1. Introduction

Let $G$ and $H$ be finite groups of order $n$. A mapping $f$ from $G$ into $H$ is called a planar function of degree $n$ if, for each element $u \in H$ and $v \in G^* = G - \{1\}$, there exists exactly one $x \in G$ such that $f(vx)f(x)^{-1} = u$. In [2] Hiramine has shown that if both $G$ and $H$ are abelian groups of order $3p$ with $p(\geq 5)$ a prime, then there exists no planar function from $G$ into $H$. To prove this he has established two results on character values. Their proofs are slightly complicated. In this note we shall give short proofs.

In section 2 we shall present Proposition 2 which is useful for the proof of Result 2. In section 3 we shall state Hiramine' results and give short proofs.

We follow the notation and terminology of [2].

2. Planar Functions and Equations in Group Algebras

Let $G$ and $H$ be finite groups of order $n$. Throughout this article elements of $G$ will be denoted by small Roman letters and elements of $H$ by small Greek letters. Let $f$ be a mapping from $G$ into $H$ and $S_\alpha = \{x \in G|f(x) = \alpha\}$, $\alpha \in H$. If $S_\alpha \neq \emptyset$, we set $\hat{S}_\alpha = \sum_{x \in S_\alpha} x \in C[G]$ and $\hat{S}_\alpha^{-1} = \sum_{x \in S_\alpha} x^{-1} \in C[G]$, otherwise $\hat{S}_\alpha = \hat{S}_\alpha^{-1} = 0$, where $C[G]$ is the group algebra of $G$ over the complex number field $C$. Let $G_0 = G \times H$ be the direct product of groups $G, H$.

To prove the results we need two propositions. The following is Proposition 2.1 [2].

**Proposition 1.** The following are equivalent.

(i) The function $f$ is planar.

(ii) In the group algebra $C[G]$ of $G$,

$$\sum_{\alpha \in H} \hat{S}_\tau \hat{S}_\alpha^{-1} = \sum_{\alpha \in H} \hat{S}_\alpha^{-1} \hat{S}_\alpha = \begin{cases} \hat{G} + n - 1 & \text{if } \tau = 1, \\ \hat{G} - 1 & \text{otherwise.} \end{cases}$$
REMARK 1. If $\tau \neq 1$, then it follows from the equation in (ii) of the proposition above that in the group algebra $C[G_0]$ of $G_0$,

$$\sum_{\alpha \in H} \hat{S}_\tau \alpha \hat{S}_\alpha^{-1} \alpha^{-1} = (\hat{G} - 1) \tau.$$  

We prove the following

**Proposition 2.** We have in $C[G_0]$,

$$(\sum_{\alpha \in H} \hat{S}_\alpha \alpha)(\sum_{\beta \in H} \hat{S}_\beta^{-1} \beta^{-1}) = \hat{G} + n - 1 + \sum_{\tau \in H, \tau \neq 1} (\hat{G} - 1) \tau.$$  

Proof of Proposition 2.

$$(\sum_{\alpha \in H} \hat{S}_\alpha \alpha)(\sum_{\beta \in H} \hat{S}_\beta^{-1} \beta^{-1}) = \sum_{\tau \in H} (\sum_{\beta \in H} \hat{S}_\tau \beta \hat{S}_\beta^{-1} \beta^{-1})$$

$$= \hat{G} + n - 1 + \sum_{\tau \in H, \tau \neq 1} (\hat{G} - 1) \tau, \text{ by Remark 1.}$$

We complete the proof of Proposition 2. \qed

3. **Proofs of Hiramine' Results**

We start with the following well-known facts about character theory. These facts play important parts in the proofs of his results.

**FACT 1.** Let $G$ be an abelian group and $\chi$ an arbitrary (linear) character of $G$. Then $\chi$ is a homomorphism from $G$ into $C^* = C - \{0\}$. So we can extend this homomorphism $\chi$ to an algebra homomorphism $\overline{\chi}$ from $C[G]$ into $C$.

**FACT 2.** Let $H_1, H_2$ be finite groups and $G_1$ the direct product of $H_1, H_2$. Then all irreducible characters of $G_1$ are obtained as follows. Let $\chi_0, ..., \chi_{s-1}$ be the irreducible characters of $H_1$, $\rho_0, ..., \rho_{t-1}$ the irreducible characters of $H_2$. Then $G_1$ has exactly $st$ irreducible characters $\Psi_{ij}(0 \leq i \leq s - 1, 0 \leq j \leq t - 1)$, satisfying $\Psi_{ij}(h_1 h_2) = \chi_i(h_1) \rho_j(h_2)$, where $h_1 \in H_1, h_2 \in H_2$.

