QUANTUM DEFORMATIONS OF CERTAIN PREHOMOGENEOUS VECTOR SPACES. II

YOSHIYUKI MORITA

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Introduction

Let $G$ be a reductive algebraic group over the complex number field $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. The quantized coordinate algebra $A_q(G)$ of $G$ is constructed as a certain dual Hopf algebra of the quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$. The Hopf algebras $U_q(\mathfrak{g})$ and $A_q(G)$ over $\mathbb{C}(q)$ tend to the ordinary enveloping algebra $U(\mathfrak{g})$ and the coordinate algebra $A(G)$ respectively when the parameter $q$ tends to 1 in a certain sense (Drinfeld [1], Jimbo [3]).

Let us consider what object we should regard as a quantum deformation of an affine variety $X$ with $G$-action.

An affine variety $X$ is endowed with an action of $G$ if and only if its coordinate algebra $A(X)$ is equipped with a right $A(G)$-comodule structure

$$\tau : A(X) \rightarrow A(X) \otimes A(G)$$

which is simultaneously an algebra homomorphism. By the duality between $U(\mathfrak{g})$ and $A(G)$ we obtain a locally finite left $U(\mathfrak{g})$-module structure

$$(\ast) \quad \gamma : U(\mathfrak{g}) \otimes A(X) \rightarrow A(X)$$

given by

$$(\ast \ast) \quad \tau(n) = \sum_i n_i \otimes f_i \Rightarrow \gamma(u \otimes n) = \sum_i (u, f_i)n_i,$$

where $(\cdot, \cdot) : U(\mathfrak{g}) \times A(G) \rightarrow \mathbb{C}$ is the dual pairing. Since $\tau$ is an algebra homomorphism, we have

$$(\ast \ast \ast) \quad u \in U(\mathfrak{g}), \quad m, n \in A(X), \quad \Delta(u) = \sum_i u_i \otimes v_i \Rightarrow u(mn) = \sum_i (u_i m)(v_i n),$$

where $\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the coproduct. Then the action of $G$ on $X$ is uniquely determined by the infinitesimal action $\gamma$. Moreover, for a locally finite left
$U(g)$-module structure $(\ast)$ on $A(X)$ satisfying $(\ast \ast \ast)$ and a certain condition on irreducible $U(g)$-modules appearing as submodules of $A(X)$, there exists a unique action of $G$ on $X$ whose infinitesimal action is given by $\gamma$.

Now we define the notion of a quantum deformation of an affine variety $X$ with $G$-action as follows. A (not necessarily commutative) $\mathbb{C}(q)$-algebra $A_q(X)$ endowed with a locally finite left $U_q(g)$-module structure

$$\gamma_q : U_q(g) \otimes A_q(X) \to A_q(X)$$

is called a quantum deformation of $X$ if $A_q(X)$ and $\gamma_q$ tend to $A(X)$ and $\gamma : U(g) \otimes A(X) \to A(X)$ respectively when $q$ tends to 1 and if it satisfies

$$\Delta(u) = \sum_i u_i \otimes v_i \Rightarrow u(mn) = \sum_i (u(m)(v)n).$$

It seems to be an interesting problem to determine in which case $X$ admits a quantum deformation. In this paper we consider the problem when $X$ is a prehomogeneous vector space, that is, when $X$ is a vector space with a linear $G$-action containing an open $G$-orbit. Such a quantum deformation was intensively studied in the case where $G = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ and $X = M_{mn}(\mathbb{C})$ (see Taft-Tower [10], Hashimoto-Hayashi [2] and Noumi-Yamada-Mimachi [7]), and also in the case where $G = GL_n(\mathbb{C})$ and $X$ is the set of skew symmetric matrices of degree $n$ (see Strickland [8]).

In our previous paper [4] we gave a general method to construct quantum deformations of prehomogeneous vector spaces of parabolic type. Moreover, for each non-open $G$-orbit $C$ on $X$, we have shown that the defining ideal of the closure $\overline{C}$ and its canonical generators admit quantum deformations inside $A_q(X)$. It includes the existence of the quantum deformation of the irreducible relative invariant when $X$ is a regular prehomogeneous vector space. Indeed, the canonical generator of the defining ideal of the closure of the one-codimensional orbit is nothing but the irreducible relative invariant.

