COLOCAL PAIRS IN PERFECT RINGS

MITSUO HOSHINO AND TAKESHI SUMIOKA

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Our main aim of the present note is to provide several sufficient conditions for a colocal module \(L\) over a left or right perfect ring \(A\) to be injective. Also, by developing the previous works [8] and [5], we will extend recent results of Baba [1, Theorems 1 and 2] to left perfect rings and provide simple proofs of them.

Throughout this note, rings are associative rings with identity and modules are unitary modules. For a ring \(A\) we denote by \(\text{Mod}A\) (resp. \(\text{Mod}A^{\text{op}}\)) the category of left (resp. right) \(A\)-modules, where \(A^{\text{op}}\) denotes the opposite ring of \(A\). Sometimes, we use the notation \(AL\) (resp. \(LA\)) to signify that the module \(L\) considered is a left (resp. right) \(A\)-module. For a module \(L\), we denote by \(\text{soc}(L)\) the socle, by \(\text{rad}(L)\) the Jacobson radical, by \(E(L)\) an injective envelope and by \(\ell(L)\) the composition length of \(L\). For a subset \(X\) of a right module \(LA\) and a subset \(M\) of \(A\), we set \(l_X(M) = \{x \in X | xM = 0\}\) and \(r_M(X) = \{a \in M | xa = 0\}\). Also, for a subset \(X\) of \(A\) and a subset \(M\) of a left module \(AL\) we set \(l_X(M) = \{a \in X | aM = 0\}\) and \(r_M(X) = \{x \in M | Xx = 0\}\).

Recall that a module \(L\) is called colocal if it has simple essential socle. We call a bimodule \(HU_R\) colocal if both \(HU\) and \(UR\) are colocal. Let \(A\) be a semiperfect ring with Jacobson radical \(J\). Let \(LA\) be a colocal module with \(H = \text{End}_A(LA)\) and \(f \in A\) a local idempotent with \(\text{soc}(LA) \cong fA/fJ\). In case \(LA\) has finite Loewy length, we will show that \(LA\) is injective if and only if \(HLf_{JAf}\) is a colocal bimodule and \(M = r_{AF}(l_{L}(M))\) for every submodule \(M\) of \(Af_{JAf}\). Also, in case \(A\) is left or right perfect and \(\ell(Af/r_{AF}(L)_{fAf}) < \infty\), we will show that the following are equivalent: (1) \(LA\) is injective; (2) \(HLf_{JAf}\) is a colocal bimodule and \(r_{AF}(L) = 0\); and (3) \(HLf_{JAf}\) is a colocal bimodule and \(M = r_{AF}(l_{L}(M))\) for every submodule \(M\) of \(Af_{JAf}\).

Recall that a module \(LA\) is called \(M\)-injective if for any submodule \(N\) of \(MA\) every \(\theta: NA \twoheadrightarrow LA\) can be extended to some \(\phi: MA \rightarrow LA\). Dually, a module \(LA\) is called \(M\)-projective if for any factor module \(N\) of \(MA\) every \(\theta: LA \rightarrow NA\) can be lifted to some \(\phi: LA \rightarrow MA\). In case \(L\) is \(L\)-injective (resp. \(L\)-projective), \(L\) is called quasi-injective (resp. quasi-projective). Let \(A\) be a left perfect ring with Jacobson radical \(J\) and \(e, f \in A\) local idempotents. Assume \(\ell(AF/r_{AF}(eA)_{fAf}) < \infty\). Then we will show that \(eA\) is quasi-projective with \(\text{soc}(eA) \cong fA/fJ\) if and only if \(Ae = E(Ae/Je)\) is quasi-projective with \(AE/JE \cong Af/Jf\) (cf. [1, Theorem 1]).
We call a pair \((eA, Af)\) of a right ideal \(eA\) and a left ideal \(Af\) in \(A\) a colocal pair if \(e, f \in A\) are local idempotents and \(eAeAf\) is a colocal bimodule. We will see that \(\ell(eAeAf/eA(Af)) = \ell(Af/rAf(eA)fAf)\) for every colocal pair \((eA, Af)\) in \(A\). In case \(\ell(eAeAf/eA(Af)) = \ell(Af/rAf(eA)fAf) < \infty\), a colocal pair \((eA, Af)\) in \(A\) is called finite. Let \(A\) be a left perfect ring with Jacobson radical \(J\) and \(e, f_1, f_2, \ldots, f_n \in A\) local idempotents. Put \(E = E(Ae/Je)\). Assume \((eA, Af_i)\) is a finite colocal pair in \(A\) for all \(1 \leq i \leq n\). Then we will show that \(\text{soc}(eAeA) = \bigoplus_{i=1}^{n} f_i A/f_i J\) if and only if \(Ae/JE \cong \bigoplus_{i=1}^{n} Af_i/Jf_i\) (cf. [1, Theorem 2]).

Following Harada [4], we call a module \(L_A\) \(M\)-simple-injective if for any submodule \(N\) of \(M_A\) every \(\theta : N_A \to L_A\) with \(\text{Im}\theta\) simple can be extended to some \(\phi : M_A \to L_A\). In case \(L\) is \(L\)-simple-injective, \(L\) is called simple-quasi-injective. We will show that a left perfect ring \(A\) is left artinian if \(A\) satisfies the ascending chain condition on annihilator right ideals and \(eA_A\) is simple-quasi-injective for every local idempotent \(e \in A\).

1. Preliminaries

In this section, we collect several basic results which we need in later sections. We refer to Bass [2] for perfect rings.

**Lemma 1.1.** Let \(A\) be a left or right perfect ring and \(f \in A\) an idempotent. Assume \(\ell(AfAf) < \infty\). Then \(AAf\) has finite Loewy length.

**Proof.** Denote by \(J\) the Jacobson radical of \(A\). Consider first the case of \(A\) being left perfect. Since the descending chain \(Af \supset Jf \supset \cdots\) terminates, there exists \(n \geq 1\) such that \(J^nf = J^{n+1}f\). Thus \(J^nf = 0\). Assume next that \(A\) is right perfect. Then, since the ascending chain \(\text{soc}(AfAf) \subset \text{soc}^2(AfAf) \subset \cdots\) terminates, there exists \(n \geq 1\) such that \(\text{soc}^n(AfAf) = Af\). Thus \(J^nf = J^n(\text{soc}^n(AfAf)) = 0\).

**Lemma 1.2.** Let \(e \in A\) be an idempotent. Then for a module \(L \in \text{Mod } A\) with \(r_L(eA) = 0\) the following hold.

1. If \(AL\) is simple, so is \(eAeL\).
2. \(eAeE(AL) \cong E(eAeL)\).
3. The canonical homomorphism \(AE(AL) \to A \text{Hom}_{eAe}(eA, eE(AL)), x \mapsto (a \mapsto ax)\), is an isomorphism.

**Proof.** (1) See e.g. [5, Lemma 1.1].
(2) See e.g. [5, Lemmas 1.2 and 1.3].
(3) See e.g. [5, Lemma 1.3].

Recall that a module \(L_A\) is called \(M\)-injective if for any submodule \(N\) of \(M_A\) every \(\theta : N_A \to L_A\) can be extended to some \(\phi : M_A \to L_A\). Dually, a module \(L_A\)
is called $M$-projective if for any factor module $N$ of $M_A$ every $\theta : L_A \to N_A$ can be lifted to some $\phi : L_A \to M_A$. In case $L$ is $L$-injective (resp. $L$-projective), $L$ is called quasi-injective (resp. quasi-projective).

**Lemma 1.3** ([6, Theorem 1.1]). Let $L \in \text{Mod } A^{\text{op}}$ and put $H = \text{End}_A(E(L_A))$. Then $L_A$ is quasi-injective if and only if $HL = L$. In particular, if $L_A$ is quasi-injective, then we have a surjective ring homomorphism $\rho_L : \text{End}_A(E(L_A)) \to \text{End}_A(L_A), h \mapsto h|_L$.

The equivalence $(1) \Leftrightarrow (2)$ of the next lemma is due to Wu and Jans [11, Propositions 2.1, 2.2 and 2.4].

