SCATTERING THEORY FOR TIME-DEPENDENT HARTREE-FOCK TYPE EQUATION

TAKESHI WADA

(Received November 4, 1997)

1. Introduction

In this paper we consider the scattering problem for the following system of nonlinear Schrödinger equations with nonlocal interaction

\[ \begin{align*}
\frac{i\partial}{\partial t} u_j &= -\frac{1}{2} \Delta u_j + f_j(u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\
u_j(0,x) &= \phi_j(x), \quad j = 1, \ldots, N.
\end{align*} \]

Here \( \Delta \) denotes the Laplacian in \( x \),

\[ f_j(u) = \sum_{k=1}^{N} (V * |u_k|^2)u_j - \sum_{k=1}^{N} [V * (u_j u_k)]u_k, \]

and * denotes the convolution in \( \mathbb{R}^n \). In this paper we treat the case \( n \geq 2 \) and \( V(x) = |x|^{-\gamma} \) with \( 0 < \gamma < n \).

The system (1)-(2) appears in the quantum mechanics as an approximation to a fermionic N-body system and is called the time-dependent Hartree-Fock type equation.

Throughout the paper we use the following notation:

\[ \begin{align*}
N &= \{1, 2, 3, \ldots, \}, \quad \nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n), \quad U(t) = \exp(it\Delta/2), \quad M(t) = \exp(i|x|^2/2t), \\
J = U(i)xU(-t) = M(t)(itV)M(-t).
\end{align*} \]

For \( 1 < p < \infty \), \( p' = p/(p-1) \), \( \delta(p) = n/2 - n/p \). \( \| \cdot \|_p \) denotes the norm of \( L^p(\mathbb{R}^n) \) (if \( p = 2 \), we write \( \| \cdot \|_2 = \| \cdot \| \)). For \( 1 \leq p < \infty \) and for the interval \( I \subset \mathbb{R} \), \( \| \cdot \|_{q,r,l} \) denotes the norm of \( L^r(I; L^q(\mathbb{R}^n)) \), namely,

\[ \| u \|_{q,r,l} = \left[ \int_I \left( \int_{\mathbb{R}^n} |u(t,x)|^q \, dx \right)^{r/q} \, dt \right]^{1/r}. \]

For positive integers \( l \) and \( m \), \( \Sigma^{l,m} \) denotes the Hilbert space defined as

\[ \Sigma^{l,m} = \{ \psi \in L^2(\mathbb{R}^n); \| \psi \|_{\Sigma^{l,m}} = \left( \sum_{|\alpha| \leq l} \| \nabla^\alpha \psi \|^2 + \sum_{|\beta| \leq m} \| x^\beta \psi \|^2 \right)^{1/2} < \infty \}. \]

When we use \( N \)'th direct sums of various function spaces, we denote them by the same symbols and denote these elements by writing arrow over the letter, like \( \tilde{f} \).

Now we state our main theorem.
Theorem 1.1. (i) Suppose that $1 < \gamma < \min(4, n)$, and $l, m \in \mathbb{N}$. Then for any $\tilde{\phi}^{(+)} \in \Sigma^{l,m}$, there exists a unique $\tilde{\phi} \in \Sigma^{l,m}$ such that

\begin{equation}
\lim_{t \to +\infty} \|\tilde{\phi}^{(+)} - U(-t)\tilde{u}(t)\|_{\Sigma^{l,m}} = 0,
\end{equation}

where $\tilde{u}(t)$ is the solution of (1)-(2) with $U(-t)\tilde{u}(t) \in C(\mathbb{R}; \Sigma^{l,m})$. For any $\tilde{\phi}^{(-)} \in \Sigma^{l,m}$, the same result as above holds valid with $+\infty$ replaced by $-\infty$ in (3).

(ii) Suppose that $4/3 < \gamma < \min(4, n)$, and $l, m \in \mathbb{N}$. And if $\gamma \leq \sqrt{2}$, suppose, in addition, that $m \geq 2$. Then for any $\tilde{\phi} \in \Sigma^{l,m}$, there exist $\tilde{\phi}^{(\pm)} \in \Sigma^{l,m}$ such that the solution of (1)-(2) with $U(-t)\tilde{u}(t) \in C(\mathbb{R}; \Sigma^{l,m})$ satisfies

\begin{equation}
\lim_{t \to \pm\infty} \|\tilde{\phi}^{(\pm)} - U(-t)\tilde{u}(t)\|_{\Sigma^{l,m}} = 0.
\end{equation}

By Theorem 1.1 (i), if $1 < \gamma < \min(4, n)$, we can define the operator $W^{+}$ in $\Sigma^{l,m}$ as

\[ W^{+} : \tilde{\phi}^{(+)} \mapsto \tilde{\phi}, \]

which is called the wave operator. The operator $W^{-}$ is defined similarly. Under the condition of $4/3 < \gamma < \min(4, n)$ ($m \geq 2$ if $\gamma \leq \sqrt{2}$), Theorem 1.1 (ii) implies the completeness of $W^{\pm}$, namely, $\text{Range} W^{\pm} = \Sigma^{l,m}$.