Quantum deformations of prehomogeneous vector spaces of commutative parabolic type associated to classical simple Lie algebras are intensively studied in Kamita [5]. In this paper we shall deal with the remaining two cases

(I) $G = \mathbb{C}^\times \times \text{Spin}(10, \mathbb{C}), X = \mathbb{C}^{16}$, the scalar multiplication and the half-spin representation,

(II) $G = \mathbb{C}^\times \times E_6, X = \mathbb{C}^{27}$, the scalar multiplication and the 27-dimensional irreducible representation of $E_6$,

which naturally arise from the exceptional simple Lie algebras of type $E_6$ and $E_7$ respectively using the method in our previous paper [4]. In Introduction we shall only state the results in case (II).

Let $g_{E_7}$ be a simple Lie algebra of type $E_7$ over $\mathbb{C}$ and let $\mathfrak{h}$ be its Cartan subalgebra. We shall use the labelling of the vertices of the Dynkin diagram 1.
Dynkin diagram 1.

Set $I_0 = \{1, 2, \ldots, 7\}$, $I = I_0 \setminus \{1\}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system of type $E_7$. We denote the set of simple roots by $\{\alpha_i\}_{i \in I_0}$ and the set of positive roots by $\Delta^+$. Let $(\ , \ ) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be a standard symmetric bilinear form. Set $D = \Delta^+ \setminus \sum_{i \in I_0} \mathbb{Z}\alpha_i$. Then we have $\#D = 27$. Set $\Lambda = \{1, 2, \ldots, 27\}$, and fix a bijection $\Lambda \ni j \mapsto \beta_j \in D$ such that $\beta_k - \beta_j \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i$ implies $j \leq k$, where $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$. Set $\delta = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7$. For each $n \in \Lambda$ there exist exactly five pairs $(i, j) \in \Lambda^2$ such that $\beta_i + \beta_j = \delta - \beta_n$, $i < j$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n), (i_5^n, j_5^n) \in \Lambda^2$ where $i_1^n < i_2^n < i_3^n < i_4^n < i_5^n$. Let $K_i, E_i, F_i (i \in I_0)$ be the canonical generators of $U_q(\mathfrak{g}_{E_7})$, and set $U_q(\mathfrak{g}) = \langle K_i^{\pm 1}, E_j, F_j \mid j \in I \rangle \subset U_q(\mathfrak{g}_{E_7})$. Then $U_q(\mathfrak{g})$ is isomorphic to the tensor product of $\mathbb{C}(q)[K, K^{-1}]$ and the quantized enveloping algebra of type $E_6$, where $K = K_1^3 K_2^3 K_3^2 K_4^3 K_5^3 K_6^3 K_7$.

**Theorem 0.1.** A quantum deformation of the 27-dimensional irreducible prehomogeneous vector space $X$ of $G = \mathbb{C}^* \times E_6$ is given by the following.

(a) $A_q(X)$ is an associative $\mathbb{C}(q)$-algebra defined by the following generators and fundamental relations:

**Generators:** $Y_i$ with $i = 1, \ldots, 27$.

**Fundamental relations:** For $i < j$

$$Y_i Y_j = \begin{cases} q Y_i Y_j & \text{if } \beta_i + \beta_j \text{ does not have another decomposition } \beta + \beta', \; \beta, \beta' \in D, \\ Y_j Y_i + q Y_i Y_j - q^{-1} Y_j Y_i & \text{if there exist } k \in I, \; a, b \in \Lambda \text{ such that } \beta_a = \beta_i + \alpha_k, \; \beta_b = \beta_j - \alpha_k, \\ Y_j Y_i & \text{otherwise}. \end{cases}$$

(b) The action $\gamma_q : U_q(\mathfrak{g}) \otimes A_q(X) \to A_q(X)$ is given by the following. For $2 \leq k \leq 7, \; 1 \leq m \leq 7$

$$\gamma_q(F_k \otimes Y_i) = \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i + \alpha_k, \\ 0 & \text{otherwise}, \end{cases}$$

$$\gamma_q(E_k \otimes Y_i) = \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i - \alpha_k, \\ 0 & \text{otherwise}, \end{cases}$$

$$\gamma_q(K_m \otimes Y_i) = q^{-(\alpha_m, \beta_i)} Y_i.$$
\[ \varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n|} Y_n \psi_n, \]

where \( |\beta| = \sum_{i \in \mathfrak{h}_0} m_i \) (\( \beta = \sum_{i \in \mathfrak{h}_0} m_i \alpha_i \)), \( \psi_n = Y_{i^n_1} Y_{i^n_2} - q Y_{i^n_2} Y_{i^n_1} + q^2 Y_{i^n_3} Y_{i^n_2} - q^3 Y_{i^n_2}^2 Y_{i^n_1} + q^4 Y_{i^n_3} Y_{i^n_1} \).