**Lemma 1.4** ([11]). Let $A$ be a left perfect ring. Then for a module $L \in \text{Mod } A$ the following are equivalent.

1. $A_L$ is indecomposable quasi-projective.
2. There exist a local idempotent $f \in A$ and a two-sided ideal $I$ of $A$ such that $A_L \cong Af/If$.
3. There exists a local idempotent $f \in A$ such that $A_L \cong Af/l_A(L)f$.

**Proof.** (1) $\Rightarrow$ (2). By [11, Proposition 2.4] there exists an epimorphism $\pi : A Af \to A L$ with $f \in A$ a local idempotent. Put $K = \text{Ker } \pi$. Then by [11, Proposition 2.2] $KfAf = K$ and $A L \cong Af/If$ with $I = KfA$ a two-sided ideal of $A$.

(2) $\Rightarrow$ (1). Since $A/I Af/If \cong A/I(A/I)f$ is projective, $A Af/If$ is quasi-projective.

(2) $\Rightarrow$ (3). Since $If = l_A(Af/If)f$, $A L \cong Af/l_A(L)f$.

(3) $\Rightarrow$ (2). Obvious. \qed

Recall that an object $L$ of an abelian category $\mathcal{A}$ in which arbitrary direct products exist is called linearly compact if for any inverse system of epimorphisms $\{\pi_\lambda : L \to L_\lambda\}_\lambda \in \Lambda$ in $\mathcal{A}$ the induced morphism $\lim \pi_\lambda : L \to \lim L_\lambda$ is epic. In case $\mathcal{A} = \text{Mod } A$, there is an equivalent definition of linear compactness. Recall that, for a family of submodules $\{L_\lambda\}_\lambda \in \Lambda$ in a module $A L$, a system of congruences $\{x \equiv x_\lambda \mod L_\lambda\}_\lambda \in \Lambda$ is said to be finitely solvable if for any nonempty finite subset $F$ of $\Lambda$ there exists $x_F \in L$ such that $x_F \equiv x_\lambda \mod L_\lambda$ for all $\lambda \in F$, and to be solvable if there exists $x_0 \in L$ such that $x_0 \equiv x_\lambda \mod L_\lambda$ for all $\lambda \in \Lambda$.

For the benefit of the reader, we include a proof of the following.

**Proposition 1.5.** For a module $L \in \text{Mod } A$ the following are equivalent.

1. $A_L$ is linearly compact.
2. For any family of submodules $\{L_\lambda\}_\lambda \in A L$, every finitely solvable system of congruences $\{x \equiv x_\lambda \mod L_\lambda\}_\lambda \in \Lambda$ is solvable.

Recall that an object $L$ of an abelian category $\mathcal{A}$ in which arbitrary direct products exist is called linearly compact if for any inverse system of epimorphisms $\{\pi_\lambda : L \to L_\lambda\}_\lambda \in \Lambda$ in $\mathcal{A}$ the induced morphism $\lim \pi_\lambda : L \to \lim L_\lambda$ is epic. In case $\mathcal{A} = \text{Mod } A$, there is an equivalent definition of linear compactness. Recall that, for a family of submodules $\{L_\lambda\}_\lambda \in \Lambda$ in a module $A L$, a system of congruences $\{x \equiv x_\lambda \mod L_\lambda\}_\lambda \in \Lambda$ is said to be finitely solvable if for any nonempty finite subset $F$ of $\Lambda$ there exists $x_F \in L$ such that $x_F \equiv x_\lambda \mod L_\lambda$ for all $\lambda \in F$, and to be solvable if there exists $x_0 \in L$ such that $x_0 \equiv x_\lambda \mod L_\lambda$ for all $\lambda \in \Lambda$.

For the benefit of the reader, we include a proof of the following.

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1. $A_L$ is linearly compact.
2. For any family of submodules $\{L_\lambda\}_\lambda \in A L$, every finitely solvable system of congruences $\{x \equiv x_\lambda \mod L_\lambda\}_\lambda \in \Lambda$ is solvable.
Proof. (1) $\Rightarrow$ (2). Let $\{L_\lambda\}_{\lambda \in \Lambda}$ be a family of submodules in $L$ and $\{x \equiv x_\lambda \mod L_\lambda\}_{\lambda \in \Lambda}$ a finitely solvable system of congruences. Denote by $\phi_\lambda : L \to L/L_\lambda$ the canonical epimorphism for each $\lambda \in \Lambda$ and set $\phi : L \to \prod_{\lambda \in \Lambda} L/L_\lambda$, $x \mapsto (\phi_\lambda(x))$. Put $\hat{x} = (\phi_\lambda(x_\lambda)) \in \prod_{\lambda \in \Lambda} L/L_\lambda$. We claim that $\hat{x} \in \text{Im } \phi$. Let $\mathcal{F}$ be the directed set of nonempty finite subsets of $\Lambda$. For each $F \in \mathcal{F}$, denote by $p_F : \prod_{\lambda \in F} L/L_\lambda \to \prod_{\lambda \in \Lambda} L/L_\lambda$ the projection and put $\hat{x}_F = p_F(\hat{x}) \in \prod_{\lambda \in F} L/L_\lambda$ and $X_F = (p_F \circ \phi)^{-1}(A\hat{x}_F)$. Note that for any $F \in \mathcal{F}$, since $\{x \equiv x_\lambda \mod L_\lambda\}_{\lambda \in \Lambda}$ is finitely solvable, $p_F \circ \phi : L \to \prod_{\lambda \in F} L/L_\lambda$ induces an epimorphism $\varphi_F : X_F \to A\hat{x}_F$. For each $F \in \mathcal{F}$, take a push-out of $\varphi_F : X_F \to A\hat{x}_F$ along with the inclusion $X_F \to L$:

\[
\begin{array}{ccc}
X_F & \longrightarrow & L \\
\varphi_F \downarrow & & \downarrow \pi_F \\
A\hat{x}_F & \longrightarrow & Y_F.
\end{array}
\]

Then we get an inverse system of epimorphisms $\{\pi_F : L \to Y_F\}_{F \in \mathcal{F}}$. Also, since $\lim$ is left exact, we get a pull-back square

\[
\begin{array}{ccc}
\lim X_F & \longrightarrow & L \\
\lim \varphi_F \downarrow & & \downarrow \lim \pi_F \\
\lim A\hat{x}_F & \longrightarrow & \lim Y_F.
\end{array}
\]

Since $L$ is linearly compact, $\lim \pi_F$ is epic, so is $\lim \varphi_F$. Note that $\lim X_F \cong \bigcap_{F \in \mathcal{F}} X_F$. Also, $\lim p_F : \prod_{\lambda \in \Lambda} L/L_\lambda \to \lim \prod_{\lambda \in F} L/L_\lambda$ is an isomorphism and hence induces an isomorphism $A\hat{x} \cong \lim A\hat{x}_F$. It follows that $\phi(\bigcap_{F \in \mathcal{F}} X_F) = A\hat{x}$. Thus $\hat{x} \in \text{Im } \phi$.

(2) $\Rightarrow$ (1). Let $\{\pi_\lambda : L \to L_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of epimorphisms in $\text{Mod } A$. We may consider $\lim L_\lambda$ as a submodule of $\prod_{\lambda \in \Lambda} L_\lambda$. Let $(x_\lambda) \in \lim L_\lambda$ and for each $\lambda \in \Lambda$ choose $y_\lambda \in L$ with $\pi_\lambda(y_\lambda) = x_\lambda$. Then, since for any nonempty finite subset $F$ of $\Lambda$ there exists $\lambda_0 \in \Lambda$ such that $\lambda_0 \geq \lambda$ for all $\lambda \in F$, the system of congruences $\{x \equiv y_\lambda \mod \text{Ker } \pi_\lambda\}_{\lambda \in \Lambda}$ is finitely solvable and thus solvable. Hence $\lim \pi_\lambda : L \to \lim L_\lambda$ is an epimorphism. \(\square\)

Let $H U_R$ be a bimodule and $K \in \text{Mod } R^{\text{op}}$. For a pair of a subset $X$ of $(K_R)^*$ and a subset $M$ of $K_R$, we set $r_M(X) = \{a \in M | h(a) = 0 \text{ for all } h \in X\}$ and $l_X(M) = \{h \in X | h(a) = 0 \text{ for all } a \in M\}$, where $(\cdot)^* = \text{Hom}_R(-, H U_R)$.

