K\), there exists \(\phi : A_A \to eA_A\) with \(\phi \circ \mu = \theta\). Then \(\phi(1)I = \phi(I) = \theta(I) \neq 0\) and \(\phi(1)K = \phi(K) = \theta(K) = 0\). Thus \(\phi(1) \in \ell_{eA}(K)\) and \(\phi(1) \not\in \ell_{eA}(I)\). Since \(\ell_{eA}(M) = \ell_{eA}(MA) \subseteq \ell_{eA}(I) \subseteq \ell_{eA}(K) \subseteq \ell_{eA}(NA) = \ell_{eA}(N)\), \(\ell_{eA}(I) \neq \ell_{eA}(K)\) implies \(\ell_{eA}(M) \neq \ell_{eA}(N)\).

(2) \(\Rightarrow\) (3). Let \(M\) be a submodule of \(Af_{Af}\) and put \(L = r_{Af}(\ell_{eA}(M))\). Then \(M \subseteq L\) and \(\ell_{eA}(L) = \ell_{eA}(r_{Af}(\ell_{eA}(M))) = \ell_{eA}(M)\). Thus \(M = L\).

(3) \(\Rightarrow\) (1). Let \(I\) be a nonzero right ideal and \(\mu : I_A \to A_A\) the inclusion. Let \(\theta : I_A \to eA_A\) with \(\text{Im} \theta = \text{soc}(eA_A)\) and put \(K = \text{Ker} \theta\). Then by Lemma 1.1 \(If/Kf_{Af} \cong (I/K)f_{Af}\) is simple, so is \(eA \ell_{eA}(Kf)/\ell_{eA}(If)\) by Lemma 2.4(1). Let \(a \in I f\) with \(a \not\in K f\). Then, since \(\ell_{eA}(Kf)a \neq 0\) and \(\ell_{eA}(If)a = 0\), \(eA \ell_{eA}(Kf)a\) is simple. Thus by Lemma 2.2(2) \(\ell_{eA}(Kf)a = \text{soc}(eA_A)f\), so that \(\theta(a) = \theta(af) = \theta(a)f = ba\) with \(b \in \ell_{eA}(Kf)\). Define \(\phi : A_A \to eA_A\) by \(1 \mapsto b\). Then, since by Lemmas 2.2(1) and 2.1(2) \(b \in \ell_{eA}(K)\), and since \(I = K + aA\), we have \(\phi \circ \mu = \theta\).

Lemma 2.8. Let \((eA, Af)\) be an i-pair in \(A\). Assume \(eA_A\) is injective. Then the canonical homomorphism \(eAeA_A \to eAe\text{Hom}_{Af}(Af, eAf)_A\), \(a \mapsto (b \mapsto ab)\), is an isomorphism and \(eAf_{Af}\) is injective.

Proof. By Lemmas 2.2(1) and 1.3.

3. Injective pairs in perfect rings

In this section, we extend results of Baba and Oshiro [1] to left perfect rings. We refer to [2] for perfect rings. We abbreviate the ascending (resp. descending) chain condition as the ACC (resp. DCC).

Remark 4. (1) Let \((eA, Af)\) be an i-pair in \(A\). Then, since \(A_r(eA, Af)\) is anti-isomorphic to \(A_r(eA, Af)\), \(A_r(eA, Af)\) satisfies the ACC (resp. DCC) if and only if \(A_r(eA, Af)\) satisfies the DCC (resp. ACC).

(2) Let \(e \in A\) be an idempotent. Then, since \(eAe\) appears as a direct sum-
mand in $e Ae e A$, $e Ae e A$ is artinian if and only if it has finite composition length.

(3) Every module $L_A$ with $\soc(L_A) = 0$ is $A$-simple-injective.

**Lemma 3.1** (cf. [1, Proposition 5]). Let $(e A, A f)$ be an $i$-pair in $A$. Assume $A_r(e A, A f)$ satisfies the ACC and $f A f$ is a left perfect ring. Then $A f A_f$ is artinian and $M \in A_r(e A, A f)$ for every submodule $M$ of $A f A_f$.