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1. Preliminaries

Let \( \mathfrak{g} \) be a simple Lie algebra of type \( E_6 \) or \( E_7 \) over the complex number field \( \mathbb{C} \), and let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \). Let \( \Delta \subset \mathfrak{h}^* \) be the root system, and let \( W \subset GL(\mathfrak{h}) \) be the Weyl group. We denote the set of positive roots by \( \Delta^+ \) and the set of simple roots by \( \{ \alpha_i \} \), \( i \in I_0 \), where \( I_0 \) is an index set. For \( i \in I_0 \) we denote the simple reflection corresponding to \( \alpha_i \) by \( s_i \in W \). Let \( (\ , \ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \) be the invariant symmetric bilinear form such that \( (\alpha, \alpha) = 2 \) for any \( \alpha \in \Delta \). Set \( a_{ij} = (\alpha_i, \alpha_j) \). The matrix \( (a_{ij})_{i,j \in I_0} \) is called the Cartan matrix of type \( E_6 \) or \( E_7 \). For \( \alpha \in \Delta \) we denote the corresponding root space by \( \mathfrak{g}_\alpha \). Set \( n^+ = \bigoplus_{\alpha \in \Delta^+, \alpha} \mathfrak{g}_\alpha, \ n^- = \bigoplus_{\alpha \in \Delta^-, \alpha} \mathfrak{g}_\alpha \). For a subset \( I \subset I_0 \) we define

\[ \Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i, \quad W_I = \langle s_i \mid i \in I \rangle. \]

We set

\[ l_I = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha \right), \quad n_I^+ = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \quad n_I^- = \bigoplus_{\alpha \in \Delta^- \setminus \Delta_I} \mathfrak{g}_\alpha. \]

Let \( G \) be a connected algebraic group with Lie algebra \( \mathfrak{g} \). We denote by \( L_I \) the subgroup of \( G \) corresponding to \( l_I \). Then \( L_I \) acts on \( n_I^\pm \) via the adjoint action.

The quantized enveloping algebra \( U_q(\mathfrak{g}) \) (Drinfel’d [1], Jimbo [3]) is an associative algebra over the rational function field \( \mathbb{C}(q) \) generated by the elements \( E_i, F_i, K_i, K_i^{-1} \) \((i \in I_0)\) satisfying the following fundamental relations:

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,
K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i,
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},
E_i E_j = E_j E_i \quad (i \neq j, \ a_{ij} = 0),
E_i^2 E_j = (q + q^{-1}) E_i E_j E_i + E_j E_i^2 \quad (i \neq j, \ a_{ij} = -1),
F_i F_j = F_j F_i \quad (i \neq j, \ a_{ij} = 0),
F_i^2 F_j = (q + q^{-1}) F_i F_j F_i + F_j F_i^2 \quad (i \neq j, \ a_{ij} = -1).
\]
A Hopf algebra structure on $U_q(g)$ is defined as follows. The comultiplication
\[ \Delta : U_q(g) \to U_q(g) \otimes U_q(g) \]
is the algebra homomorphism satisfying
\[ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i. \]
The counit $\epsilon : U_q(g) \to \mathbb{C}(q)$ is the algebra homomorphism satisfying
\[ \epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0. \]
The antipode $S : U_q(g) \to U_q(g)$ is the algebra antiautomorphism satisfying
\[ S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i, \quad S(F_i) = -K_i^{-1}F_i. \]

Using the Hopf algebra structure, we define the adjoint action of $U_q(g)$ on $U_q(g)$ as follows. For $x, y \in U_q(g)$ write \( \Delta(x) = \sum_k x_k^1 \otimes x_k^2 \) and set \( \text{ad}(x)y = \sum_k x_k^1yS(x_k^2) \).

Then \( \text{ad} : U_q(g) \to \text{End}_{\mathbb{C}(q)}(U_q(g)) \) is an algebra homomorphism. For $x, y, z \in U_q(g)$ we have \( \text{ad}(x)(yz) = \sum_k \text{ad}(x_k^1)y \text{ad}(x_k^2)z \), where \( \Delta(x) = \sum_k x_k^1 \otimes x_k^2 \).