The next lemma is due essentially to [7, Lemma 4].

**Lemma 1.6.** Let $H U_R$ be a bimodule and $K \in \text{Mod } R^{\text{op}}$ a module such that $U_R$ is $K$-injective. Assume $X = l_K(r_K(X))$ for every submodule $X$ of $(K_R)^*$. Then $(K_R)^*$ is linearly compact.
Proof. Let \( \{ \pi_\lambda : K^* \to X_\lambda \}_{\lambda \in \Lambda} \) be an inverse system of epimorphisms in \( \text{Mod} \, H \). For \( \lambda \in \Lambda \), put \( Y_\lambda = \text{Ker} \, \pi_\lambda \) and \( M_\lambda = r_K(Y_\lambda) \), and let \( j_\lambda : M_\lambda \to K \) be the inclusion. Then for each \( \lambda \in \Lambda \), since \( \text{Ker} \, j_\lambda^* \cong l_{K^*}(M_\lambda) = Y_\lambda \), and since \( j_\lambda^* : K^* \to M_\lambda^* \) is epic, there exists an isomorphism \( \phi_\lambda : M_\lambda^* \to X_\lambda \) with \( \pi_\lambda = \phi_\lambda \circ j_\lambda^* \). Since \( \varinjlim \, j_\lambda \) is monic, \( \varinjlim \, j_\lambda^* \cong (\varinjlim \, \phi_\lambda)^* \) is epic. Also, \( \varinjlim \, \phi_\lambda \) is an isomorphism. Thus \( \varinjlim \pi_\lambda = (\varinjlim \, \phi_\lambda) \circ (\varinjlim \, j_\lambda^*) \) is epic.

**Corollary 1.7.** Let \( A \) be a left or right perfect ring. Assume \( A_A \) is injective and \( I = I_A(\varinjlim \, r_A(I)) \) for every left ideal \( I \) of \( A \). Then \( A \) is quasi-Frobenius.

Proof. It follows by Lemma 1.6 that \( A_A \) is linearly compact. Thus by [10, Propositions 2.9 and 2.12] \( A \) is left noetherian.

### 2. Bilinear maps into colocal bimodules

In this section, as further preliminaries, we modify results of [8, Section 1]. For a left \( H \)-module \( H_L \), a right \( R \)-module \( K_R \) and an \( H \)-\( R \)-bimodule \( HUR \), we call a map \( \varphi : H_L \times K_R \to HUR \) \( H \)-\( R \)-bilinear if \( K_R \to HUR, a \mapsto \varphi(x, a) \), is \( R \)-linear for every \( x \in L \) and \( H_L \to H \text{Hom}_R(K_R, HUR), x \mapsto (a \mapsto \varphi(x, a)) \), is \( H \)-linear.

Throughout this section, \( \varphi : H_L \times K_R \to HUR \) is a fixed \( H \)-\( R \)-bilinear map. For a pair of a subset \( X \) of \( L \) and a subset \( M \) of \( \Lambda \), we set \( V_\Lambda(X) = \{ \alpha \in M | \varphi(x, \alpha) = 0 \text{ for all } x \in X \} \) and \( Z_\Lambda(M) = \{ x \in X | \varphi(x, \alpha) = 0 \text{ for all } \alpha \in M \} \). We denote by \( \Lambda(L, K) \) the lattice of submodules \( X \) of \( H_L \) with \( X = \varinjlim \, r_K(X) \) and by \( \Lambda(L, K) \) the lattice of submodules \( M \) of \( K_R \) with \( M = \varinjlim \, r_K(\varinjlim \, r_K(M)) \).

**Remarks** (see e.g. [3, Part I] for details). (1) Let \( X \) be a subset of \( L \). Then \( \varphi(X, r_K(X)) = 0 \) implies \( X \subset \varinjlim \, r_K(X) \) and thus \( r_K(\varinjlim \, r_K(X)) \subset r_K(X) \). Also, \( \varphi(\varinjlim \, r_K(X), r_K(X)) = 0 \) implies \( r_K(X) \subset \varinjlim \, r_K(\varinjlim \, r_K(X)) \). Thus \( r_K(X) = \varinjlim \, r_K(\varinjlim \, r_K(X)) \) and \( r_K(X) \in \Lambda(L, K) \).

(2) Let \( X \) be a subset of \( L \). For any \( Y \in \Lambda(L, K) \) with \( X \subset Y \), \( l_L(r_K(Y)) \subset l_L(r_K(X)) = Y \). Thus \( l_L(r_K(X)) \) is the smallest module in \( \Lambda(L, K) \) containing \( X \).

(3) Let \( \{ X_\lambda \}_{\lambda \in \Lambda} \) be a family of submodules of \( H_L \). For any \( \mu \in \Lambda \), since \( \bigcap_{\lambda \in \Lambda} X_\lambda \subset X_\mu \subset \sum_{\lambda \in \Lambda} X_\lambda \), \( r_K(\sum_{\lambda \in \Lambda} X_\lambda) \subset r_K(X_\mu) \subset r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) \). Thus \( r_K(\sum_{\lambda \in \Lambda} X_\lambda) \subset \varinjlim \, r_K(X_\lambda) \) and \( \sum_{\lambda \in \Lambda} r_K(X_\lambda) \subset r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) \). Let \( a \in \bigcap_{\lambda \in \Lambda} r_K(X_\lambda) \). Since \( \varphi(X_\lambda, a) = 0 \) for all \( \lambda \in \Lambda \), and since \( H_L \to HU, x \mapsto \varphi(x, a) \), is \( H \)-linear, \( \varphi(\sum_{\lambda \in \Lambda} X_\lambda, a) = 0 \) and \( a \in r_K(\sum_{\lambda \in \Lambda} X_\lambda) \). Thus \( r_K(\sum_{\lambda \in \Lambda} X_\lambda) = \bigcap_{\lambda \in \Lambda} r_K(X_\lambda) \).

(4) Let \( \{ X_\lambda \}_{\lambda \in \Lambda} \) be a family of submodules of \( H_L \) with the \( X_\lambda \in \Lambda(L, K) \). Then by (3) \( \bigcap_{\lambda \in \Lambda} X_\lambda = \bigcap_{\lambda \in \Lambda} l_L(r_K(X_\lambda)) = l_L(\sum_{\lambda \in \Lambda} r_K(X_\lambda)) \). Thus \( r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) = r_K(l_L(\sum_{\lambda \in \Lambda} r_K(X_\lambda))) \) and by (2) \( r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) \) is the smallest module in \( \Lambda(L, K) \) containing \( \sum_{\lambda \in \Lambda} r_K(X_\lambda) \), so that \( r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) = \sum_{\lambda \in \Lambda} r_K(X_\lambda) \) whenever \( \sum_{\lambda \in \Lambda} r_K(X_\lambda) \in \Lambda(L, K) \).
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(5) We have an anti-isomorphism of lattices \( \mathcal{A}_L(L, K) \rightarrow \mathcal{A}_r(L, K), X \mapsto r_K(X) \).
In particular, \( \mathcal{A}_L(L, K) \) satisfies the ACC (resp. DCC) if and only if \( \mathcal{A}_r(L, K) \) satisfies the DCC (resp. ACC).

Recall that a module is called colocal if it has simple essential socle. We call a bimodule \( HUR \) colocal if both \( HU \) and \( UR \) are colocal modules.

**Lemma 2.1.** Let \( HUR \) be a colocal bimodule. Then \( \soc(HU) = \soc(UR) \).

**Proof.** Since \( \soc(HU) \) is a subbimodule of \( HUR \), \( \soc(UR) \subset \soc(HU) \). Similarly, \( \soc(HU) \subset \soc(UR) \). Thus \( \soc(HU) = \soc(UR) \).

Throughout the rest of this section, \( HUR \) is assumed to be a colocal bimodule with \( HS = \soc(HU) = \soc(UR) \), and \( (\ )^* \) denotes both the \( U \)-dual functors.