Proof. It follows by Lemma 2.5 that there exists a maximal element $M$ in the set of submodules of $A f A_f$ of finite composition length. We claim $M = A f A_f$. Otherwise, there exists a submodule $L$ of $A f A_f$ with $M \subseteq L$ and $L/M$ simple, a contradiction. Thus $A f A_f$ has finite composition length and again by Lemma 2.5 the last assertion follows.

**Proposition 3.2.** Let $(e A, A f)$ be an $i$-pair in a left perfect ring $A$. Then the following are equivalent.

1. $e Ae e A$ is artinian.
2. $A_r(e A, A f)$ satisfies both the ACC and the DCC.
3. $A_r(e A, A f)$ satisfies the ACC.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1). Since the ascending chain $\ell_{e A}(A f) \subseteq \ell_{e A}(J f) \subseteq \ell_{e A}(J^2 f) \subseteq \cdots$ in $A_r(e A, A f)$ terminates, $\ell_{e A}(J^n f) = \ell_{e A}(J^{n+1} f)$ for some $n \geq 0$. We claim $\ell_{e A}(J^n f) = e A$. Suppose otherwise. Then there exists a submodule $M$ of $e A A$ with $\ell_{e A}(J^n f) \subseteq M$ and $M/\ell_{e A}(J^n f)$ simple. Since $M J \subseteq \ell_{e A}(J^n f)$, $M J^{n+1} f \subseteq \ell_{e A}(J^n f) J^n f = 0$ and $M \subseteq \ell_{e A}(J^{n+1} f) = \ell_{e A}(J^n f)$, a contradiction. Thus $\ell_{e A}(J^n f) = e A$ and by Lemma 2.2(1) $J^n f \subseteq r_{A f}(\ell_{e A}(J^n f)) = 0$. Then by Lemma 2.3 $e J^n f = 0$ and $e Ae$ is a semiprimary ring. Thus by Lemma 3.1 $e Ae e A$ is artinian.

**Lemma 3.3.** Let $e \in A$ be a local idempotent. Assume $e A A$ is $A$-simple-injective and has nonzero socle. Then $\soc(e A A)$ is simple.

Proof. Let $S$ be a simple submodule of $\soc(e A A)_A$. We claim $S = \soc(e A A)$. Suppose otherwise. Let $\pi : \soc(e A A) \to S_A$ be a projection and $\mu : \soc(e A A) \to e A A$, $\nu : S_A \to e A A$ inclusions. There exists $\phi : e A A \to e A A$ with $\phi \circ \mu = \nu \circ \pi$. Since $\pi$ is not monic, $\phi$ is not an isomorphism. Thus $\phi(e) \in e A e$ and $(e - \phi(e))$ is a unit in $e A e$. For any $x \in S$, since $\phi(e) x = \phi(x) = \pi(x) = x$, $(e - \phi(e)) x = 0$ and thus $x = 0$, a contradiction.

**Lemma 3.4** (cf. [1, Proposition 2]). Let $A$ be a semiperfect ring and $e \in A$ a local idempotent. Assume $e A A$ is $A$-simple-injective and has finite Loewy length. Then
$eA_A$ is injective.

Proof. Let $I$ be a nonzero right ideal and $μ : I_A → A_A$ the inclusion. Let $θ : I_A → eA_A$. We make use of induction on the Loewy length of $θ(I)$ to show the existence of $φ : A_A → eA_A$ with $θ = φ o μ$. Let $n = \min\{k ≥ 0 | θ(I)J^k = 0\}$. We may assume $n > 0$. Since $eA_A$ has nonzero socle, by Lemma 3.3 $soc(eA_A)$ is simple and $soc(eA_A) = θ(I)J^{n-1} = θ(IJ^{n-1})$. Let $μ_1$ and $θ_1$ denote the restrictions of $μ$ and $θ$ to $IJ^{n-1}$, respectively. Then $Im θ_1 = soc(eA_A)$ and there exists $φ_1 : A_A → eA_A$ with $φ_1 o μ_1 = θ_1$. Since $(θ - φ_1 o μ)(IJ^{n-1}) = 0$, by induction hypothesis there exists $φ_2 : A_A → eA_A$ with $φ_2 o μ = θ - φ_1 o μ$. Then $θ = (φ_1 + φ_2) o μ$.