We define subalgebras $U_q(n^-)$ and $U_q(l_I)$ for $I \subset I_0$ by
\[ U_q(n^-) = \langle F_i \mid i \in I_0 \rangle, \quad U_q(l_I) = \langle E_i, F_i, K_j, K_j^{-1} \mid i \in I, j \in I_0 \rangle. \]

For $i \in I_0$ we define an algebra automorphism $T_i$ of $U_q(g)$ by
\[ T_i(K_j) = K_jK_i^{-a_{ij}}, \quad T_i(E_j) = \begin{cases} -F_iK_i & (i = j) \\ E_j & (i \neq j, \ a_{ij} = 0) \\ E_iE_j - q^{-1}E_jE_i & (i \neq j, \ a_{ij} = -1), \end{cases} \quad T_i(F_j) = \begin{cases} -K_i^{-1}E_i & (i = j) \\ F_j & (i \neq j, \ a_{ij} = 0) \\ F_jF_i - qF_iF_j & (i \neq j, \ a_{ij} = -1), \end{cases} \]
(see Lusztig [6]). For $w \in W$ choose a reduced expression $w = s_i \cdots s_t$, and set $T_w = T_t \cdots T_i$. It is known that $T_w$ does not depend on the choice of a reduced expression.

We shall use the following later (see Lusztig [6]).

**Lemma 1.1.** If $w(\alpha_i) = \alpha_j$ for $w \in W$ and $i, j \in I_0$, then we have $T_w(F_i) = F_j$.

For $I \subset I_0$ let $w_I$ be the longest element of $W_I$ and let $w_0$ be the longest element of $W$. Choose a reduced expression $w_Iw_0 = s_i \cdots s_t$ of $w_Iw_0$ and set
\[ \beta_j = s_i s_2 \cdots s_{j-1}(\alpha_i), \quad Y_j = Y_\beta_j = T_{i_1} \cdots T_{i_{j-1}}(F_{i_j}) \]
for $1 \leq j \leq r$. Then it is known that $\{\beta_j \mid 1 \leq j \leq r\} = \Delta^+ \setminus \Delta_I$. Set

$$U_q(n^-) = \sum_{d_j \geq 0} \mathbb{C}(q)Y_{d_1}^1 \cdots Y_{d_r}^r.$$ 

Then $\{Y_{d_1}^1 \cdots Y_{d_r}^r \mid d_j \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq r\}$ is a basis of $U_q(n^-)$ and $U_q(n^-)$ is a subalgebra of $U_q(n^-)$. We have

$$U_q(n^-) = U_q(n^-) \cap T_{w_I}^{-1} U_q(n^-)$$

and $U_q(n^-)$ does not depend on the choice of a reduced expression of $w_I w_0$ (see Lusztig [6]).

If $n_\beta^+ \neq \{0\}$, then $Y_\beta$ for $\beta \in \Delta^+ \setminus \Delta_I$ does not depend on the choice of a reduced expression of $w_I w_0$ (see [4]). In this case we denote the $\mathbb{C}(q)$-algebra $U_q(n^-)$ by $A_q$. We can regard it as a quantum deformation of the coordinate algebra $A = \mathbb{C}[n_\beta^+]$ of $n_\beta^+$ as explained in [4].

2. Case of type $E_6$

Let $\mathfrak{g}$ be a simple Lie algebra of type $E_6$. We shall use the labelling of the vertices of the Dynkin diagram 2.

\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
  \node (4) at (3,0) {$4$};
  \node (5) at (4,0) {$5$};
  \node (6) at (5,0) {$6$};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (5);
  \draw (5) -- (6);
\end{tikzpicture}
\end{center}

Dynkin diagram 2.

Hence we have $I_0 = \{1, 2, 3, 4, 5, 6\}$. Set $I = \{2, 3, 4, 5, 6\}$. In this case we have $n_\beta^+ \neq \{0\}$, $[n_\beta^+, n_\beta^+] = \{0\}$. Then $l_I$ is isomorphic to $\mathbb{C} \oplus o(10, \mathbb{C})$ and $n_\beta^+$ is a 16-dimensional irreducible prehomogeneous vector space. There are three $L_I$-orbits $\{0\}, C_0, O$ on $n_\beta^+$ satisfying $\{0\} \subset O \subset C_0 \subset \bar{O}$. Let $J_{C_0} \subset \mathbb{C}[n_\beta^+]$ be the defining ideal of the closure of $C_0$, and let $J_{C_0}^0$ denote the subspace of $J_{C_0}$ consisting of the polynomials in $J_{C_0}$ with homogeneous degree 2. Then $J_{C_0}^0$ is a ten-dimensional irreducible $l_I$-module, and it generates the ideal $J_{C_0}$.