Proof. See[1, p.54]. \qed

**REMARK 2.** In Fact 2 if both $\chi_i$ and $\rho_j$ are linear characters, then $\Psi_{ij}$ is a homomorphism from $G_1$ to $C^*$. As in Fact 1, we have an algebra homomorphism $\overline{\Psi}_{ij}$ from $C[G_1]$ into $C$ which is an extension of $\Psi_{ij}$. 
Now we shall start to state Hiramine' results and prove them. In the remainder of this section we assume that \( f \) is a planar function and that \( G \) is an abelian group of order \( n \).

Let \( \chi_0(= 1_G), \ldots, \chi_{n-1} \) be the irreducible (linear) characters of \( G \), where \( 1_G \) denote the principal character of \( G \). We set

\[
d_i^{(\alpha)} = \begin{cases} \sum_{x \in S_\alpha} \chi_i(x) & \text{if } S_\alpha \neq \emptyset, \\ 0 & \text{if } S_\alpha = \emptyset \end{cases}
\]

for each \( 0 \leq i \leq n - 1 \) and for each \( \alpha \in H \). Now we state one of Hiramine' results [2].

**RESULT 1.** The following hold

(i) \( d_0^{(\alpha)} = |S_\alpha| \) and

\[
\sum_{\alpha \in H} d_0^{(\tau \alpha)} d_0^{(\alpha)} = \sum_{\alpha \in H} d_0^{(\alpha \tau)} d_0^{(\alpha)} = \begin{cases} 2n - 1 & \text{if } \tau = 1, \\ n - 1 & \text{otherwise.} \end{cases}
\]

(ii) For \( i \neq 0 \),

\[
\sum_{\alpha \in H} d_i^{(\tau \alpha)} \overline{d_i^{(\alpha)}} = \sum_{\alpha \in H} d_i^{(\alpha \tau)} d_i^{(\alpha)} = \begin{cases} n - 1 & \text{if } \tau = 1, \\ -1 & \text{otherwise.} \end{cases}
\]

(Here \( \overline{d} \) denotes the complex conjugate of \( d \in \mathbb{C} \).)

**Proof.** Since \( G \) is abelian, from Fact 1 we note that for each \( 0 \leq i \leq n - 1 \), \( \chi_i \) is an algebra homomorphism from \( C[G] \) into \( C \). We shall prove (i). It is immediate that \( d_0^{(\alpha)} = |S_\alpha| \). If \( \tau = 1 \), then the equation in (ii) of Proposition 1 becomes

\[
\sum_{\alpha \in H} \hat{S}_\alpha \hat{S}_\alpha^{-1} = \sum_{\alpha \in H} \hat{S}_\alpha^{-1} \hat{S}_\alpha = \hat{G} + n - 1.
\]

We apply \( \chi_0 \) to this equation. Then

\[
\sum_{\alpha \in H} \overline{\chi_0(\hat{S}_\alpha)} \overline{\chi_0(\hat{S}_\alpha^{-1})} = \sum_{\alpha \in H} \overline{\chi_0(\hat{S}_\alpha^{-1})} \overline{\chi_0(\hat{S}_\alpha)} = \overline{\chi_0(\hat{G} + n - 1)} ,
\]

which implies the equation for \( \tau = 1 \) in (i). Similarly we can prove the equation for \( \tau \neq 1 \) in (i). We have proved (i). Next we shall prove (ii). By applying the algebra homomorphism \( \chi_i \) \( (i \neq 0) \) to the equation in (ii) of Proposition 1 we can prove (ii). This completes the proof of Result 1.

We state another result on character values in [2].
RESULT 2. With the same notation and assumption as in Result 1, suppose that $H$ is abelian and let $\rho_0(= 1_H), \ldots, \rho_{n-1}$ be the irreducible characters of $H$. Set $z_{ij} = \sum_{\alpha \in H} d_i^{(\alpha)} \rho_j(\alpha)$. Then,

(i) $z_{0,0} = n$, and $z_{i,0} = 0 (i \neq 0)$

(ii) For $j \neq 0$, $z_{ij} \overline{z}_{ij} = n$.

Proof. Since $\chi_i$ and $\rho_j$ are linear, from Remark 2 we see that $\overline{\Psi}_{ij}$ is an algebra homomorphism from $C[G_0]$ into $C$. First we shall prove (ii). We apply $\overline{\Psi}_{ij}$ ($j \neq 0$) to the equation in Proposition 2. Then we get (ii). We have proved (ii). Next we shall prove (i). Similarly by using $\overline{\Psi}_{i0}$ we can prove (i). This completes the proof of Result 2.


References