**Lemma 2.2.** The following hold.

(1) The canonical ring homomorphisms \( H \rightarrow \text{End}_R(S_R) \) and \( R \rightarrow \text{End}_H(HS)^{op} \)
are surjective.

(2) \( (HS)^* \cong SR \) and \( (SR)^* \cong HS \).

**Proof.** (1) Let \( 0 \neq u \in S \). Then \( S = Hu = uR \). For any \( h \in \text{End}_R(S_R) \), \( h(u) = au \) for some \( a \in H \) and \( h(ub) = h(u)b = (au)b = a(ub) \) for all \( b \in R \).
Thus the canonical ring homomorphism \( H \rightarrow \text{End}_R(S_R) \) is surjective. Similarly, the canonical ring homomorphism \( R \rightarrow \text{End}_H(HS)^{op} \) is surjective.

(2) Let \( \pi : R_R \rightarrow S_R \) be an epimorphism. We have a monomorphism \( \mu : (S_R)^* \rightarrow HU \) such that \( \mu(h) = (\pi^*(h))(1) \) for \( h \in (S_R)^* \). Put \( u = \pi(1) \). Then \( \mu(h) = h(u) \in S \) for all \( h \in (S_R)^* \) and \( \text{Im} \mu = HS \), so that \( (S_R)^* \cong HS \). Similarly, \( (HS)^* \cong SR \).

**Lemma 2.3.** Let \( N \subset M \) be submodules of \( K_R \) with \( N = r_K(l_L(N)) \) and \( M/N_R \) simple. Then the following hold.

(1) \( M/N \cong SR \) and \( l_L(N)/l_L(M) \cong (M/N)^* \cong HS \).

(2) \( M = r_K(l_L(M)) \).

**Proof.** (1) Since \( M \neq N = r_K(l_L(N)) \), \( l_L(M) \subset l_L(N) \) with \( l_L(N)/l_L(M) \neq 0 \). Let \( j : N_R \rightarrow M_R \) be the inclusion. Then we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & l_L(M) & \longrightarrow & L & \longrightarrow & M^* \\
& & \downarrow & & \| & & \downarrow j^* \\
0 & \longrightarrow & l_L(N) & \longrightarrow & L & \longrightarrow & N^*.
\end{array}
\]
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Thus $0 \neq l_L(N)/l_L(M)$ embeds in $\text{Ker } j^* \cong (M/N)^*$. Hence $(M/N)^* \neq 0$, so that $M/N \cong S_R$ and by Lemma 2.2(2) $(M/N)^* \cong H S$.

(2) Since $l_L(M) \subseteq l_L(N)$ with $l_L(N)/l_L(M)$ simple, one can apply the part (1) to conclude that $r_K(l_L(M))/r_K(l_L(N))$ is simple. Thus, since $r_K(l_L(N)) = N \subseteq M \subseteq r_K(l_L(M))$ with both $M/N$ and $r_K(l_L(M))/r_K(l_L(N))$ simple, it follows that $M = r_K(l_L(M))$.

Lemma 2.4. Let $M$ be a submodule of $K_R$ with $r_K(L) \subseteq M$ and $\ell(M/r_K(L)_R) < \infty$. Then the following hold.

(1) Every composition factor of $M/r_K(L)_R$ is isomorphic to $S_R$.
(2) $M = r_K(l_L(M))$.

Proof. Since $r_K(L) = r_K(l_L(r_K(L)))$, Lemma 2.3 enables us to make use of induction on $\ell(M/r_K(L)_R)$.

Lemma 2.5 ([8, Lemma 1.3]). $\ell(H L/l_L(K)) = \ell(K/r_K(L)_R)$.

Proof. By symmetry we may assume $\ell(H L/l_L(K)) < \infty$. Let $l_L(K) = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L$ be a chain of submodules of $H L$ with the $L_{i+1}/L_i$ simple. Then by Lemma 2.3 we get a chain of submodules $r_K(L) = r_K(L_n) \subseteq \cdots \subseteq r_K(L_1) \subseteq r_K(L_0) = K$ in $K_R$ with the $r_K(L_i)/r_K(L_{i+1})$ simple.

Lemma 2.6. Assume $R$ is left perfect. Then the following are equivalent.

(1) $\ell(K/r_K(L)_R) < \infty$.
(2) $A_r(L, K)$ satisfies both the ACC and the DCC.
(3) $A_r(L, K)$ satisfies the ACC.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). It follows by Lemma 2.4 that there exists a maximal element $K_0$ in the set of submodules $M$ of $K_R$ with $r_K(L) \subseteq M$ and $\ell(M/r_K(L)_R) < \infty$. We claim $K_0 = K$. Otherwise, there exists a submodule $M$ of $K_R$ with $K_0 \subseteq M$ and $M/K_0$ simple, a contradiction.

3. Simple-injective colocal modules

Throughout the rest of this note, $A$ stands for a ring with Jacobson radical $J$. For any pair of a right module $L_A$ and a left ideal $K$ of $A$, we have a canonical bilinear map $H L \times K_R \to H L K_R$, $(x, a) \mapsto xa$, where $H = \text{End}_A(L_A)$ and $R = \text{End}_A(A K)^{\text{op}}$, so that, in case $H L K_R$ is a colocal bimodule, we can apply results of the preceding section.
Lemma 3.1. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module and $f \in A$ a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Then the following hold.

1. $l_{L}(Af) = 0$.
2. $l_{L}(I) = l_{L}(I)$ for every right ideal $I$ of $A$.
3. $Lf_{fAf}$ is colocal with $\text{soc}(Lf_{fAf}) = \text{soc}(L_A)f$.

Proof. (1) For any $0 \neq x \in L$, since $\text{soc}(L_A) \subset xA$, $0 \neq \text{soc}(L_A)f \subset xAf$ and thus $x \notin l_{L}(Af)$.

(2) We have $l_{L}(I) \subset l_{L}(I)$. For any $x \in l_{L}(I)$, since $xIAf = xI$, by the part (1) $xI \subset l_{L}(Af) = 0$ and $x \in l_{L}(I)$. Thus $l_{L}(I) \subset l_{L}(I)$.

(3) Let $0 \neq x \in \text{soc}(L_A)f$. For any $0 \neq y \in Lf$, since $xA \subset yA$, $xfAf = xAf \subset yAf = yfAf$. Thus $Lf_{fAf}$ is colocal and $\text{soc}(Lf_{fAf}) = \text{soc}(L_A)f$. \hfill \Box

Lemma 3.2. Let $L \in \text{Mod } A^{\text{op}}$ and $f \in A$ a local idempotent. Then the following are equivalent.

1. $L_A$ is colocal with $\text{soc}(L_A) \cong fA/fJ$.
2. $Lf_{fAf}$ is colocal and $l_{L}(Af) = 0$.

Proof. (1) $\Rightarrow$ (2). By (3) and (1) of Lemma 3.1.

(2) $\Rightarrow$ (1). Since by Lemma 1.2(2) $E(L_A)f_{fAf} \cong E(Lf_{fAf}) \cong (fAf/fJ_{fAf}) \cong E(fA/fJ_{A})$, by Lemma 1.2(3) $E(L_A) \cong \text{Hom}_{fAf}(Af, (E(L_A)f) \cong \text{Hom}_{fAf}(Af, E(fA/fJ_{A})f) \cong E(fA/fJ_{A})$. Thus $L_A$ is colocal with $\text{soc}(L_A) \cong fA/fJ$. \hfill \Box

Corollary 3.3. Let $e, f \in A$ be local idempotents. Then the following are equivalent.

1. $eA/l_{eA}(Af)_A$ is colocal with $\text{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ$.
2. $eAf_{fAf}$ is colocal.

Proof. Put $L = eA/l_{eA}(Af)_A$. Then $l_{L}(Af) = 0$ and, since $l_{eA}(Af)f = 0$, $Lf_{fAf} \cong eAf_{fAf}$. Thus Lemma 3.2 applies. \hfill \Box

Following Harada [4], we call a module $L_A$ $M$-simple-injective if for any submodule $N$ of $M_A$ every $\theta : N_A \rightarrow L_A$ with $\text{Im} \theta$ simple can be extended to some $\bar{\theta} : M_A \rightarrow L_A$. In case $L$ is $L$-simple-injective, $L$ is called simple-quasi-injective.