There are many papers for the following equation

\begin{alignat}{2}
\frac{\partial u}{\partial t} &= -\frac{1}{2}\Delta u + f(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^{n}, \\
u(0, x) &= \phi(x),
\end{alignat}

where

\[ f(u) = [V \ast |u|^2]u = \int_{\mathbb{R}^{n}} |x - y|^{-\gamma} |u(t, y)|^{2} dy \ u(t, x) \]

(see, for example, [5, 7, 8, 9, 12]). The equation (5)-(6) is called the Hartree type equation. For the scattering problem for (5)-(6), the following results are known (see [9]).

[A] Suppose that $1 < \gamma < \min(4, n)$, and $l, m \in \mathbb{N}$. Then, for any $\phi^{(+)} \in \Sigma^{l,m}$, there exists a unique $\phi \in \Sigma^{l,m}$ such that

\begin{equation}
\lim_{t \to +\infty} \|\phi^{(+)} - U(-t)\phi(t)\|_{\Sigma^{l,m}} = 0,
\end{equation}

where $\phi$ is the solution of (1)-(2) with $U(-t)\phi(t) \in C(\mathbb{R}; \Sigma^{l,m})$. For any $\phi^{(-)} \in \Sigma^{l,m}$, the same result as above holds valid with $+\infty$ replaced by $-\infty$ in (3).
where \( u(t) \) is the solution of (5)-(6) with \( U(-t)u(t) \in C(\mathbb{R}; \Sigma^{l,m}) \). For any \( \phi^{(-)} \in \Sigma^{l,m} \), the same result as above holds valid with \( +\infty \) replaced by \( -\infty \) in (7).

[B] Suppose that \( 4/3 < \gamma < \min(4, n) \), and \( l, m \in \mathbb{N} \). Then, for any \( \phi \in \Sigma^{l,m} \), there exist unique \( \phi^{(\pm)} \in \Sigma^{l,m} \) such that the solution \( u(t) \) of (5)-(6) with \( U(-t)u(t) \in C(\mathbb{R}; \Sigma^{l,m}) \) satisfies

\[
\lim_{t \to \pm \infty} \| \phi^{(\pm)} - U(-t)u(t) \|_{\Sigma^{l,m}} = 0.
\]

Our main Theorem is the analogous results to [A], [B].

Since \( U(t) \) is unitary in \( H^l \), (4) implies that the asymptotic profiles of \( \bar{u}(t) \) as \( t \to \pm \infty \) are \( U(t)\phi^{(\pm)} \); and by the estimates

\[
\| U(t)\phi^{(\pm)} \|_p \leq (2\pi|t|)^{-\delta(p)}\| \phi^{(\pm)} \|_{p'}, \quad 2 \leq p \leq \infty,
\]

it is expected that

\[
\| \bar{u}(t) \|_p = O(|t|^{-\delta(p)})
\]

as \( t \to \pm \infty \). Indeed, in Corollary 4.1, we shall prove (9) for \( p = \infty \) under the suitable condition for \( \phi \).

Conversely, if (9) holds for some \( p \) sufficiently large, we can prove Theorem 1.1(ii). Actually, in Propositions 3.1 and 3.2, we prove (9) for some \( p > 2 \). This decay estimate is the key point of our proof of the main theorem.

The proof of Theorems [B] is much more simple than our proof of Theorem 1.1 (ii). But we cannot apply the method in [9] for (5)-(6) to prove Theorem 1.1 (ii). So we shall use the method in our work [15] to prove the main theorem.

2. Preliminaries

First, we collect various inequalities which will be used in later sections.

Lemma 2.1. (The Gagliardo-Nirenberg inequality) Let \( 1 \leq q, r \leq \infty \) and \( j, m \) be any integers satisfying \( 0 \leq j < m \). If \( u \) is any function in \( W^{m,q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n) \), then

\[
\sum_{|\alpha| = j} \| \nabla^\alpha u \|_p \leq C \left( \sum_{|\beta| = m} \| \nabla^\beta u \|_q \right)^\alpha \| u \|_r^{1-\alpha}
\]
where
\[ \frac{1}{p} - \frac{j}{n} = a\left(\frac{1}{q} - \frac{m}{n}\right) + (1 - a)\frac{1}{r} \]
for all \( a \) in the interval \( j/m \leq a \leq 1 \), where the constant \( C \) is independent of \( u \), with the following exception: if \( m - j - (n/q) \) is a nonnegative integer, then (10) is asserted for \( j/m \leq a < 1 \).

