0. Introduction

In the present paper we show a commutator estimate of pseudo-differential operators in the framework of $L^2(\mathbb{R}^n)$. As an application we give a sharp form of Gårding’s inequality for sesqui-linear forms with coefficients in $B^2$. There has been similar kinds of commutator estimates. In [5], Kumano-go and Nagase obtain a result on commutator estimates and used it to show a sharp form of Gårding’s inequality for sesqui-linear form defined by elliptic differential operators of the form

$$B[u, v] = \sum_{|\alpha| \leq m, |\beta| \leq m} (a_{\alpha\beta}(x) D_x^\alpha u, D_x^\beta v)$$

where the coefficients $a_{\alpha\beta}(x)$ are $B^2(\mathbb{R}^n)$ functions.

In [3], Koshiba shows a sharp form of Gårding’s inequality for the form

$$B[u, v] = (p(X, D_x) u, v)$$

where the symbol $p(x, \xi)$ of the operator $p(X, D_x)$ is $B^2$ smooth in space variable $x$ and homogeneous in covariable $\xi$, and used the sharp form of Gårding’s inequality to the study of the stability of difference schemes for hyperbolic initial problems. On the other hand in [2], N. Jacob shows Gårding’s inequality for the form

$$B[u, v] = \sum_{i,j=1}^m \int_{\mathbb{R}^n} a_{i,j}(x) Q_j(D) u(x) P_i(D) v(x) dx$$

where $P_i(D)$ and $Q_j(D)$ are pseudo-differential operators, and $a_{i,j}(x)$ are non-smooth functions. The symbol class of the present paper is similar to the one in [2].

In section 1, as a preliminary we give definitions and fundamental facts of pseudo-differential operators. In section 2 we treat commutator estimates and give the main theorem relative to the commutator estimate. Finally in section 3 we give the sharp form of Gårding’s inequalities for our class of operators.
1. Preliminaries

Let \( \alpha = (\alpha_1 , \ldots , \alpha_n ) \) and \( \beta = (\beta_1 , \ldots , \beta_n ) \) be multi-integers. We denote
\[
|\alpha| = \alpha_1 + \ldots + \alpha_n
\]
We denote \( n \)-dimensional partial differential operators by
\[
\partial_\xi = \left( \frac{\partial}{\partial \xi_1} , \ldots , \frac{\partial}{\partial \xi_n} \right) \quad \text{and} \quad D_x = \frac{1}{i} \partial_x = \frac{1}{i} \left( \frac{\partial}{\partial x_1} , \ldots , \frac{\partial}{\partial x_n} \right)
\]
Then for a function \( f(x, \xi) \), we denote
\[
\partial_\xi^\alpha D_x^\beta f(x, \xi) = f^{(\alpha)}_{(\beta)}(x, \xi)
\]
and
\[
\partial_\xi^\alpha D_x^\beta D_x'^{\beta'} f(x, \xi, x') = f^{(\alpha)}_{(\beta, \beta')}(x, \xi, x')
\]
for a function \( f(x, \xi, x') \). We denote by \( B^k = B^k(\mathbb{R}^n) \) the set of \( k \)-times continuously differentiable functions on \( \mathbb{R}^n \) which are bounded with all upto \( k \)-th derivatives. We denote by \( C_0^\infty(\mathbb{R}^n) \) the set of \( C^\infty \)-smooth functions with compact support. Moreover \( S \) denotes the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^n \). Let \( \lambda \) be a real valued smooth function on \( \mathbb{R}^n \) satisfying
\[
(i) \quad \lambda(\xi) \geq 1 \\
(ii) \quad |\lambda^{(\alpha)}(\xi)| \leq C_\alpha \lambda(\xi)^{1-|\alpha|}
\]
for any \( \alpha \). Then we say that the function \( \lambda(\xi) \) is a basic weight function (see [5]).

Let \( \lambda(\xi) \) be a basic weight function. Then we say that a function \( p(x, \xi, x') \) on \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) belongs to \( S^m_{p, \delta, \lambda} \) if
\[
|p_{(\beta, \beta')}(x, \xi, x')| \leq C_{\alpha, \beta, \beta'} \lambda(\xi)^{m-p|\alpha|+\delta|\beta+\beta'|}
\]
for any multi-integers \( \alpha, \beta \) and \( \beta' \). For any \( p(x, \xi, x') \) in \( S^m_{p, \delta, \lambda} \), we define the pseudo-differential operator \( p(X, D_x, X') \) by
\[
p(X, D_x, X')u(x) = \frac{1}{(2\pi)^n} \int \int e^{i(x-x')\cdot \xi} p(x, \xi, x') u(x') dx' d\xi
\]
for any \( u \) in \( S \). In the present paper the integrations \( \int \) are taken on \( \mathbb{R}^n \). In particular if \( p(x, \xi, x') \in S^m_{p, \delta, \lambda} \) is independent in \( x' \), that is, \( p(x, \xi) \in S^m_{p, \delta, \lambda} \), the operator \( p(X, D_x) \) is defined, as usual, by
\[
p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi
\]
where \( \hat{u}(\xi) \) denotes the Fourier transform of \( u(x) \), that is,

\[
\hat{u}(\xi) = \int e^{-ix' \cdot \xi} u(x') dx'
\]