Lemma 3.4. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Then the following hold.

1. If $L_A$ is $A$-simple-injective, then $M = r_{Af}(l_{L}(M))$ for every submodule $M$ of $Af_{fAf}$.
2. If $Hf_{fAf}$ is a colocal bimodule and $M = r_{Af}(l_{L}(M))$ for every submodule $M$ of $Af_{fAf}$, then $L_A$ is $A$-simple-injective.
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Proof. (1) Let $M$ be a submodule of $A f_{A f}$ and put $N = r_{A f}(l_M(A))$. We claim $M = N$. Suppose otherwise. Note first that $l_M(N) = l_M(M)$. Since $(N A/M A)f \cong N/M \neq 0$, there exist right ideals $K$, $I$ of $A$ such that $M A \subseteq K \subseteq I \subseteq N A$ and $I/K \cong f_A/f_J \cong \text{soc}(L_A)$. Then we have $\theta : I_A \rightarrow L_A$ with $\text{Im} \theta = \text{soc}(L_A)$ and $\text{Ker} \theta = K$. Let $\mu : I_A \rightarrow A_A$ be the inclusion. There exists $\phi : A_A \rightarrow A_A$ with $\phi \circ \mu = \theta$. Then $\phi(1)I = \phi(I) = \theta(I) \neq 0$ and $\phi(1)K = \phi(K) = \theta(K) = 0$. Thus $\phi(1) \in l_M(K)$ and $\phi(1) \neq l_M(I)$. Since $l_M(N) = l_M(N A) \subset l_M(I) \subset l_M(K) \subset l_M(M A) = l_M(M)$, $l_M(K) \neq l_M(I)$ implies $l_M(M) \neq l_M(N)$, a contradiction.

(2) Let $I$ be a nonzero right ideal of $A$ and $\mu : I_A \rightarrow A_A$ the inclusion. Let $\theta : I_A \rightarrow L^A$ with $\text{Im} \theta = \text{soc}(L^A)$ and put $\Lambda = \text{Ker} \theta$. Since by Lemma 1.2(1) $I f/K f_{A f} \cong (I/K)f_{A f}$ is simple, by Lemma 2.3(1) so is $h l_M(K f)/l_M(I f)$. Let $a \in I f$ with $a \notin K f$. Then, since $l_M(K f)a \neq 0$ and $l_M(I f)a = 0$, $h l_M(K f)a$ is simple. Thus by Lemmas 2.1 and 3.1(3) $l_M(K f)a = soc(L f_{A f}) = soc(L^A f)$, so that $\theta(a) = \theta(af) = \theta(a) = xa$ for some $x \in l_M(K f)$. Define $\phi : A_A \rightarrow L_A$ by $1 \mapsto x$. Then, since by Lemma 3.1(2) $x \in l_M(K f) = l_M(K)$, and since $I = K + aA$, we have $\phi \circ \mu = \theta$.

Lemma 3.5. Let $L \in \text{Mod} \ A^{op}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong f_A/f_J$. Then the following hold.

(1) If $L_A$ is simple-quasi-injective, then $H f_{f_{A f}}$ is a colocal bimodule and $l_M(A f) = 0$.

(2) If $L_A$ is $A$-simple-injective, then $r_{A f}(L) = 0$ and $r_{A}(L/L_J A) \subset l_A(soc(A A f))$.

Proof. (1) By Lemma 3.2 $L f_{A f}$ is colocal and $l_M(A f) = 0$. Let $0 \neq x \in soc(L_A)f$. We claim that $x \in H y$ for all $0 \neq y \in L f$. Note that $r_{f A}(x) = f J$. Let $0 \neq y \in L f$. Then $r_{f A}(y) \subset f J = r_{f A}(x)$ and we have $\theta : y A_A \rightarrow x A_A = soc(L_A)$, $ya \mapsto xa$. Let $\mu : soc(L_A) \rightarrow L_A$ and $\nu : y A_A \rightarrow L_A$ be inclusions. There exists $h \in H$ with $h \circ \nu = \mu \circ \theta$, so that $x = h(y) \in H y$. Thus $H L f$ is colocal.

(2) By Lemma 3.4(1) $r_{A f}(L) = r_{A f}(l_M(0)) = 0$. Next, let $a \in r_{A}(L/L_J)$. Since $L a \subset L_J$, $L a(soc(A A f)) \subset L_J(soc(A A f)) = 0$. Thus $a(soc(A A f)) \subset r_{A f}(L) = 0$ and $a \in l_A(soc(A A f))$.

Lemma 3.6 ([5, Lemma 3.3]). Let $L \in \text{Mod} \ A^{op}$ be a simple-quasi-injective module with $soc(L_A) \neq 0$. Assume $\text{End}_A(L_A)$ is a local ring. Then $soc(L_A)$ is simple.

Proof. Let $S$ be a simple submodule of $soc(L_A)$. Suppose to the contrary that $S \neq soc(L_A)$. Let $\pi : soc(L_A) \rightarrow S_A$ be a projection and $\mu : soc(L_A) \rightarrow L_A$, $\nu : S_A \rightarrow L_A$ inclusions. There exists $\phi : L_A \rightarrow L_A$ with $\phi \circ \mu = \nu \circ \pi$. Since $\pi$ is not monic, $\phi$ is not an isomorphism. Thus $\phi \in \text{rad End}_A(L_A)$ and $(id_L - \phi)$ is a unit in $\text{End}_A(L_A)$, so that for $0 \neq x \in S$, since $\phi(x) = \pi(x) = x$, $(id_L - \phi)(x) = 0$ and $x = 0$, a contradiction.
4. Injectivity of colocal modules

In this section, by extending the previous results [8, Theorems 2.7, 2.8 and Proposition 2.9], we provide several sufficient conditions for a colocal module over a left or right perfect ring \( A \) to be injective.

Lemma 4.1 ([5, Lemma 3.4]). Let \( A \) be a semiperfect ring and \( L \in \text{Mod} \ A^{\text{op}} \) an \( A \)-simple-injective colocal module of finite Loewy length. Then \( LA \) is injective.

Proof. Let \( I \) be a right ideal of \( A \) and \( \mu : I_A \to A \) the inclusion. Let \( \theta : I_A \to L_A \). We make use of induction on the Loewy length of \( \theta(I) \) to show the existence of \( \phi : A_A \to L_A \) with \( \theta = \phi \circ \mu \). Let \( n = \min \{ k \geq 0 | \theta(I)J^k = 0 \} \). We may assume \( n > 0 \). Since \( \text{soc}(L_A) \) is simple, \( \text{soc}(L_A) = \theta(I)J^{n-1} = \theta(IJ^{n-1}) \). Let \( \mu_1 \) and \( \theta_1 \) denote the restrictions of \( \mu \) and \( \theta \) to \( IJ^{n-1} \), respectively. Then \( \text{Im} \theta_1 = \text{soc}(L_A) \) and there exists \( \phi_1 : A_A \to L_A \) with \( \phi_1 \circ \mu_1 = \theta_1 \). Since \( (\theta - \phi_1 \circ \mu)(IJ^{n-1}) = 0 \), by induction hypothesis there exists \( \phi_2 : A_A \to L_A \) with \( \phi_2 \circ \mu = \theta - \phi_1 \circ \mu \). Thus \( \theta = (\phi_1 + \phi_2) \circ \mu \). \( \square \)

Theorem 4.2. Let \( A \) be a semiperfect ring. Let \( L \in \text{Mod} \ A^{\text{op}} \) be a colocal module of finite Loewy length and put \( H = \text{End}_A(L_A) \). Let \( f \in A \) be a local idempotent with \( \text{soc}(L_A) \cong fA/fJ \). Then the following are equivalent.

(1) \( LA \) is injective.
(2) \( HfAfL \) is a colocal bimodule and \( M = r_{AfL}L(M) \) for every submodule \( M \) of \( AfAfL \).