For the proof of Lemma 2.1, see [3, 14].

**Lemma 2.2.** Let \( \alpha > 0 \). Then
\[ \|(-\Delta)^{\alpha/2}fg\| \leq C\|\|(-\Delta)^{\alpha/2}f\| \|g\|_{\infty} + \|f\|_{\infty}\|(-\Delta)^{\alpha/2}g\|\). \]

This lemma is essentially due to [4, 6]. The lemma is obtained as in the proof of Lemma 3.4 in [4] and Lemma 3.2 in [6], by using the theory of Besov space (for Besov space, see [1]).

**Lemma 2.3.** (The Hardy-Littlewood-Sobolev inequality) Let \( 0 < \gamma < n, 1 < p, q < \infty \) and \( 1 + 1/p = \gamma/n + 1/q \). Then
\[ \| |x|^{-\gamma} \ast \phi\|_p \leq C\|\phi\|_q. \]

For the proof, see [10, 13].

A pair \((q, r)\) of real numbers is called admissible, if it satisfies the condition \( 0 \leq \delta(q) = 2/r < 1 \). Then

**Lemma 2.4.** If a pair \((q, r)\) is admissible, then for any \( \psi \in L^2(\mathbb{R}^n) \), we have
\[ \|U(t)\psi\|_{q, r, \mathbb{R}} \leq C\|\psi\|. \]

**Lemma 2.5.** We put \((Gu)(t) = \int_{t_0}^t U(t - \tau)u(\tau)d\tau\). Let \( I \subset \mathbb{R} \) be an interval containing \( t_0 \), and let pairs \((q_j, r_j)\), \( j = 1, 2 \), be admissible. Then \( G \) maps \( L^{q_j'}(I; L^{r_j'}) \) into \( L^{q_2}(I; L^{r_2}) \) and satisfies
\[ \|Gu\|_{q_2, r_2, I} \leq C\|u\|_{q_j', r_j', I}, \]
where \( C \) is independent of \( I \).
For the proof of Lemmas 2.4 and 2.5, see [11, 16].

Next, we summarize the results for the Cauchy problem to (1)-(2). We convert (1)-(2) into the integral equations

\begin{equation}
U_j(t) = U(t)\phi_j - i \int_0^t U(t - \tau) f_j(\bar{u}(\tau)) d\tau, \quad j = 1, \ldots, N,
\end{equation}

then

**Proposition 2.1.** (i) Suppose that \( n \geq 2, 0 < \gamma < \min(4, n), \) and \( l, m \in \mathbb{N} \).

Then for any \( \widehat{\phi} \in H^l \), there exists a unique solution \( \bar{u}(t) \in C(\mathbb{R}; H^l) \) of (15). The solution \( \bar{u}(t) \) satisfies following equalities.

\begin{align}
(u_j(t), u_k(t)) &= (\phi_j, \phi_k), \quad j, k = 1, \ldots, N, \\
\|u_j(t)\| &= \|\phi_j\|, \quad j = 1, \ldots, N;
\end{align}

and

\begin{equation}
E(\bar{u}(t)) = E(\widehat{\phi}),
\end{equation}

where

\[ E(\psi) = \sum_{j=1}^N \|\nabla \psi_j\|^2 + P(\psi), \]

\[ P(\psi) = \sum_{j,k=1}^N \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^{-\gamma} (|\psi_j(x)|^2 |\psi_k(y)|^2 - \psi_j(x) \bar{\psi}_k(x) \psi_k(y) \bar{\psi}_j(y)) dx dy; \]

(ii) Furthermore, if \( \widehat{\phi} \in \Sigma^{l,m} \), then \( U(-t)\bar{u}(t) \in C(\mathbb{R}; \Sigma^{l,m}) \), and the solution \( \bar{u}(t) \) satisfies

\begin{equation}
\sum_{j=1}^N \|x U(-t) u_j(t)\|^2 + i^2 P(\bar{u}(t)) = \sum_{j=1}^N \|x \psi_j\|^2 + (2 - \gamma) \int_0^t \tau P(\bar{u}(\tau)) d\tau.
\end{equation}

**Remark.** (i) By the Cauchy-Schwarz inequality, \( P(\hat{\psi}) \geq 0 \).

(ii) The equalities (17), (18) and (19) are called the \( L^2 \)-norm, the energy, and the pseudo-conformal conservation laws, respectively.