For any functions \( f(x) \) and \( g(x) \) on \( \mathbb{R}^n \), we define the inner product of \( L^2(\mathbb{R}^n) \) by

\[
(f, g) = \int f(x) \overline{g(x)} dx
\]

and denote the usual \( L^2 \) norm of function \( f(x) \) by

\[
||f|| = \left\{ \int |f(x)|^2 dx \right\}^{\frac{1}{2}}
\]

For any real number \( s \) and \( u \in S \), we define the norm \( ||u||_{s,\lambda} \) by

\[
||u||_{s,\lambda} = \left\{ \int |\lambda(D_x)^s u(x)|^2 dx \right\}^{\frac{1}{2}} = \left\{ \int |\lambda(\xi)^s \hat{u}(\xi)|^2 d\xi \right\}^{\frac{1}{2}}
\]

In particular if \( s = 0 \), the norm \( || \cdot ||_{s,\lambda} \) coincides with usual \( L^2 \) norm \( || \cdot || \).

The space \( H_{s,\lambda} \) is defined by the completion of the space \( S \) by the norm \( || \cdot ||_{s,\lambda} \). It is not difficult to see that the space \( H_{s,\lambda} \) is a Hilbert space.

Let \( s \) and \( m \) be real numbers. For a symbol \( p(x, \xi, x') \) in \( S^{m}_{\rho,\delta,\lambda} \), we have

\[
||p(X, D_x, X') u||_{s,\lambda} \leq C ||u||_{s+m,\lambda}
\]

for any \( u \) in \( S \) (see [5]).

2. Estimates of commutators

Let us consider commutators of pseudo-differential operators in \( L^2(\mathbb{R}^n) \). The estimates is essential for the proof of sharp Gårding's inequality. However the estimates itself are interesting subject.

Let \( 0 \leq \delta < 1 \) and \( \lambda(\xi) \) be a basic weight function and \( \phi(x) \) be an even function in \( C_0^\infty(\mathbb{R}^n) \) satisfying \( \int \phi(x) dx = 1 \). For a function \( b(x) \) on \( \mathbb{R}^n \), we define

\[
\tilde{b}(x, \xi) = \int \phi(z) b(x - \lambda(\xi)^{-\delta} z) dz
\]

Then in [5], the following approximation theorem is shown.
Lemma 2.1. If \( b(x) \) is a bounded function, then \( \tilde{b}(x, \xi) \) belongs to \( S^{0}_{1, \delta, \lambda} \).

(i) If \( b(x) \) is a function in \( B^{1} \), then \( \tilde{b}_{(\alpha)}(x, \xi) \) belongs to \( S^{0}_{1, \delta, \lambda} \) for \( |\alpha| \leq 1 \) and we have

\[
||\{b(X) - \tilde{b}(X, D_{x})\}u|| \leq C||u||_{-\delta, \lambda}
\]

for any \( u \) in \( S \).

(ii) If \( b(x) \) is a function in \( B^{2} \), then \( \tilde{b}_{(\alpha)}(x, \xi) \) belongs to \( S^{0}_{1, \delta, \lambda} \) for \( |\alpha| \leq 2 \) and we have

\[
||\{b(X) - \tilde{b}(X, D_{x})\}u|| \leq C||u||_{-2\delta, \lambda}
\]

for any \( u \) in \( S \).

From Lemma 2.1 we can prove the following lemma 2.2.

Lemma 2.2. (i) If \( b(x) \) is in \( B^{1} \), then we have

\[
||\{b(X) - \tilde{b}(X, D_{x})\}u||_{\delta, \lambda} \leq C||u||
\]

for any \( u \) in \( S \).

(ii) If \( b(x) \) is a \( B^{2} \), then we have

\[
||\{b(X) - \tilde{b}(X, D_{x})\}u||_{2\delta, \lambda} \leq C||u||
\]

for any \( u \) in \( S \).

Proof. We prove (i), and (ii) can be shown in a similar way.