Proof. (1) \( \Rightarrow \) (2). By Lemmas 3.5(1) and 3.4(1).
(2) \( \Rightarrow \) (1). By Lemmas 3.4(2) and 4.1. \( \square \)

Corollary 4.3. Let \( A \) be a semiperfect ring. Let \( L \in \text{Mod} \ A^{\text{op}} \) be a colocal module of finite Loewy length and put \( H = \text{End}_A(L_A) \). Let \( f \in A \) be a local idempotent with \( \text{soc}(L_A) \cong fA/fJ \). Assume \( HfAfL \) is a colocal bimodule and \( M = r_{AfL}L(M) \) for every submodule \( M \) of \( AfAfL \) with \( r_{AfL}(L) \subset M \). Then \( LA \) is quasi-injective.

Proof. Put \( I = r_{AfL}(L) \). Then by Theorem 4.2 \( L_A/I \) is injective, so that \( L_A \) is quasi-injective. \( \square \)

Theorem 4.4. Let \( A \) be a left or right perfect ring. Let \( L \in \text{Mod} \ A^{\text{op}} \) be a colocal module and put \( H = \text{End}_A(L_A) \). Let \( f \in A \) be a local idempotent with \( \text{soc}(L_A) \cong fA/fJ \). Assume \( (Af/r_{AfL}(L))_{fAf} \) is an \( A \)-module and \( r_{AfL}(L) < \infty \). Then the following are equivalent.

(1) \( LA \) is injective.
(2) \( HfAfL \) is a colocal bimodule and \( r_{AfL}(L) = 0 \).
(3) \( H L f_{fAf} \) is a colocal bimodule and \( M = r_{Af}(l_L(M)) \) for every submodule \( M \) of \( A f_{fAf} \).

Proof. (1) \( \Rightarrow \) (2). By Lemma 3.5.
(2) \( \Rightarrow \) (3). By Lemma 2.4.
(3) \( \Rightarrow \) (1). By Lemma 3.4(2) \( L_A \) is \( A \)-simple-injective. Note that \( r_{Af}(L) = 0 \). Thus \( \ell(Af_{fAf}) < \infty \) and by Lemma 1.1 \( J^n f = 0 \) for some \( n \geq 1 \), so that \( LJ^n Af = LJ^n f = 0 \) and by Lemma 3.1(1) \( LJ^n \subseteq l_L(Af) = 0 \). Hence by Lemma 4.1 \( L_A \) is injective.

**Corollary 4.5.** Let \( A \) be a left or right perfect ring. Let \( L \in \text{Mod } A^{\text{op}} \) be a colocal module and put \( H = \text{End}_A(L_A) \). Let \( f \in A \) be a local idempotent with \( \text{soc}(L_A) \cong fA/fJ \). Assume \( H L f_{fAf} \) is a colocal bimodule and \( \ell(Af/r_{Af}(L)fAf) < \infty \). Then \( L_A \) is quasi-injective.

Proof. Put \( I = r_A(L) \). Then \( r_{Af/If}(L) = 0 \) and by Theorem 4.4 \( L_{A/I} \) is injective, so that \( L_A \) is quasi-injective.

**Proposition 4.6.** Let \( A \) be a left or right perfect ring. Let \( L \in \text{Mod } A^{\text{op}} \) be a colocal module and put \( H = \text{End}_A(L_A) \). Let \( f \in A \) be a local idempotent with \( \text{soc}(L_A) \cong fA/fJ \). Then the following are equivalent.

(1) \( L_A \) is injective and \( X = l_L(r_{Af}(X)) \) for every submodule \( X \) of \( HL \).
(2) \( H L f_{fAf} \) is a colocal bimodule, \( r_{Af}(L) = 0 \) and \( \ell(Af_{fAf}) < \infty \).

Proof. (1) \( \Rightarrow \) (2). By Lemma 3.5(1) \( H L f_{fAf} \) is a colocal bimodule, and by Lemma 3.5(2) \( r_{Af}(L) = 0 \). It remains to show \( \ell(Af_{fAf}) < \infty \). Put \( K_n = Af(fJf)^n \) for \( n \geq 0 \). We claim \( \ell(K_n/K_{n+1}fAf) < \infty \) for all \( n \geq 0 \). Let \( n \geq 0 \). Note that by Lemma 3.4(1) the lattice of submodules of \( Af_{fAf} \) is anti-isomorphic to the lattice of submodules of \( HL \). Thus \( \ell(K_n/K_{n+1}fAf) = \ell(Hl_L(K_{n+1})/l_L(K_n)) \). Also, since \( \text{rad}(K_n/K_{n+1}fAf) = 0 \), \( Hl_L(K_{n+1})/l_L(K_n) \) is semisimple. For any submodule \( X \) of \( HL \), since \( r_{Af}(X) = r_A(X)f \), by Lemma 3.1(2) \( X = l_L(r_{Af}(X)) = l_L(r_A(X)f) = l_L(r_A(X)) \). Thus by Lemma 1.6 \( HL \cong \text{Hom}_A(A_A, H L_A) \) is linearly compact, so is \( HL(K_{n+1})/l_L(K_n) \) by [10, Proposition 2.2]. Hence by [10, Lemma 2.3] \( \ell(K_n/K_{n+1}fAf) = \ell(Hl_L(K_{n+1})/l_L(K_n)) < \infty \). Since \( \ell(fJf/(fJf)^2fAf) < \ell(K_1/K_2fAf) < \infty \), by [9, Lemma 11] \( fAf \) is right artinian. Then \( \ell(K_0/K_1fAf) < \infty \) implies \( \ell(Af_{fAf}) < \infty \).

(2) \( \Rightarrow \) (1). By Theorem 4.4 \( L_A \) is injective. Since by Lemma 3.1(1) \( l_L(Af) = 0 \), by Lemma 2.5 \( \ell(HL) = \ell(Af_{fAf}) < \infty \) and thus by Lemma 2.4 \( X = l_L(r_{Af}(X)) \) for every submodule \( X \) of \( HL \).
5. Colocal pairs

We call a pair \((eA, Af)\) of a right ideal \(eA\) and a left ideal \(Af\) in \(A\) a colocal pair if \(e, f \in A\) are local idempotents and \(eAeAfAf\) is a colocal bimodule. Note that by Lemma 2.5 \(\ell(eAeAfAf) = \ell(Af/rAf(eA)fAf)\) for every colocal pair \((eA, Af)\) in \(A\). In case \(\ell(eAeAfAf) = \ell(Af/rAf(eA)fAf) < \infty\), a colocal pair \((eA, Af)\) in \(A\) is called finite.

In [5], a pair \((eA, Af)\) of a right ideal \(eA\) and a left ideal \(Af\) in \(A\) is called an i-pair if \(e, f \in A\) are local idempotents, \(eA_A\) is colocal with \(\text{soc}(eA_A) \cong fA/fJ\) and \(A Af\) is colocal with \(\text{soc}(A Af) \cong Ae/Je\).

**Lemma 5.1.** Let \(e, f \in A\) be local idempotents. Then the following are equivalent.

1. \((eA, Af)\) is an i-pair in \(A\).
2. \((eA, Af)\) is a colocal pair in \(A\) with \(l_{eA}(Af) = 0\) and \(r_{Af}(eA) = 0\).

Proof. (1) \(\Rightarrow\) (2). By (1) and (3) of Lemma 3.1.

(2) \(\Rightarrow\) (1). By Corollary 3.3.

The equivalence (1) \(\Leftrightarrow\) (2) of the next lemma has been established in [5, Theorem 3.7]. Here we provide another proof of the implication (2) \(\Rightarrow\) (1) which does not appeal to Morita duality.

**Lemma 5.2 ([5, Theorem 3.7]).** Let \((eA, Af)\) be an i-pair in a left or right perfect ring \(A\). Then the following are equivalent.

1. \((eA, Af)\) is finite.
2. Both \(eA_A\) and \(A Af\) are injective.
3. \(eA_A\) is injective and \(A Af\) is \(A\)-simple-injective.

Proof. (1) \(\Rightarrow\) (2). By Theorem 4.4.

(2) \(\Rightarrow\) (3). Obvious.

(3) \(\Rightarrow\) (1). It follows by Lemma 3.4(1) that \(X = l_{eA}(r_{Af}(X))\) for every submodule \(X\) of \(eAeA\). Thus by Proposition 4.6 \(\ell(AfAf) < \infty\).  