We fix a reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_1 s_6 s_5 s_3 s_2 s_4 s_3 s_5 s_6$$

of $w_I w_0$ and define the elements $Y_i$ ($i \in \Lambda = \{1, 2, \ldots, 16\}$) as in Section 1.

Set $I'_0 = \{1, 2, 3, 4, 5\}$, $I' = \{2, 3, 4, 5\}$, $\Lambda' = \{1, 2, \ldots, 8\}$. Then $\{a_i\}_{i \in I'_0}$ is a set of simple roots of type $D_5$. Let $g'$ be the simple subalgebra of $\mathfrak{g}$ corresponding to $I'_0$. We choose a reduced expression $w_{I'} w_{I'_0} = s_1 s_2 s_3 s_4 s_5 s_2 s_1$ of $w_{I'} w_{I'_0}$. The elements $Y_i$ ($i \in \Lambda'$) can be computed inside $U_q(g')$.

Let $\beta_j = \sum_{i \in I'_0} m_i^j a_i$ and set $m^j = (m_1^j, \ldots, m_6^j)$ for $j \in \Lambda$. Then we have
Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ $(k \in I)$. Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_p}$ and $p \in \Lambda$ satisfying

\begin{align*}
\beta &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_p}), \\
\beta' &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p} (\alpha_{i_{p+1}}), \\
\alpha_k &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_p}), \\
\alpha_k &= -1. 
\end{align*}

Proof. Among the 20 triplets $(k, j, l)$ satisfying $\beta_j + \alpha_k = \beta_l$ $(k \in I, j, k \in \Lambda)$, the 12 triplets satisfy $l = j + 1$, $(\alpha_l, \alpha_{l+1}) = -1$. Therefore it is sufficient to deal with the remaining 8 cases. In the cases $(k, j, l) = (5, 3, 5), (5, 4, 6), (5, 11, 13), (5, 12, 14), (2, 7, 8), (6, 5, 9), (4, 9, 10), (3, 10, 11), (2, 11, 12), (5, 11, 13), (5, 12, 14), (2, 13, 14), (6, 10, 11)$, $(6, 11, 12), (6, 8, 12)$.

For $k \in I, j \in \Lambda$, we have $\beta_j - 2\alpha_k, \beta_j + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$.

Lemma 2.1. Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ $(k \in I)$. Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_p}$ and $p \in \Lambda$ satisfying

\begin{align*}
\beta &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_p}), \\
\beta' &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p} (\alpha_{i_{p+1}}), \\
\alpha_k &= s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_p}), \\
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Proof. Among the 20 triplets $(k, j, l)$ satisfying $\beta_j + \alpha_k = \beta_l$ $(k \in I, j, k \in \Lambda)$, the 12 triplets satisfy $l = j + 1$, $(\alpha_l, \alpha_{l+1}) = -1$. Therefore it is sufficient to deal with the remaining 8 cases. In the cases $(k, j, l) = (5, 3, 5), (5, 4, 6), (5, 11, 13), (5, 12, 14), (2, 7, 8), (6, 5, 9), (4, 9, 10), (3, 10, 11), (2, 11, 12), (5, 11, 13), (5, 12, 14), (2, 13, 14), (3, 14, 15), (4, 15, 16), (6, 6, 10), (6, 7, 11), (6, 8, 12)$.

It is known that $U_q(n_I^+) = \bigoplus_{\beta \in \Delta^+ \setminus \Delta_I} C(q)Y_{\beta}$ is an irreducible $U_q(l_I)$-module. (see [4])

Lemma 2.2. For $k \in I, j \in \Lambda$, we have

\[\text{ad}(F_k)Y_j = \begin{cases} Y_l & \text{if there exists } l \in \Lambda \text{ such that } \beta_l = \beta_j + \alpha_k, \\
0 & \text{otherwise}, \end{cases}\]
ad(E_k)Y_j = \begin{cases} Y_i & \text{if there exists } l \in \Lambda \text{ such that } \beta_l = \beta_j - \alpha_k, \\ 0 & \text{otherwise.} \end{cases}

Proof. Since \( \bigoplus_{j \in \Lambda} C(q)Y_j \) is a \( U_q(I_1) \)-module, we have \( \text{ad}(F_k)Y_j = 0 \) if \( \beta_j + \alpha_k \notin \Delta^+ \setminus \Delta_I \), and we have \( \text{ad}(E_l)Y_j = 0 \) if \( \beta_j - \alpha_k \notin \Delta^+ \setminus \Delta_I \).