For any \( u \) and \( v \) in \( S \), we have

\[
(\{b(X) - \tilde{b}(X, D_{x})\}u, v) = (u, \{\tilde{b}(X) - \tilde{b}(D_{x}, X')\}v)
\]

where \( \tilde{b}(\xi, x') = \tilde{b}(x', \xi) \). Then by using the asymptotic expansion formula of pseudo-differential operators (see, for example [4]), we have

\[
\tilde{b}(D_{x}, X') = \tilde{b}(X, D_{x}) + b_{1}(X, D_{x})
\]

where

\[
b_{1}(x, \xi) = \sum_{j=1}^{N} \sum_{|\alpha|=j} \frac{1}{\alpha!} \tilde{b}_{(\alpha)}(x, \xi) + R_{N}(x, \xi) \quad \text{and} \quad R_{N}(x, \xi) \in S^{0}_{1, \delta, \lambda(1-\delta)}
\]
SHARP FORM OF GÅRDING’S INEQUALITY

Since $b(x) \in B^1$, we can see by Lemma 2.1 (i) that

$$\tilde{b}^{(\alpha)}(x, \xi) \in S^{-1}_{1, \delta, \lambda}$$

for $|\alpha| \neq 0$

Hence taking $N$ sufficiently large we can see

$$b_1(x, \xi) \in S^{-1}_{1, \delta, \lambda}$$

Using Lemma 2.1 (i) and the boundedness of pseudo-differential operators we have

$$\|\{b(X) - \tilde{b}(X, D_x)\}v\| \leq C\|\{b(X) - \tilde{b}(X, D_x)\}v\| + \|b_1(X, D_x)\|$$

$$\leq C\|v\|_{-\delta, \lambda}$$

Therefore by Schwarz inequality and duality argument of the spaces $H_{s, \lambda}$, we have the estimate.

In order to show the main estimate in this section, the following theorem plays an essential role.

**Theorem 2.3.** Let $b(x)$ be a function in $B^2$, and let $0 \leq \delta < 1$. For a basic weight function $\lambda(\xi)$ we define a symbol $\tilde{b}(x, \xi)$ by

$$\tilde{b}(x, \xi) = \int \phi(z)b(x - \lambda(\xi)^{-\delta}z)dz$$

where $\phi(x)$ is an even function in $S$ with $\int \phi(x)dx = 1$. Then for any $s \in [0, 2\delta]$ we have

$$\|\{b(X) - \tilde{b}(X, D_x)\}u\|_{s, \lambda} \leq C\|u\|_{s-2\delta, \lambda}$$

for any $u$ in $S$.

Proof. For the proof, we use the three line theorem in complex analysis. Let $u$ and $v$ be functions in $S$ and we consider the complex function

$$f(z) = (\lambda(D_x)^{2\delta(1-z)}\{b(X) - \tilde{b}(X, D_x)\}^2\lambda(D_x)^{2\delta z}u, v)$$

Since $u$ and $v$ are in $S$ and $\lambda(\xi) \geq 1$, it is clear that the function $f(z)$ is holomorphic in the complex $z = \sigma + i\tau$-plane $\mathbb{C}$. Since the symbol $\lambda(\xi)^s$ is in $S^{1+\delta}_{0, \lambda}$, independent of $x$ and $|\lambda(\xi)| = 1$, we can see from the Lemma 2.1 and 2.2 that

$$|f(i\tau)| \leq \|\lambda(D_x)^{2\delta}\{b(X) - \tilde{b}(X, D_x)\}\lambda(D_x)^{2i\delta\tau}u\| \|v\| \leq C\|u\| \|v\|$$

$$|f(1 + i\tau)| \leq \|\{b(X) - \tilde{b}(X, D_x)\}\lambda(D_x)^{2\delta(1+i\tau)}u\| \|v\| \leq C\|u\| \|v\|$$
Hence from the three line theorem (see [7]), we have

$$|f(\sigma)| \leq C||u|| ||v||$$

for $0 < \sigma < 1$. Taking $\sigma = 1 - \frac{s}{2\delta}$, we have

$$|(\lambda(D_x)^s\{b(X) - \tilde{b}(X, D_x)\}\lambda(D_x)^{-s+2\delta}u, v)| \leq C||u|| ||v||$$

for any $u$ and $v$ in $S$. Hence using the usual duality argument, we can get the inequality.

Theorem 2.3 implies the following estimate.

**Theorem 2.4.** Let $b(x)$ be a function in $B^2$ and let $a(\xi)$ in $S^\nu_{1,0,\lambda}$ with $0 < s < 1$. Then we have

$$|||b(D_x), b(X)||0, \lambda \leq C||u||_{s-1, \lambda}$$

for any $u$ in $S$.