**Lemma 5.3.** Let \((eA, Af)\) be a finite colocal pair in a left or right perfect ring \(A\). Then the following hold.

1. \(eA/l_{eA}(Af)_A\) is a quasi-injective colocal module with \(\text{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ\).
2. If \(r_{Af}(eA) = 0\), then \(E(fA/fJa) \cong eA/l_{eA}(Af)\), so that \(E(fA/fJa)\) is quasi-projective and \(eA/l_{eA}(Af)_A\) is injective.
Proof. Put \( I = l_A(Af) \) and \( L = eA/eI_A \). Then \( l_{eA}(Af) = eI \) and \( l_L(Af) = 0 \).

Note that, since \( If = 0 \), \( Lf_{fA} \cong eAf_{fA} \). Thus by Lemma 3.2 \( L_A \) is colocal with \( \text{soc}(L_A) \cong fA/fJ \). Since \( Lf_{fA} \cong eAf_{fA} \) and \( H = \text{End}_A(L_A) \cong eAee/eIe \), \( Hf_{fA} \) is a colocal bimodule. Note also that \( \ell(Af/r_{Af}(L)_{fAf}) = \ell(Af/r_{Af}(eA)_{fAf}) < \infty \).

(1) By Corollary 4.5 \( L_A \) is quasi-injective.
(2) By Theorem 4.4 \( L_A \) is injective. Thus, since \( \text{soc}(L_A) \cong fA/fJ \), \( E(fA/fJ_A) \cong L \). Since \( L_A/I \cong e(A/I)_{A/I} \) is projective, \( L_A \) is quasi-projective. \( \square \)

**Proposition 5.4.** Let \((eA, Af)\) be a colocal pair with \( l_{eA}(Af) = 0 \) in a left or right perfect ring \( A \). Put \( \overline{A} = A/r_{A}(eA) \). Let \( \pi : A \to \overline{A} \) be the canonical ring homomorphism and put \( \bar{e} = \pi(e), \bar{f} = \pi(f) \). Then the following are equivalent.

1. \((eA, Af)\) is finite.
2. \( eA_A \) is quasi-injective, \( eAeA \) is finitely generated and \( AAf/r_{Af}(eA) \) is injective.
3. \((\bar{e}A, \bar{f}A)\) is a finite \( \iota \)-pair in \( \overline{A} \).

Proof. Note first that \( \overline{A} \) is left or right perfect and \( \bar{e}, \bar{f} \in \overline{A} \) are local idempotents. Put \( I = r_A(eA) \). Then \( Ie = 0 \) and \( If = r_{Af}(eA) \). Thus \( \ell(eAe\overline{A}) = \ell(eAeA) \) and, since \( eAeA_{fAf} = eAeAf_{fAf} \) is a colocal bimodule, \((\bar{e}A, \bar{f}A)\) is a colocal pair in \( \overline{A} \).

(1) \( \Rightarrow \) (2). By Lemma 5.3(1) \( eA_A \) is quasi-injective, and by Lemma 5.3(2) \( AAf/r_{Af}(eA) \) is injective. Also, since \( \ell(eAeA) < \infty \), \( eAeA \) is finitely generated.

(2) \( \Rightarrow \) (3). By [3, Corollary 5.6A] \( \overline{eA} \cong eA \overline{A} \) is injective. Also, since \( A\overline{A} = \overline{AAf} \) \( r_{Af}(eA) \) is injective, so is \( \overline{Af} \). It is obvious that \( r_{Af}(\overline{eA}) = 0 \). For any \( a \in l_{eA}(\overline{A}) \), since \( Af \subseteq If, aAf = eaAf \subseteq eIf = 0 \) and \( a \in l_{eA}(Af) = 0 \). It follows that \( l_{eA}(\overline{Af}) = 0 \). Thus by Lemmas 5.1 and 5.2 \((\bar{e}A, \bar{f}A)\) is a finite \( \iota \)-pair in \( \overline{A} \).

(3) \( \Rightarrow \) (1). Obvious. \( \square \)

**Corollary 5.5.** Let \((eA, Af)\) be an \( \iota \)-pair in a left or right perfect ring \( A \). Then the following are equivalent.

1. \((eA, Af)\) is finite.
2. \( eA_A \) is quasi-injective, \( eAeA \) is finitely generated and \( AAf \) is injective.

6. Applications of colocal pairs I

In this section, as applications of colocal pairs, we extend recent results of Baba [1, Theorems 1 and 2] to left perfect rings and provide simple proofs of them.

**Lemma 6.1.** Let \( A \) be a left perfect ring and \( e \in A \) a local idempotent. Assume \( A_E = E(Ae/JE) \) is quasi-projective. Then \( A_E/JE \) is simple and for a local idempotent \( f \in A \) with \( A_E/JE \cong Af/Jf \) the following hold:

Proof.
(a) \( _AE \cong Af/r_Af(eA) \);
(b) \( eAeAf \cong eAeE \) is injective; and
(c) \((eA, Af)\) is a colocal pair in \(A\) with \(l_{eA}(Af) = 0\).

Proof. Put \( I = l_A(E) \). By Lemma 1.4 there exists a local idempotent \( f \in A \) such that \( _AE \cong Af/If \). We claim \( If = r_Af(eA) \). Since by Lemma 3.5(2) \( eAf = eIf \subseteq l_{eA}(E) = 0 \), \( If \subseteq r_Af(eA) \). Conversely, let \( a \in r_Af(eA) \). Then \( eA(a+If) = 0 \) and by Lemma 3.1(3) \( (a+If) \in r_Af/eA(eA) = 0 \), so that \( a \in If \). Next, since \( e(r_Af(eA)) = 0 \), \( _AeAf \cong eAeAf \) is colocal by Lemma 3.1(3) and injective by Lemma 1.2(2). Also, since \( \text{End} _A(AnAf/If) \cong fAf/fIf \), by Lemma 3.5(1) \( eAeAf \) is colocal. Finally, by Lemma 3.5(2) \( l_{eA}(Af) \subseteq l_{eA}(Af/r_Af(eA)) = l_{eA}(E) = 0 \).

**Theorem 6.2** (cf. [1, Theorem 1]). Let \( A \) be a left perfect ring and \( e, f \in A \) local idempotents. Put \( E = E(Ae/Je) \). Assume \( \ell(Af/r_Af(eA)f) < \infty \). Then the following are equivalent.

1. \( eA_A \) is quasi-injective with \( \text{soc}(eA_A) \cong fA/fJ \).
2. \( _AE \) is quasi-projective with \( _AE/JE \cong Af/Jf \).
3. \((eA, Af)\) is a colocal pair in \(A\) with \( l_{eA}(Af) = 0 \).
4. \( eAeAf \) is colocal and \( \text{soc}(eA_A) \cong fA/fJ \).

Proof. (1) \( \Rightarrow \) (3). By Lemma 3.5(1).
(3) \( \Rightarrow \) (1). By Lemma 5.3(1).
(2) \( \Rightarrow \) (3). By Lemma 6.1.
(3) \( \Rightarrow \) (2). By Lemma 5.3(2).
(3) \( \Rightarrow \) (4). By Corollary 3.3.
(4) \( \Rightarrow \) (3). By (3) and (1) of Lemma 3.1.

**Lemma 6.3.** Let \((eA, Af)\) be a colocal pair in a left or right perfect ring \(A\). Put \( E = E(Ae/Je) \) and \( H = \text{End} _A(AE)^{op} \). Assume \( \text{soc}(eA_A)f \neq 0 \). Then the following hold.

1. \( \text{soc}(eA_A)fA \) is the unique simple submodule of \( eA_A \) which is isomorphic to \( fA/fJ_A \).
2. If \((eA, Af)\) is finite, then \( AEX_X \) contains a submodule \( X \) such that \( AX \cong Af/r_Af(eA)f \), \( eAeAX_H \) is a colocal bimodule, \( \text{soc}(eA_A)fA \cap l_{eA}(X) = 0 \) and \( \ell(eAeA/l_{eA}(X)) < \infty \).