We shall show \( \text{ad}(F_k)Y_\beta = Y_{\beta'} \) for \( \beta, \beta' \in \Delta^+ \setminus \Delta_I \) and \( k \in I \) satisfying \( \beta' = \beta + \alpha_k \). By Lemma 2.1 we can choose a reduced expression of \( w_lw_0 = s_{i_1}s_{i_2} \cdots s_{i_0} \) satisfying \( \beta = s_{i_1}s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}), \beta' = s_{i_1}s_{i_2} \cdots s_{i_{p-1}}s_{i_p}(\alpha_{i_{p+1}}), (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1 \). Then we can write \( Y_\beta = T_{i_1}T_{i_2} \cdots T_{i_{p-1}}(F_{i_p}), Y_{\beta'} = T_{i_1}T_{i_2} \cdots T_{i_{p-1}}T_{i_p}(F_{i_{p+1}}) \). Since \( (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1 \), we have \( T_{i_p}(F_{i_{p+1}}) = F_{i_{p+1}}F_{i_p} - qF_{i_p}F_{i_{p+1}} \). Moreover, since \( \alpha_k = s_{i_1}s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_{p+1}}) \), we have \( T_{i_1}T_{i_2} \cdots T_{i_{p-1}}(F_{i_{p+1}}) = F_k \) by Lemma 1.1, and hence

\[
Y_{\beta'} = T_{i_1}T_{i_2} \cdots T_{i_{p-1}}T_{i_p}(F_{i_{p+1}}) = F_k Y_\beta - qY_\beta F_k.
\]

Since \( (\beta, \alpha_k) = -1 \), we have \( \text{ad}(F_k)Y_\beta = F_k Y_\beta - qY_\beta F_k \). Hence we have \( \text{ad}(F_k)Y_{\beta'} = Y_{\beta'} \).

Let us show \( \text{ad}(E_k)Y_\beta = Y_{\beta'} \) for \( \beta \in \Delta^+ \setminus \Delta_I \) and \( k \in I \) satisfying \( \beta' = \beta - \alpha_k \).

By the above argument we have \( Y_\beta = \text{ad}(F_k)Y_{\beta'} = F_k Y_{\beta'} - qY_{\beta'} F_k \). Since \( \beta' - \alpha_k = \beta - \alpha_k \), we have \( \text{ad}(F_k)Y_\beta = 0 \), and hence \( E_k Y_{\beta'} = Y_{\beta'} E_k \). Since \( (\beta', \alpha_k) = -1 \), we have \( K_k Y_{\beta'} = qY_{\beta'} K_k \). Hence we have

\[
\text{ad}(E_k)Y_\beta = (E_k Y_\beta - Y_\beta E_k)K_k = (E_k(F_k Y_{\beta'} - qY_{\beta'} F_k) - (F_k Y_{\beta'} - qY_{\beta'} F_k)E_k)K_k
= \left( \frac{K_k - K_k^{-1}}{q - q^{-1}} - qY_{\beta'} - q^{-1} \right) K_k = (Y_{\beta'} K_k^{-1})K_k = Y_{\beta'}.
\]

Next we shall consider quadratic fundamental relations among the elements \( Y_i \). Since we have

\[
\sum_{i,j \in \Lambda} C(q)Y_i Y_j = \bigoplus_{s \leq t} C(q)Y_s Y_t,
\]

we can write

\[
Y_i Y_j = \sum_{s \leq t} a_{s,i,j}^{l,j}Y_s Y_t \quad (a_{s,i,j}^{l,j} \in C(q))
\]