**Proof.** We take an even function $\phi(x)$ in $C_0^\infty(\mathbb{R}^n)$ such that $\int \phi(x)dx = 1$. For $b(x)$ we define a symbol $\tilde{b}(x, \xi)$ by

$$\tilde{b}(x, \xi) = \int \phi(z)b(x - \lambda(\xi)^{-\delta}z)dz$$

for $s < \delta = \frac{1+s}{2} (< 1)$. We write

$$[a(D_x), b(X)]u = a(D_x)\{b(X) - \tilde{b}(X, D_x)\}u(x)$$

$$+ \lambda(D_x)\tilde{b}(X, D_x)u(x) - \tilde{b}(X, D_x)a(D_x)u(x)$$

$$+ \{b(X) - \tilde{b}(X, D_x)\}a(D_x)u(x)$$

Since $a(\xi) \in S^\nu_{1,0,\lambda}$, we can see that the first term can be estimated by

$$||a(D_x)\{b(X) - \tilde{b}(X, D_x)\}u(x)||0, \lambda \leq C||\lambda(D_x)^s\{b(X) - \tilde{b}(X, D_x)\}u(x)||0, \lambda$$

and therefore by Theorem 2.3 we have

$$||a(D_x)\{b(X) - \tilde{b}(X, D_x)\}u(x)||0, \lambda \leq C||u||_{s-25, \lambda}$$

$$\leq C||u||_{s-1, \lambda}$$
The third term is estimated by Lemma 2.1(ii) and we have

\[(2.3)\]

\[\|\{b(X) - \tilde{b}(X, D_x)\}a(D_x)u(x)\| \leq C\|a(D_x)u\|_{-2\delta, \lambda} \leq C\|a(D_x)u\|_{s-1, \lambda}\]

The second term is estimated by the usual asymptotic expansion formula for pseudo-differential operators (see [4]), that is, we have

\[a(D_x)b(X, D_x) = b_L(X, D_x)\]

and

\[b_L(x, \xi) \sim \tilde{b}(x, \xi)a(\xi) + \sum_{j=1}^{\infty} b_j(x, \xi)\]

\[b_j(x, \xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} \tilde{b}_{(\alpha)}(x, \xi)a^{(\alpha)}(\xi)\]

Since the symbols \(\tilde{b}_{(\alpha)}(x, \xi)\) belong to \(S^{a-1}_{1, \delta, \lambda}\) for \(|\alpha| \leq 2\), we can see that \(b_1(x, \xi) \in S^{a-1}_{1, \delta, \lambda}\)

and \(b_j(x, \xi) \in S^{a-j+(j-2)\delta}_{1, \delta, \lambda}\) for \(j \geq 2\). Hence we can write

\[a(D_x)b(X, D_x)u - \tilde{b}(X, D_x)a(D_x)u = B_1(X, D_x)u\]

where \(B_1(x, \xi)\) belongs to \(S^{a-1}_{1, \delta, \lambda}\). Therefore we have

\[(2.4)\]

\[\|\{a(D_x)b(X, D_x) - \tilde{b}(X, D_x)a(D_x)\}u(x)\| \leq \|B_1(X, D_x)u\| \leq C\|u(x)\|_{s-1, \lambda}\]

From the estimates (2.2), (2.3) and (2.4), we have the estimate (2.1).

In particular we have

**Corollary 2.5.** Let \(b(x)\) be a function in \(B^2\) and let \(a(\xi)\) be in \(S^{\frac{1}{2}}_{1, 0, \lambda}\). Then we have

\[\|a(D_x), b(X)\|_{0, \lambda} \leq C\|u\|_{-\frac{1}{2}, \lambda}\]

for any \(u\) in \(S\).

**Remark 1.** We note that if the basic weight function \(\lambda(\xi) = \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}\),

we can get more general results for \(L^p(\mathbb{R}^n)\) \((1 < p < \infty)\) than the ones in this section. Because in case of \(\langle \xi \rangle\), we can use the kernel representations of the operators(see [5]). We can see sharper results in [1] than in [5] in the case \(\lambda(\xi) = \langle \xi \rangle\).
3. A sharp form of Gårding's inequality

Let us begin with the following inequality, which we can say a sharp form of Gårding's inequality.