The proof of Propositions 2.1 is similar to that of the corresponding result for (5)-(6), so we shall omit it (see, for example, [8, 9, 12]).
3. Decay estimates for some norm of the solution

In this section we shall estimate the $L^p$-norm of the solution $\tilde{u}(t)$ of (1)-(2) to prove the main theorem. We use the following transform

$$v_j(t) = \mathcal{F}M(t)u(-t)u_j(t) = (it)^{n/2} \exp(-it|x|^2/2)u_j(t,tx),$$

where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^n$. This transform was introduced by N. Hayashi and T. Ozawa [7]. Then the equations (1) are transformed into the equations

$$i \frac{\partial}{\partial t} v_j = -\frac{1}{2t^2} \Delta v_j + \frac{1}{t^\gamma} f_j(\tilde{v}), \quad j = 1, \ldots, N,$$

and if $\bar{\phi} \in \Sigma_{1,1}$, then $\bar{v}(t) \in C((0, \infty); \Sigma_{1,1})$. The relations (17) and (19) are equivalent to

$$\frac{d}{dt} \|v_j(t)\| = 0, \quad j = 1, \ldots, N$$

and

$$t^{-2} \frac{d}{dt} \sum_{j=1}^{N} \|\nabla v_j(t)\|^2 + t^{-\gamma} \frac{d}{dt} P(\bar{v}(t)) = 0,$$

respectively. Using the relation (22), we show

**Lemma 3.1.** Suppose that $n \geq 2$, $0 < \gamma < \min(4, n)$, and $\bar{\phi} \in \Sigma_{1,1}$. Then, for $t \geq 1$,

$$\sum_{j=1}^{N} \|\nabla v_j(t)\|^2 \leq \begin{cases} C t^{2-\gamma} & \text{if } \gamma \leq 2, \\ C & \text{if } \gamma > 2. \end{cases}$$

Here, the constants $C$ depend on $\|\bar{\phi}\|_{\Sigma_{1,1}}$.

**Proof.** If $\gamma < 2$,

$$\frac{d}{dt} \left( t^{\gamma-2} \sum_{j=1}^{N} \|\nabla v_j(t)\|^2 + P(\bar{v}(t)) \right) = (\gamma - 2) t^{\gamma-3} \sum_{j=1}^{N} \|\nabla v_j(t)\|^2 \leq 0,$$
and if $\gamma \geq 2$,
\[
\frac{d}{dt} \left( \sum_{j=1}^{N} \| \nabla v_j(t) \|^2 + t^{2-\gamma} P(\tilde{v}(t)) \right) = (2 - \gamma) t^{1-\gamma} P(\tilde{v}(t)) \leq 0.
\]

Hence
\[
\sum_{j=1}^{N} \| \nabla v_j(t) \|^2 \leq \begin{cases} 
C t^{2-\gamma} & \text{if } \gamma < 2, \\
C & \text{if } \gamma \geq 2.
\end{cases}
\]

So we shall prove (23) when $\sqrt{2} < \gamma < 2$. We multiply (20) by $\Delta \tilde{v}_j$, and integrate the imaginary part over $\mathbb{R}^n$. Then
\[
\frac{1}{2} \frac{d}{dt} \| \nabla v_j(t) \|^2 = t^{-\gamma} \text{Im} \int_{\mathbb{R}^n} f_j(\tilde{v}) \Delta \tilde{v}_j dx.
\]

Since $\text{Im} \int_{\mathbb{R}^n} V * |v_k|^2 |\nabla v_j|^2 dx$ and $\text{Im} \sum_{j,k=1}^{N} \int_{\mathbb{R}^n} V * (v_j \tilde{v}_k) \nabla v_k \cdot \nabla \tilde{v}_j dx$ are equal to zero, we have, by Hölder’s inequality and Lemma 2.3,
\[
\frac{1}{2} \frac{d}{dt} \sum_{j=1}^{N} \| \nabla v_j(t) \|^2 = t^{-\gamma} \text{Im} \left[ \int_{\mathbb{R}^n} v_j \nabla (V * |v_k|^2) \cdot \nabla \tilde{v}_j dx + \int_{\mathbb{R}^n} v_k \nabla (V * (v_j \tilde{v}_k)) \cdot \nabla \tilde{v}_j dx \right] \leq C t^{-\gamma} \| \tilde{v}(t) \|^2 \rho \sum_{j=1}^{N} \| \nabla v_j(t) \|^2,
\]

where $\rho = 2n/(n - \gamma)$. By Lemma 2.1 and (24), we have
\[
\| v_j(t) \|_\rho \leq C \| v_j \|^{1-\gamma/2} \| \nabla v_j \|^{\gamma/2} \leq C t^{(2\gamma - \gamma^2)/4}.
\]

Therefore,
\[
\frac{d}{dt} \sum_{j=1}^{N} \| \nabla v_j(t) \|^2 \leq C t^{-\gamma/2} \sum_{j=1}^{N} \| \nabla v_j(t) \|^2.
\]

Since $\gamma^2/2 > 1$ if $\gamma > \sqrt{2}$, (25) and Gronwall’s inequality yield (23).