for \( i > j \) (see [4]). Hence if \( \beta_i + \beta_j \) does not have another decomposition \( \beta + \beta' \) (\( \beta, \beta' \in \Delta^+ \setminus \Delta_I \), \( \beta_i + \beta_j = \beta + \beta' \)) then we have \( Y_i Y_j = a_{i,j} Y_j Y_i \) for some \( a_{i,j} \in C(q) \). We denote the set of weights of the ten-dimensional irreducible highest weight \( I_1 \)-module \( J_{\mathfrak{q}_0}^0 \) with highest weight \( -\beta_i - \beta_j \) by \( \Gamma \). For \( \beta, \beta' \in \Delta^+ \setminus \Delta_I \), a weight \( \beta + \beta' \) has another decomposition if and only if we have \( -(\beta + \beta') \in \Gamma \). We fix a bijection
{1, 2, \ldots, 10} \ni n \mapsto -\delta_n \in \Gamma \text{ such that if } \delta_m - \delta_n \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i, \text{ then } n \leq m.  
For each \( n \) there exist exactly four pairs \((i, j)\) \( \in \Lambda^2 \) such that \( i < j \), \( \beta_i + \beta_j = \delta_n \). We 
denote them by \((i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n)\) \( \in \Lambda^2 \) where \( i_4^n < i_3^n < i_2^n < i_1^n \). Set 
\( A(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_4^n, j_3^n, j_2^n, j_1^n) \in \Lambda^8 \) \( (1 \leq n \leq 10) \). Then we have

\[
\begin{align*}
A(1) & = (1, 2, 3, 4, 5, 6, 7, 8), & A(2) & = (1, 2, 3, 4, 9, 10, 11, 12), \\
A(3) & = (1, 2, 5, 6, 9, 10, 13, 14), & A(4) & = (1, 3, 5, 7, 9, 11, 13, 15), \\
A(5) & = (2, 3, 5, 8, 9, 12, 14, 15), & A(6) & = (1, 4, 6, 7, 10, 11, 13, 16), \\
A(7) & = (2, 4, 6, 8, 10, 12, 14, 16), & A(8) & = (3, 4, 7, 8, 11, 12, 15, 16), \\
A(9) & = (5, 6, 7, 8, 13, 14, 15, 16), & A(10) & = (9, 10, 11, 12, 13, 14, 15, 16).
\end{align*}
\]

We denote the set \( \{i_4^n, i_3^n, i_2^n, i_1^n, j_4^n, j_3^n, j_2^n, j_1^n\} \) by \( |A(n)| \) for \( 1 \leq n \leq 10 \). For any \( i, j \in \Lambda \) there exists \( n \) satisfying \( i, j \in |A(n)| \).

Set

\[ A = \{(k, n, n') \in I \times \Lambda \times \Lambda \mid \delta_n + \alpha_k = \delta_{n'}\}. \]

Then

\[ A = \{(6, 1, 2), (5, 2, 3), (3, 3, 4), (2, 4, 5), (4, 4, 6), \\
(2, 6, 7), (4, 5, 7), (3, 7, 8), (5, 8, 9), (6, 9, 10)\}. \]

For any \( n \in \{2, 3, \ldots, 10\} \) we can take a sequence \((k_1, n_1, n'_1), \ldots, (k_s, n_s, n'_s)\) of \( A \) satisfying \( n_1 = 1, n'_s = n, n'_s = n_{s+1} \) \((1 \leq j \leq s - 1)\).

For \((k, n, n') \in A\) and \( m \in \{1, 2, 3, 4\}\), we have either

\[ (P_m^1) \quad (\beta_{i_m}^n, \alpha_k) = 0, \quad i_m^n = i_m^n, \quad (\beta_{j_m}^n, \alpha_k) = -1, \quad \beta_{j_m}^n = \beta_{j_m}^n + \alpha_k \]

or

\[ (P_m^2) \quad (\beta_{i_m}^n, \alpha_k) = -1, \quad \beta_{i_m}^n = \beta_{i_m}^n + \alpha_k, \quad (\beta_{j_m}^n, \alpha_k) = 0, \quad j_m^n = j_m^n. \]

**Proposition 2.3.** For any \( i, j \in \Lambda \) satisfying \( i < j \), we have

\[
(Y_i Y_j \begin{cases}
Y_j Y_i & \text{if there exists } n \text{ such that } i = i_1^n, \; j = j_1^n, \\
Y_{j_2} Y_{i_2} + (q - q^{-1}) Y_{i_2} Y_{j_2} & \text{if there exists } n \text{ such that } i = i_2^n, \; j = j_2^n,
\end{cases}
\]

\[
q Y_i Y_j \begin{cases}
Y_{j_m} Y_{i_m} + q Y_{j_m} Y_{i_m - 1} Y_{i_m - 1} - q^{-1} Y_{i_m} Y_{j_m - 1} & \text{if there exist } n, \; m \in \{3, 4\} \text{ such that } i = i_m^n, \; j = j_m^n, \\
q Y_i Y_j & \text{otherwise}.\end{cases}
\]