**Theorem 3.1.** Let \( \delta < 1 \). We assume that a symbol \( p(x, \xi) \in S^m_{1, \delta, \lambda} \) satisfies that \( p(\beta)(x, \xi) \) are in \( S^m_{1, \delta, \lambda} \) for \( |\beta| \leq 2 \) and

\[
\text{Re } p(x, \xi) \geq 0
\]

for some constant \( c_0 \). Then we have

\[
\text{Re } (p(X, D_x)u, u) \geq -C||u||^2_{m-1, \lambda}
\]

for any \( u \) in \( S \).

For the proof of this theorem 3.1 we use the following lemma

**Lemma 3.2.** (see [5]) Let \( \tau \) be a real number. Let \( \psi(x) \) be an infinitely smooth function on \( \mathbb{R}^n \), and let \( \lambda(\xi) \) be a basic weight function. Then for any \( \alpha \) we have

\[
\partial_{\xi}^\alpha \{\psi(\lambda(\xi)\tau x)\} = \sum_{|\alpha'| \leq |\alpha|} \phi_{\alpha', \alpha}(\xi) \{\lambda(\xi)\tau x\}^{\alpha'} \psi^{(\alpha')}(\lambda(\xi)\tau x)
\]

where \( \phi_{\alpha', \alpha}(\xi) \) belong to \( S_{\lambda, 1, 0}^{-|\alpha|} \) for all \( \alpha' \) with \( |\alpha'| \leq |\alpha| \).

**Proof of Theorem 3.1.** First we note that we can assume that the symbol \( p(x, \xi) \) be real-valued. In fact, if a real-valued symbol \( r(x, \xi) \in S^m_{1, \delta, \lambda} \) satisfies the same assumption of \( p(x, \xi) \) in Theorem 3.1, then we have

\[
\text{Re}(ir(X, D_x)u, u) = \frac{1}{2} \text{Im}\{r(X, D_x) - r^*(X, D_x)\}u, u)
\]

Since the symbol \( r(x, \xi) \) is real-valued, by using the expansion formula for the symbol of the formal adjoint operator \( r^*(X, D_x) \) we have

\[
r^*(x, \xi) \sim r(x, \xi) + \sum_{j=1}^{\infty} \sum_{|\alpha|=j} \frac{1}{\alpha!} r^{(\alpha)}(x, \xi)
\]

Therefore using the assumption we have

\[
r(x, \xi) - r^*(x, \xi) = R(x, \xi) \in S^{m-1}_{1, \delta, \lambda}
\]
From this relation we have

\[
|\text{Re}(ir(X, D_x)u, u)| = \frac{1}{2}|\text{Im}\left\{\left(r(X, D_x) - r^*(X, D_x)\right)u, u\right\}|
\]
\[
= \frac{1}{2}|(R(X, D_x)u, u)|
\]
\[
\leq C||u||_{2, -\frac{1}{2}, \lambda}
\]

Now we assume that the symbol \(p(x, \xi)\) is real and non-negative. We take an even and real-valued function \(\psi(x)\) in \(S\) such that \(\int \psi(x)^2dx = 1\) and we put

\[
p_G(x, \xi, x') = \int \psi(\lambda(\xi)\frac{1}{2}(x - z))\psi(\lambda(\xi)\frac{1}{2}(x' - z))p(z, \xi)dz\lambda(\xi)^{\frac{3}{2}}
\]

Then using the lemma 3.2 we can see that \(p_G(x, \xi, x')\) is in \(S_{1, \frac{1}{2}, \lambda}^m\) and changing the order of integrations we have

\[
\text{Re}(p_G(X, D_x, X')w, w) \geq 0
\]

for any \(w \in S\) (see [6]). Moreover by using the formula of simplified symbols in [4] for the operator with double symbols we can see that the operator \(p_G(X, D_x, X')\) can be written asymptotically as

\[
p_G(X, D_x, X') \sim \sum_{j=0}^{\infty} p_j(X, D_x)
\]

where \(p_j(x, \xi)\) is in \(S_{1, \frac{1}{2}, \lambda}^{m-\frac{1}{2}}\) for any \(j\) and has the form

\[
p_j(x, \xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} p_{G, (\alpha)}^{(\alpha)}(x, \xi, x)
\]