Lemma 3.1 immediately implies
Proposition 3.1. Suppose that $\sqrt{2} < \gamma < \min(4, n)$, and $\tilde{\phi} \in \Sigma^{1, 1}$. Then for the number $p$ satisfying $0 \leq \delta(p) \leq 1$ if $n \geq 3$ and $0 \leq \delta(p) < 1$ if $n = 2$, the solution of (1)-(2) has the estimate

$$\|\tilde{u}(t)\|_p \leq C(1 + |t|)^{-\delta(p)}.$$  

Proof. Since $\|\tilde{u}(t)\|_p = t^{-\delta(p)}\|\tilde{\nu}(t)\|_p$, Lemma 2.1 and Lemma 3.1 yield (26).

Now we show the $L^p$ decay estimate of the solution in case $1 < \gamma \leq \sqrt{2}$.

Lemma 3.2. Suppose that $1 < \gamma \leq \sqrt{2}$ and $\tilde{\phi} \in \Sigma^{1, 2}$. Then we have for $t \geq 1$,

$$\sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \leq \begin{cases} 
Ct^{(\gamma^2 - 8\gamma + 10)/(2 - \gamma)} & \text{if } n \geq 3, \\
Ct^{(\gamma^2 - 8\gamma + 10)/(2 - \gamma) + \varepsilon} & \text{if } n = 2.
\end{cases}$$

Here $\varepsilon$ is a positive number which can be chosen arbitrarily small, and the constant $C$ depends on $\|\tilde{\phi}\|_{\Sigma^{1, 2}}$, and $\varepsilon$ (the case $n = 2$).

Proof. We apply $\Delta$ to the both side of (20) and obtain

$$i \frac{\partial}{\partial t} \Delta v_j = -\frac{1}{2t^2} \Delta^2 v_j + \frac{1}{t^\gamma} \Delta f_j(\nu), \quad j = 1, \cdots, N.$$  

Multiplying (28) by $\Delta \bar{v}_j$, integrating the imaginary part over $\mathbb{R}^n$, we have

$$\frac{1}{2} \frac{d}{dt} \|\Delta v_j(t)\|^2 = \frac{1}{t^\gamma} \text{Im} \int_{\mathbb{R}^n} \Delta f_j(\nu) \Delta \bar{v}_j dx.$$  

Since $\text{Im} \int_{\mathbb{R}^n} V * |v_k|^2 \Delta v_j dx$ and $\text{Im} \sum_{j,k=1}^{N} \int_{\mathbb{R}^n} V * (v_j \bar{v}_k) \Delta v_k \Delta \bar{v}_j dx$ are equal to zero,

$$\frac{1}{2} \frac{d}{dt} \sum_{j=1}^{N} \|\Delta v_j(t)\|^2$$

$$= t^{-\gamma} \text{Im} \sum_{j,k=1}^{N} \left[ \int_{\mathbb{R}^n} \Delta(V * |v_k|^2)v_j \Delta \bar{v}_j dx + 2 \int_{\mathbb{R}^n} \nabla(V * |v_k|^2) \cdot \nabla v_j \Delta \bar{v}_j dx \\
+ \int_{\mathbb{R}^n} \Delta(V *(v_j \bar{v}_k)) v_k \Delta \bar{v}_j dx + 2 \int_{\mathbb{R}^n} \nabla(V *(v_j \bar{v}_k)) \cdot \nabla v_k \Delta \bar{v}_j dx \right].$$
(i) Case \( n \geq 3 \). Hölder's inequality, Lemma 2.1 and Lemma 2.3 imply that the first term in the brackets of the right of (29) is dominated by

\[
C \int_{\mathbb{R}^N} |x|^{-\gamma-1} * (|\nabla v_k| |v_k| |v_j| |\Delta v_j|) dx 
\]

\[
\leq C \|\nabla v_k\| \|v_k\|_{2n/(n-2\gamma)} \|v_j\|_{2n/(n-2)} \|\Delta_j\| 
\]

\[
\leq C \left( \sum_{j=1}^{N} \|\nabla v_j\| \right)^{-\gamma} \left( \sum_{j=1}^{N} \|\Delta v_j\|^2 \right)^{\gamma/2}. 
\]