Proof. (1) Since \( \text{soc}(eA_A)f \neq 0 \), \( eA_A \) contains a simple submodule \( K \cong fA/fJ \).
On the other hand, by Corollary 3.3 \( eA/l_{eA}(Af)_A \) is colocal with \( \text{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ \). Thus \( K \) is the unique simple submodule of \( eA_A \) which is isomorphic to \( fA/fJ \). It follows that \( K = \text{soc}(eA_A)fA \).
(2) Put \( I = r_A(eA) \). Then \( If = r_Af(eA) \) and by Lemma 5.3(1) \( AAff/If \) is a quasi-injective colocal module with \( \text{soc}(AAff/If) \cong Ae/Je \). Thus \( A\text{E} \) contains a submodule \( X \cong AAff/If \). Then \( // = r_Af Af(eA) \) and by Lemma 5.3(1) \( AAff/If \) is a quasi-injective colocal module with \( \text{soc}(Aff/If) = Ae/eA(X) = 0 \). Finally, since \( l_A(X) = l_A(Af) \), \( (eA Af)/l_A(X) = (eA Af)/l_A(Af) < \infty \). \( \square \)

Lemma 6.4. Let \( A \) be a left perfect ring and \( e \in A \) a local idempotent. Put \( E = E(Ae/Ae) \) and \( H = \text{End}_A(Ae)^{op} \). Assume \( \text{soc}(eA) \cong \bigoplus_{i=1}^n f_iA/f_iJ \) with the \( (eA, Af_i) \) finite colocal pairs in \( A \). Then \( f_iA/f_iJ \not\cong f_jA/f_jJ \) for \( i \neq j \), \( \ell(EH) = \ell(eAeA/eA) < \infty \) and \( A\text{E}/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i \).

Proof. By Lemma 6.3(1) \( f_iA/f_iJ \not\cong f_jA/f_jJ \) for \( i \neq j \). Also, for each \( 1 \leq i \leq n \), by Lemma 6.3(2) \( A\text{E}/X_i \) contains a submodule \( X_i \) such that \( A\text{E}/X_i \cong Af_i/r_A(eA)f_i \), \( eAeX_i \) is a colocal bimodule, \( \text{soc}(eA)f_i \cap l_A(X_i) = 0 \) and \( \ell(eAeA/eA(X_i)) < \infty \). Put \( A\text{E}/X_i = \sum_{i=1}^n X_i \). Then, by Lemmas 3.1(1) and 2.5 \( \ell(X_i) = \ell(eAeA/eA(X_i)) < \infty \) for all \( 1 \leq i \leq n \), so that \( \ell(X_i) < \infty \). Also, since \( \text{soc}(eA)f_i \cap l_A(X_i) = 0 \) for all \( 1 \leq i \leq n \), by Lemma 6.3(1) \( \text{soc}(eA) \cap l_A(X_i) = 0 \). Thus, since \( eA \) has essential socle, \( eA(X_i) = 0 \). Since by Lemma 3.5(1) \( eAeX_i \) is a colocal bimodule, thus by Lemma 2.5 \( \ell(eAeA/eA) = \ell(X_i) < \infty \). Since by Lemma 1.3 we have a surjective ring homomorphism \( \rho_X : H \twoheadrightarrow \text{End}_A(A\text{E})^{op} \), \( h \mapsto h|_X \), it follows by Theorem 4.4 that \( A\text{E}/X_i \) is injective. Thus \( E = eAe \) and we have an epimorphism \( \bigoplus_{i=1}^n Af_i/Jf_i \to A\text{E}/JE \). On the other hand, since \( f_iA/f_iJ \not\cong f_jA/f_jJ \) for \( i \neq j \), it follows by Lemma 3.5(2) that \( A\text{E}/JE \) has a direct summand which is isomorphic to \( \bigoplus_{i=1}^n Af_i/Jf_i \). Thus \( A\text{E}/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i \). \( \square \)

Theorem 6.5 (cf. [1, Theorem 2]). Let \( A \) be a left perfect ring and \( e, f_1, f_2, \ldots, f_n \in A \) local idempotents. Put \( E = E(Ae/Ae) \). Assume \( (eA, Af_i) \) is a finite colocal pair in \( A \) for all \( 1 \leq i \leq n \). Then the following are equivalent.

(1) \( \text{soc}(eA) \cong \bigoplus_{i=1}^n f_iA/f_iJ \).

(2) \( A\text{E}/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i \).

Proof. (1) \( \Rightarrow \) (2). By Lemma 6.4.

(2) \( \Rightarrow \) (1). It follows by Lemmas 3.5(2) and 6.3(1) that \( \text{soc}(eA) \) is isomorphic to a direct summand of \( \bigoplus_{i=1}^n f_iA/f_iJ \). We may assume \( \text{soc}(eA) \cong \bigoplus_{i=1}^r f_iA/f_iJ \) for some \( 1 \leq r \leq n \). Then by Lemma 6.4 \( A\text{E}/JE \cong \bigoplus_{i=1}^r Af_i/Jf_i \), so that \( r = n \). \( \square \)

7. Applications of colocal pairs II

In this section, we provide some other applications of colocal pairs. Recall that a set \( \{e_1, \ldots, e_n\} \) of orthogonal local idempotents in a semiperfect ring \( A \) is called
Lemma 7.1 ([5, Lemma 3.5]). Let $A$ be a semiperfect ring and $\{e_1, \cdots, e_n\}$ a basic set of orthogonal local idempotents in $A$. Assume every $e_i A_A$ is $A$-simple-injective and has essential socle. Then there exists a permutation $\nu$ of the set $\{1, \cdots, n\}$ such that $(e_i A, A e_{\nu(i)})$ is an $i$-pair in $A$ for all $1 \leq i \leq n$.

Proof. By [5, Lemma 3.5] there exists a mapping $\nu: \{1, \cdots, n\} \to \{1, \cdots, n\}$ such that $(e_i A, A e_{\nu(i)})$ is an $i$-pair in $A$ for all $1 \leq i \leq n$. Then by the definition of $i$-pairs $\nu$ is injective.

Corollary 7.2. Let $A$ be a left perfect ring. Assume $A_A$ is simple-quasi-injective. Then $E(A A)$ and $E(A e_A)$ are injective cogenerators in $\text{Mod } A$ and $\text{Mod } A^{\text{op}}$, respectively.

Lemma 7.3. Let $A$ be a left perfect ring. Assume $A_r(A, A)$ satisfies the ACC and $e A_A$ is simple-quasi-injective for every local idempotent $e \in A$. Then $A$ is left artinian.

Proof. It suffices to show that $\ell(e e_A e A) < \infty$ for every local idempotent $e \in A$. Let $e \in A$ be a local idempotent. Since by Lemma 3.6 $e A_A$ is colocal, there exists a local idempotent $f \in A$ with $\text{soc}(e A_A) \cong f A/f J$. By Lemma 3.5(1) $(e A, A f)$ is a colocal pair in $A$ with $l_A(A f) = 0$. For each $M \in A_r(e A, A f)$, put $\tilde{M} = r_A(l_A(M)) \in A_r(A, A)$. Then $\tilde{M} f = r_A(f A l_A(M)) = M$ for every $M \in A_r(e A, A f)$. Thus, for $M, N \in A_r(e A, A f)$ with $M \subset N$, $M \subset \tilde{N}$ and $\tilde{M} = \tilde{N}$ implies $M = \tilde{M} f = \tilde{N} f = N f$. It follows that $A_r(e A, A f)$ satisfies the ACC. Thus by Lemmas 2.5 and 2.6 $\ell(e e_A e A) = \ell(A f r_A(f A) f A f) < \infty$.

Corollary 7.4. Let $A$ be a left perfect ring. Assume $A_r(A, A)$ satisfies the ACC and $A_A$ is simple-quasi-injective. Then $A$ is quasi-Frobenius.

Proof. By Lemma 7.3 $A$ is left artinian. Then it follows by Lemmas 3.6 and 4.1 that $A_A$ is injective.

References


M. Hoshino  
Institute of Mathematics  
University of Tsukuba  
Ibaraki, 305-8571  
Japan

T. Sumioka  
Department of Mathematics  
Osaka City University  
Osaka, 558-8585  
Japan