In particular, \(p_0(x, \xi)\) can be written as

\[
p_0(x, \xi) = p_G(x, \xi, x)
\]
\[
= \lambda(\xi)^{\frac{3}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}(x - z))^2 p(z, \xi)dz
\]
\[
= \lambda(\xi)^{\frac{3}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 p(x - z, \xi)dz
\]
\[
= \int \psi(z)^2 p(x - \lambda(\xi)^{-\frac{1}{2}}z, \xi)dz
\]
By using the Taylor expansion for the second expression, we have
\[
p(x - z, \xi) = p(x, \xi) + \sum_{|\beta|=1} ip_{(\beta)}(x, \xi)z^\beta + R_2(z, x, \xi)
\]
where the remainder term \(R_2(z, x, \xi)\) is
\[
R_2(z, x, \xi) = \sum_{|\beta|=2} \frac{-2}{\beta!} \int_0^1 (1 - t)z^\beta p_{(\beta)}(x - tz, \xi) dt
\]
Since \(\psi(z)\) is an even function we see that
\[
\int \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 z^\beta dz = 0
\]
for \(|\beta| = 1\). Therefore we have
\[
p_0(x, \xi) = \lambda(\xi)^{\frac{3}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 p(x, \xi) dz + \lambda(\xi)^{\frac{3}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 R_2(z, x, \xi) \, dz
\]
\[
= p(x, \xi) + \lambda(\xi)^{\frac{3}{2}} \sum_{|\beta|=2} \frac{-2}{\beta!} \int_0^1 (1 - t) \int z^\beta \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 p_{(\beta)}(x - tz, \xi) \, dz
\]
\[
= p(x, \xi) + r_2(x, \xi)
\]
From the assumption of the symbol \(p(x, \xi)\), the symbols \(p_{(\beta)}(x - tz, \xi)\) belong to \(S_{m\lambda,1,\delta}^m\) for \(|\beta| = 2\). Hence using Lemma 3.2, we can see that
\[
r_2(x, \xi) = \lambda(\xi)^{\frac{3}{2}} \sum_{|\beta|=2} \frac{-2}{\beta!} \int_0^1 (1 - t) \int z^\beta \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 p_{(\beta)}(x - tz, \xi) \, dz
\]
is in \(S_{\lambda,1,\delta}^{m-1}\). Thus we can write
\[
p_0(x, \xi) = p(x, \xi) + r_2(x, \xi)
\]
with symbol \(r_2(x, \xi)\) in \(S_{\lambda,1,\delta}^{m-1}\).

Similarly for \(|\alpha| = 1\), since
\[
p^{(\alpha)}_{G,(\alpha)}(x, \xi, x) = \partial^\alpha \{ \lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)\psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}}z)p(x - z, \xi) dz \}
\]
we can see that
\[
p^{(\alpha)}_{G,(\alpha)}(x, \xi, x) = \partial^\alpha \{ \lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)\psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}}z)\{p(x, \xi) + R_1(z, x, \xi)\} dz \}
\]
where the remainder term $R_1(z, x, \xi)$ is

$$R_1(z, x, \xi) = \sum_{|\beta|=1} i \int_0^1 (1 - t) z^{\beta} p_{(\beta)}(x - tz, \xi) dt$$

Since $\int \psi(z) \psi^{(\alpha)}(z) dz = 0$ for $|\alpha| = 1$, we can see that

$$p^{(\alpha)}_{G, (\alpha)}(x, \xi, x) = \partial_\xi^{\alpha} \sum_{|\beta|=1} i \int_0^1 (1 - t) dt \times \psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}} z) \psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}} z) R_1(z, x, \xi) dz$$

for $|\alpha| = 1$. In a similar way to the estimate of $R_2(z, x, \xi)$, we can see from Lemma 3.2 and the assumption of the symbol $p(x, \xi)$ that $p_{1, \alpha}(x, \xi) = p^{(\alpha)}_{G, (\alpha)}(x, \xi, x)$ belongs to $S_{\lambda, 1, \delta}^{m-1}$ for $|\alpha| = 1$. Therefore we can see that

$$p_1(x, \xi) \in S_{\lambda, 1, \delta}^{m-1}$$

Thus we can write

$$p_G(X, D_x, X') = p(X, D_x) + R(X, D_x) + Q(X, D_x)$$

where $Q(x, \xi) \in S_{\lambda, 1, \delta}^{m-1}$ and $R(x, \xi) \in S_{\lambda, 1, \delta}^{m-1}$. Now from the $L^2$-boundedness theorems and the algebra of pseudo-differential operators with symbols in $\cup_{m \in \mathbb{R}} S_{\lambda, 1, \delta}^{m}$ for any $\delta$ with $0 \leq \delta < 1$, we can see that

$$\|(Q(X, D_x)u, u)\| \leq \||(Q(X, D_x)u)\|_{m+1, \lambda} \|u\|_{m+1, \lambda} \leq C\|u\|_{m+1, \lambda}^2$$

$$\|(R(X, D_x)u, u)\| \leq \||(R(X, D_x)u)\|_{m+1, \lambda} \|u\|_{m+1, \lambda} \leq C\|u\|_{m+1, \lambda}^2$$