The other terms are estimated similarly. Therefore, it follows from (23) that for \( t \geq 1 \),

\[
\frac{d}{dt} \sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \leq C t^{-\gamma} \left( \sum_{j=1}^{N} \|\nabla v_j\| \right)^{4-\gamma} \left( \sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \right)^{\gamma/2} 
\]

\[
\leq C t^{(8-8\gamma+\gamma^2)/2} \left( \sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \right)^{\gamma/2}. 
\]

Integrating this differential inequality, we have

\[
\left( \sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \right)^{1-\gamma/2} \leq C t^{(10-8\gamma+\gamma^2)/2} + \left( \sum_{j=1}^{N} \|\Delta v_j(1)\|^2 \right)^{1-\gamma/2}, 
\]

which implies (27). Since \( \|\Delta v_j(1)\| = ||x|^2 U(-1)u(1)|| \leq C||\phi||_{L^1} \), the constant \( C \) in (23) depends on \( ||\phi||_{L^1} \).

(ii) Case \( n = 2 \). Since

\[
V* = \frac{2^{n-\gamma} \pi^{n/2} \Gamma \left( \frac{n-\gamma}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} (-\Delta)^{(\gamma-n)/2}, \quad 0 < \gamma < n, 
\]

we have for \( n = 2, -\Delta V* = C (-\Delta)^{\gamma/2} \). Hence, by using Hölder's inequality, Lemma 2.1 and Lemma 2.2, we can estimate the first term in the brackets of the right of (29) by

\[
C \|(-\Delta)^{\gamma/2} v_k\| \|v_j\|_{\infty} \|\Delta v_j\| 
\]

\[
\leq C \|(-\Delta)^{\gamma/2} v_k\| \|\nabla v_j\|_{\infty} \|\Delta v_j\| 
\]

\[
\leq C \|\nabla v_j\|^{2-\gamma} \left( \sum_{j=1}^{N} \|\nabla v_j\| \right)^{\gamma/2}. 
\]

Since Lemma 2.1 implies

\[
\|v_k\|_{\infty} \leq C\|\Delta v_k\|^{2/(\theta+2)} \|v_k\|^{\theta/(\theta+2)} 
\]

\[
\leq C\|v_k\|^{2/(\theta+2)} \|\nabla v_k\|^{(\theta-2)/(\theta+2)} \|\Delta v_k\|^{2/(\theta+2)}, 
\]
where \(2 \leq \theta < \infty\), the right of (32) is dominated by

\[
C \|v\|^n \left( \sum_{j=1}^{N} \|\nabla v_j\| \right)^{4-\gamma-2a} \left( \sum_{j=1}^{N} \|\Delta v_j\|^2 \right)^{(\gamma+a)/2}
\]

with \(a = 2/(\theta + 2)\). The second term in the brackets of the right of (29) is estimated by

\[
\|V \cdot (|\nabla v_k| |v_k|)\|_{n/(\gamma-1)} \|\nabla v_j\|_{2n/(n-\gamma+1)} \|\Delta v_j\|
\]

\[
\leq C \|\vec{\varphi}\|_{\infty} \left( \sum_{j=1}^{N} \|\nabla v_j\| \right)^{3-\gamma} \left( \sum_{j=1}^{N} \|\Delta v_j\|^2 \right)^{\gamma/2}
\]

\[
\leq C \|\vec{\varphi}\|^{a} \left( \sum_{j=1}^{N} \|\nabla v_j\| \right)^{4-\gamma-2a} \left( \sum_{j=1}^{N} \|\Delta v_j\|^2 \right)^{(\gamma+a)/2}
\]

The other terms are estimated similarly. Therefore, we have

\[
\frac{d}{dt} \sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \leq C t^{(8-8\gamma+\gamma^2)/2} \left( \sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \right)^{(\gamma+a)/2}
\]

Since the number \(a\) can be chosen arbitrarily small, this differential equation implies (27).

**Lemma 3.3.** Suppose that \(n \geq 2\), \(1 < \gamma \leq \sqrt{2}\) and \(\vec{\varphi} \in \Sigma^{1,2}\). Then we have for \(t \geq 1\),

\[
\|\vec{\varphi}(t)\|_p \leq C.
\]

*Here, \(p\) satisfies \(0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma)\), and the constant \(C\) depends on \(\|\vec{\varphi}\|_{\Sigma^{1,2}}\).*