**Lemma 3.5** (cf. [1, Proposition 4]). Let $A$ be a semiperfect ring and $e ∈ A$ a local idempotent. Assume $eA_A$ is $A$-simple-injective and has essential socle. Then there exists a local idempotent $f ∈ A$ such that $(eA, Af)$ is an i-pair in $A$.

Proof. By Lemma 3.3 $S_A = soc(eA_A)$ is simple. Let $f ∈ A$ be a local idempotent with $S_f ≠ 0$. We claim that $(eA, Af)$ is an i-pair in $A$. Let $0 ≠ a ∈ S_f$. It suffices to show $a ∈ Ab$ for all $0 ≠ b ∈ Af$. Let $0 ≠ b ∈ Af$. Define $α : fA_A → aA_A$ by $x ↦ ax$ and $β : fA_A → bA_A$ by $x ↦ bx$. Since $Ker β = r_A(b) ⊂ fJ = r_A(a) = Ker α$, we have $θ : bA_A → aA_A = S_A$ with $α = θ o β$. Let $μ : S_A → eA_A$, $ν : bA_A → A_A$ be inclusions. Then there exists $φ : A_A → eA_A$ with $φ o ν = μ o θ$ and $a = α(f) = θ(β(f)) = θ(b) = φ(b) = φ(1)b ∈ Ab$.

**Theorem 3.6** (cf. [1, Theorem 1]). Let $A$ be a left perfect ring and $e ∈ A$ a local idempotent. Then the following are equivalent.

1. $eA_A$ is $A$-simple-injective.
2. There exists a local idempotent $f ∈ A$ such that $(eA, Af)$ is an i-pair in $A$ and $M ∈ M_{A_e}(eA_A)$ for every submodule $M$ of $AfAf$.

Proof. By Lemmas 3.5 and 2.7.

**Theorem 3.7** (cf. [1, Theorem 2]). Let $(eA, Af)$ be an i-pair in a left perfect ring $A$. Then the following are equivalent.

1. $eA_eeA$ is artinian.
2. $AfAf$ is artinian.
3. Both $eA_A$ and $AfAf$ are injective.

Proof. (1) $↔$ (2). By Lemma 2.6.

(2) $⇒$ (3). By Lemmas 2.6, 2.5 and 2.7 both $eA_A$ and $AfAf$ are $A$-simple-injective. Also, by Lemma 2.3 both $eA_A$ and $AfAf$ have finite Loewy length. Thus by Lemma 3.4 both $eA_A$ and $AfAf$ are injective.
(3) $\Rightarrow$ (1). By Lemma 2.8 the canonical homomorphism
\[ e_AeA \to e_Ae \text{Hom}_{eAf}(Af, eAf) \]
is an isomorphism and $eAf_{eAf}$ is injective. Similarly, the canonical homomorphism $eAf_{eAf} \to e \text{Hom}_{eAf}(eA, eAf)_{eAf}$ is an isomorphism and $eAeAf$ is injective. It follows that $eAeAf_{eAf}$ defines a Morita duality. Thus by [7, Theorem 3] $eAe$ is left artinian and $eAeA$ has finite Loewy length. Since the canonical homomorphism $eAeA \to eAe \text{Hom}_{eAf}(eA, eAf, eAf)$ is an isomorphism, it follows by [7, Lemma 13] that $eAeA$ has finite composition length.

REMARK 5. In Theorem 3.7 the assumption that $A$ is left perfect cannot be replaced by a weaker condition that $A$ is semiperfect (see [7, Example 1]).

References