Therefore we have

$$\text{Re}(p(X, D_x)u, u)$$

$$= \text{Re}(p_G(X, D_x, X')u, u) - \text{Re}(R(X, D_x)u, u) - \text{Re}(Q(X, D_x)u, u)$$

$$\geq -\|(Q(X, D_x)u, u)\| - \|(R(X, D_x)u, u)\|$$

$$\geq -C\|u\|_{m+1, \lambda}^2$$

If a function $\lambda(\xi)$ is a basic weight function, then we can see that for $0 < \rho \leq 1$ the fractional power $\lambda(\xi)^\rho$ is also a basic weight function. Using this fact and Theorem 3.1 we have
**Corollary 3.3.** Let $0 \leq \delta < \rho \leq 1$. We assume that a symbol $p(x, \xi)$ in $S_{\rho, \delta, \lambda}^m$ satisfies that $p_{(\beta)}(x, \xi)$ are in $S_{\rho, \delta, \lambda}^m$ for $|\beta| \leq 2$ and

$$\text{Re } p(x, \xi) \geq 0$$

Then we have

$$\text{Re } (p(X, D_x)u, u) \geq -C||u||_{m-\delta, \lambda}^2$$

for any $u$ in $S$.

**Proof.** Since $\lambda(\xi)^\rho$ is a basic weight function, we see that

$$S_{\rho, \delta, \lambda}^m = S_{1, \frac{\delta}{2}, \lambda^\rho}^m$$

Hence we can see that the symbol $p(x, \xi)$ in Corollary satisfies the assumptions of the one in Theorem 3.1 as the class of symbols in $S_{1, \frac{\delta}{2}, \lambda^\rho}^m$. Therefore we see that

$$\text{Re } (p(X, D_x)u, u) \geq -C||u||_{m-\delta, \lambda}^2$$

$$= -C||u||_{m-\delta, \lambda}^2$$

**REMARK 2.** Let $0 < \rho < 1$. Then we can show a similar sharp form of Garding’s inequality to Theorem 3.1, under the assumption that the symbol $p(x, \xi)$ belongs to $S_{\rho, \rho, \lambda}^m$ and $p_{(\beta)}(x, \xi)$ belongs to $S_{\rho, \rho, \lambda}^m$ for any $\beta$ with $|\beta| \leq 2$, by using the similar approximation $p_G(x, \xi, x')$ defined by

$$p_G(x, \xi, x') = \int \psi(\lambda(\xi)^\frac{\rho}{2}(x - z))\psi(\lambda(\xi)^\frac{\rho}{2}(x' - z))p(z, \xi)dz\lambda(\xi)^{\frac{\rho}{2}}$$

and $L^2$-boundedness theorem(Theorem of Calderon and Vaillancourt, see [4]) of operators with symbols in $S_{\rho, \rho, \lambda}^0$.

Now using the commutator estimates in section 2 we can show the following sharp form of Garding’s inequality.

**Theorem 3.4.** Let $a_j(\xi)$ and $c_j(\xi)$ be in $S_{1,0, \lambda}^m$ and let $b_j(x)$ be $\mathcal{B}^2$ functions for $j = 1, \ldots, N$. We assume that

$$\text{Re } \sum_{j=1}^{N} a_j(\xi)b_j(x)c_j(\xi) \geq 0$$
Then there exists a positive constant $C$ such that

$$\text{Re} \sum_{j=1}^{m} (b_j(X)a_j(D_x)u, c_j(D_x)u) \geq -C||u||_{m-\frac{1}{2},\lambda}^2$$

for any $u \in S$.

Proof. We set $\tilde{a}_j(\xi) = \lambda(\xi)^{-m+\frac{1}{2}}a_j(\xi)$ and $\tilde{c}_j(\xi) = \lambda(\xi)^{-m+\frac{1}{2}}c_j(\xi)$. From the assumption we see that $\tilde{a}_j(\xi)$ and $\tilde{c}_j(\xi)$ are in $S_{1,0,\lambda}^{\frac{1}{2}}$. So writing

$$\sum_{j=1}^{N} (b_j(X)a_j(D_x)u, c_j(D_x)u) = \sum_{j=1}^{N} (b_j(X)\tilde{a}_j(D_x)\lambda(D_x)^{m-\frac{1}{2}}u, \tilde{c}_j(D_x)\lambda(D_x)^{m-\frac{1}{2}}u)$$

we put

$$\sum_{j=1}^{N} (b_j(X)a_j(D_x)u, c_j(D_x)u) = \sum_{j=1}^{N} (b_j(X)\tilde{c}_j(D_x)v, v)$$

$$+ \sum_{j=1}^{N} ([\tilde{c}_j(D_x), b_j(X)]\tilde{a}_j(D_x)v, v)$$

(3.1)

where $v = \lambda(D_x)^{m-\frac{1}{2}}u$. Since $\tilde{a}_j(\xi)$ and $\tilde{c}_j(\xi)$ are in $S_{1,0,\lambda}^{\frac{1}{2}}$, using the commutator estimate in Corollary 2.5 we can see that

$$||[\tilde{c}_j(D_x), b_j(X)]\tilde{a}_j(D_x)v|| \leq C||v|| = C||u||_{m-\frac{1}{2},\lambda}$$