**Proof.** For simplicity, we prove the lemma in case \(n \geq 3\). We put \(\|\vec{\varphi}\|_{p,*} = \left( \int_{\mathbb{R}^n} \left( \sum_{l=1}^{N} |v_l|^2 \right)^{p/2} \right)^{1/p}\), which is equivalent to the norm \(\|\vec{\varphi}\|_p = \sum_{l=1}^{N} \|v_l\|_p\). We multiply the equation (20) by \(\sum_{l=1}^{N} |v_l|^{2(p-2)/2} \vec{v}_j\), integrate their imaginary part over \(\mathbb{R}^n\), and add them. Then we have

\[
\frac{1}{p} \frac{d}{dt} \|\vec{\varphi}(t)\|_{p,*} = -\frac{1}{2t^2} \text{Im} \sum_{j=1}^{N} \int_{\mathbb{R}^n} \Delta v_j \left( \sum_{l=1}^{N} |v_l|^2 \right)^{(p-2)/2} \vec{v}_j dx,
\]

\[
\frac{1}{p} \frac{d}{dt} \|\vec{\varphi}(t)\|_{p,*} = -\frac{1}{2t^2} \text{Im} \sum_{j=1}^{N} \int_{\mathbb{R}^n} \Delta v_j \left( \sum_{l=1}^{N} |v_l|^2 \right)^{(p-2)/2} \vec{v}_j dx
\]
since \( \text{Im} \int_{\mathbb{R}^n} V \ast|v_k|^2 \left( \sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} |v_j|^2 \, dx \) and

\[
\text{Im} \sum_{j,k=1}^N \int_{\mathbb{R}^n} V \ast (v_j \bar{v}_k)v_k \bar{v}_j \left( \sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} \, dx
\]

are equal to zero. By the integration by parts and Hölder’s inequality,

\[
\frac{1}{p} \frac{d}{dt} \| \vec{v}(t) \|_{p,*}^p = \frac{1}{2t^2} \text{Im} \sum_{j=1}^N \int_{\mathbb{R}^n} \nabla v_j \cdot \nabla \left( \sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} \bar{v}_j \, dx
\leq Ct^{-2} \sum_{j=1}^N \int_{\mathbb{R}^n} |\nabla v_j|^2 \left( \sum_{l=1}^N |v_l|^2 \right)^{(p-2)/2} \, dx
\leq Ct^{-2} \sum_{j=1}^N \| \nabla v_j \|_p^2 \| \vec{v}(t) \|_{p,*}^{(p-2)}.
\]

We note that when \( 1 < \gamma \leq \sqrt{2} \), we have \( 0 < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma) < 1 \), and so \( 2 < p < 2n/(n - 2) \). Then, Lemma 2.1, Lemma 3.1 and Lemma 3.2 yield

\[
\| \nabla v_j \|_p \leq C\| \nabla v_j \|^{1-\delta(p)} \| \Delta v_j \|^{\delta(p)}
\leq Ct^\eta.
\]

Here

\[
\eta = 2 - \gamma + \frac{6 - 4\gamma}{2 - \gamma} \delta(p),
\]

and the constant \( C \) depends on \( \| \vec{\phi} \|_{\Sigma^{1,2}} \). Therefore,

\[
\frac{d}{dt} \| \vec{v}(t) \|_{p,*}^p \leq Ct^{-2+\eta} \| \vec{v}(t) \|_{p,*}^{(p-2)}.
\]

Since \( \eta < 1 \) for \( p \) satisfying \( 0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma) \), the estimate (34) follows by integrating the differential inequality (36).

By this lemma, we have

**Proposition 3.2.** Suppose that \( n \geq 2, 1 < \gamma \leq \sqrt{2} \) and \( \vec{\phi} \in \Sigma^{1,2} \). Then the solution of (1)-(2) has the following estimate

\[
\| \vec{u}(t) \|_p \leq C(1 + |t|)^{-\delta(p)},
\]

where \( p \) satisfies \( 0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma) \).
4. Proof of the main theorem

In this section, we shall prove Theorem 1.1. Since we can prove part (i) of the Theorem similar to the Theorem [A] for Hartree type equation, we omit the proof. Throughout this section, we put \( q = 4n/(2n - \gamma) \) and \( r = 8/\gamma \). Then the pair \((q, r)\) is admissible. To prove part (ii), we introduce the following Banach space:

\[
X^{l,m}(I) = \{ u \in C(I; H^l); \| u \|_{X^{l,m}(I)} < \infty \},
\]

where

\[
\| u \|_{X^{l,m}(I)} = \sum_{|\alpha| \leq l} (\| \nabla^\alpha u \|_{2,\infty,I} + \| \nabla^\alpha u \|_{q,r,I}) + \sum_{|\beta| \leq m} (\| J^\beta u \|_{2,\infty,I} + \| J^\beta u \|_{q,r,I}).
\]

Let \( I = [T, \infty) \), where \( T \) will be defined later. Using Hölder’s inequality, Lemma 2.1 and Lemma 2.3, we have

\[
\sum_{|\alpha| = l} \| \nabla^\alpha f_j(\vec{u}) \|_{q'} \leq C \| \vec{u} \|_q^2 \sum_{k=1}^N \sum_{|\alpha| = l} \| \nabla^\alpha u_k \|_q
\]

and

\[
\sum_{|\beta| = m} \| J^\beta f_j(\vec{u}) \|_{q'} \leq C \| \vec{u} \|_q^2 \sum_{k=1}^N \sum_{|\beta| = m} \| J^\beta u_k \|_q.
\]

So we have, by Lemma 2.1 and Lemma 2.5,

\[
\sum_{|\alpha| \leq l} \| \nabla^\alpha u_j \|_{2,\infty,I} \leq \sum_{|\alpha| \leq l} \| \nabla^\alpha U(-T)u_j(T) \| + C \sum_{|\alpha| \leq l} \| \nabla^\alpha f_j(\vec{u}) \|_{q',r',I}.
\]

Under the assumption of the theorem, Proposition 3.1 or Proposition 3.2 implies \( \| \vec{u}(t) \|_q \leq C t^{-\gamma/4} \). Therefore, by using (38) and Hölder’s inequality, the second term in the right of (40) is dominated by

\[
C \sum_{k=1}^N \sum_{|\alpha| \leq l} \left[ \int_T^\infty \left( \| \vec{u}(\tau) \|_q^2 \| \nabla^\alpha u_k(\tau) \|_q \right)^{r'} d\tau \right]^{1/r'}
\]

\[
\leq C \sum_{k=1}^N \sum_{|\alpha| \leq l} \left[ \int_T^\infty \left( \tau^{-\gamma/2} \| \nabla^\alpha u_k(\tau) \|_q \right)^{r'} d\tau \right]^{1/r'}
\]

\[
\leq C \left( \int_T^\infty \tau^{-2\gamma/(4-\gamma)} d\tau \right)^{(4-\gamma)/4} \sum_{k=1}^N \sum_{|\alpha| \leq l} \| \nabla^\alpha u_k \|_{q,r,I}.
\]
If \( \gamma > 4/3 \), the integral in the right of (41) converges. Hence,

\[
\sum_{|\alpha| \leq l} \| \nabla^\alpha u_j \|_{2,\infty,I} \leq \| U(-T)u_j(T) \|_{\Sigma^{l,m}} + C T^{(4-3\gamma)/4} \| \tilde{u} \|_{\Sigma^{l,m}(I)}.
\]

We can estimate

\[
\sum_{|\alpha| \leq l} \| \nabla^\alpha u_j \|_{q,r,I}, \sum_{|\beta| \leq m} \| J^\beta u_j \|_{2,\infty,I}, \text{ and } \sum_{|\beta| \leq m} \| J^\beta u_j \|_{q,r,I}
\]
similarly. Therefore,

\[
\| \tilde{u} \|_{\Sigma^{l,m}(I)} \leq C \| U(-T)\tilde{u}(T) \|_{\Sigma^{l,m}} + C T^{(4-3\gamma)/4} \| \tilde{u} \|_{\Sigma^{l,m}(I)}.
\]

If we choose \( T \) sufficiently large so that \( C T^{(4-3\gamma)/4} \leq 1/2 \), (43) implies

\[
\| \tilde{u} \|_{\Sigma^{l,m}(I)} \leq C \| U(-T)\tilde{u}(T) \|_{\Sigma^{l,m}}.
\]

Therefore, \( \| \tilde{u} \|_{\Sigma^{l,m}(R)} \) is finite. Once this has been proved, by the similar argument, for \( t > s > 0 \), we have

\[
(44) \| U(-t)\tilde{u}(t) - U(-s)\tilde{u}(s) \|_{\Sigma^{l,m}} \leq C \left( \frac{t - s}{2^{(4-3\gamma)/4}} \right)^{(4-\gamma)/4} \| \tilde{u} \|_{\Sigma^{l,m}(R)}
\]

\[
\leq C \left( t^{(4-3\gamma)/4} - s^{(4-3\gamma)/4} \right).
\]

The right of (44) tends to zero as \( s, t \) tend to infinity. Thus the theorem has been proved.

**Corollary 4.1.** Suppose that \( 4/3 < \gamma < \min(4, n) \), and \( l, m \geq 1 + [n/2] \). Then for any \( \tilde{\varphi} \in \Sigma^{l,m} \), the solution \( \tilde{u}(t) \) of (1)-(2) satisfies

\[
\| \tilde{u}(t) \|_{\infty} \leq C(1 + |t|)^{-n/2}.
\]

**Proof.** By the relation \( J^\beta \tilde{u}(t) = M(t) x^\beta M(-t)\tilde{u}(t) \) and Lemma 2.1.
References


Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560, Japan