Hence the second term $II$ of (3.1) can be estimated by

$$|II| \leq \sum_{j=1}^{N} ||([\tilde{c}_j(D_x), b_j(X)]\tilde{a}_j(D_x)v, v)|| \leq C||u||_{m-\frac{1}{2},\lambda}^2$$

Now we consider the operator

$$p(X, D_x) = \sum_{j=1}^{N} b_j(X)\tilde{c}_j(D_x)\tilde{a}_j(D_x)$$

with symbol

$$p(x, \xi) = \sum_{j=1}^{N} b_j(x)\tilde{c}_j(\xi)\tilde{a}_j(\xi)$$
For the symbol \( p(x, \xi) \) we define a new symbol \( \tilde{p}(x, \xi) \) by
\[
\tilde{p}(x, \xi) = \int \phi(y)p(x - \lambda(\xi)^{-\frac{1}{2}}y, \xi)dy
\]
\[
= \int \phi(\lambda(\xi)^{\frac{1}{2}}(x - y))p(y, \xi)dy\lambda(\xi)^{\frac{3}{2}}
\]
where \( \phi(x) \) is a non-negative function in \( C_0^\infty(\mathbb{R}^n) \) with \( \int \phi(x)dx = 1 \). Then by Lemma 2.1 (ii) we can see that the symbol \( \tilde{p}(x, \xi) \) belongs to \( S^1_{1, \frac{1}{2}, \lambda} \), \( \tilde{p}(\beta)(x, \xi) \) belongs to \( S^1_{1, \frac{1}{2}, \lambda} \) for any \( |\beta| \leq 2 \) and satisfies
\[
||\{p(X, D_x) - \tilde{p}(X, D_x)\}w|| \leq \sum_{j=1}^{N} ||\{b_j(X) - \tilde{b}_j(X, D_x)\}c_j(D_x)\tilde{a}_j(D_x)v||
\]
where
\[
\tilde{b}_j(x, \xi) = \int \phi(y)b_j(x - \lambda(\xi)^{-\frac{1}{2}}y)dy
\]
Then by Lemma 2.1 (ii) we have
\[
||\{b_j(X) - \tilde{b}_j(X, D_x)\}w|| \leq C||w||_{-1, \lambda}
\]
for any \( w \in S \) and therefore we have
\[
||\{p(X, D_x) - \tilde{p}(X, D_x)\}v|| \leq C \sum_{j=1}^{N} ||\tilde{c}_j(D_x)\tilde{a}_j(D_x)v||_{-1, \lambda}
\]
(3.2)
Moreover \( \tilde{p}(x, \xi) \) satisfies
\[
\text{Re} \: \tilde{p}(x, \xi) \geq 0
\]
Thus the symbol \( \tilde{p}(x, \xi) \) satisfies the assumption in the Theorem 3.1 with \( m = 1 \) and \( \delta = \frac{1}{2} \). Therefore we have
\[
\text{Re} \: (\tilde{p}(X, D_x)v, v) \geq -C||u||_{\frac{1}{2}, \lambda}^2
\]
From (3.1) we see
\[
\text{Re} \sum_{j=1}^{N} (b_j(X)a_j(D_x)u, c_j(D_x)u) = \text{Re}\{I + II\}
\]
\[
= \text{Re}(\tilde{p}(X, D_x)v, v) + \text{Re}(\{p(X, D_x) - \text{Re}(\tilde{p}(X, D_x))v, v\} + II
\]
\[
\geq -||\{p(X, D_x) - \text{Re}(\tilde{p}(X, D_x))v|| \cdot ||v|| - |II|
\]
\[
\geq -C||v||^2
\]
Hence we have the theorem.

---

References


M. Nagase
Department of Mathematics, Graduate School of Science,
Osaka University, Toyonaka,
560, Japan
e-mail: nagase@math.wani.osaka-u.ac.jp

M. Yoshida
Isogo engineering center, Toshiba Co.,
Yokohama, Japan
e-mail: manabu@rdec.iec.toshiba.co.jp