**Proof.** Since there exists some \( n \) satisfying \( i, j \in |A(n)| \) for any \( i, j \in \Lambda \), it is sufficient to show that for any \( 1 \leq n \leq 10 \) the elements \( Y_{i_m^n}, Y_{j_m^n} \) \((1 \leq m \leq 4)\) satisfy
the following relations.

\[
\begin{align*}
\text{(Rn)} & \quad \left\{ \begin{array}{l}
Y_{i_1}^{m_1} Y_{j_1}^{m_2} = Y_{j_1}^{m_2} Y_{i_1}^{m_1} \quad \text{(Rn, 1)} \\
Y_{i_2}^{m_1} Y_{j_2}^{m_2} = Y_{j_2}^{m_2} Y_{i_2}^{m_1} + q Y_{j_2}^{m_2} Y_{i_2}^{m_1} - q^{-1} Y_{i_2}^{m_1} Y_{j_2}^{m_2} \quad \text{(Rn, 2)} \\
Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1} \\
\quad \quad \quad \quad (l_1, l_2 \in \{A(n)\}, l_1 < l_2, (l_1, l_2) \neq (i_m^n, j_m^n) \quad (1 \leq m \leq 4)) \quad \text{(Rn, 3)}
\end{array} \right.
\end{align*}
\]

When \( n = 1 \), the elements \( Y_i \) \((1 \leq i \leq 8)\) satisfy the same relations as those for type \( D_5 \), hence the relations (R1) hold.

For any \( m > 1 \) there exists a sequence \(((k_1, n_1, n_1^1), \ldots, (k_s, n_s, n_s^1))\) of \( A \) satisfying \( n_1 = 1, n_j^1 = m, n_j^1 = n_j+1 \) \((1 \leq j \leq s - 1)\), and hence it is sufficient to show the relations (Rn') for \((k, n, n') \in A\) assuming the relations (Rn).

Let \((k, n, n') \in A\). Assume that the relations (Rn) hold.

We first show that the relation (Rn',1) holds. If the condition \( (P_1^+) \) is satisfied, then we have \( Y_{i_1}^{m_1} = Y_{i_1}^{m_1}, F_k Y_{i_1}^{m_1} = Y_{i_1}^{m_1} Y_k, Y_{j_1}^{m_1} = ad(F_k) Y_{j_1}^{m_1} = F_k Y_{j_1}^{m_1} - q Y_{j_1}^{m_1} F_k\). Since \( Y_{i_1}^{m_1} Y_{j_1}^{m_1} = Y_{j_1}^{m_1} Y_{i_1}^{m_1}, we have

\[
Y_{i_1}^{m_1} Y_{j_1}^{m_1} = Y_{i_1}^{m_1} \quad \text{ad}(F_k) Y_{j_1}^{m_1} = Y_{i_1}^{m_1}(F_k Y_{j_1}^{m_1} - q Y_{j_1}^{m_1} F_k)
\]

If the condition \( (P_1^-) \) is satisfied, then we can prove the formula (Rn',1) similarly.

Next we prove the formula (Rn',2). Assume the condition \( (P_2^+) \) is satisfied, then we have

\[
Y_{i_2}^{m_2} Y_{j_2}^{m_2} = Y_{i_2}^{m_2} (F_k Y_{j_2}^{m_2} - q Y_{j_2}^{m_2} F_k)
\]

If the condition \( (P_2^-) \) is satisfied, then we have

\[
F_k Y_{j_2}^{m_2} Y_{i_2}^{m_2} - q Y_{j_2}^{m_2} Y_{i_2}^{m_2} F_k = (F_k Y_{j_2}^{m_2} - q Y_{j_2}^{m_2} F_k) Y_{i_2}^{m_2} = Y_{j_2}^{m_2} Y_{i_2}^{m_2},
\]

and if the condition \( (P_{m-1}^-) \) is satisfied, then we have

\[
F_k Y_{j_2}^{m_2} Y_{i_2}^{m_2} - q Y_{j_2}^{m_2} Y_{i_2}^{m_2} F_k = (F_k Y_{j_2}^{m_2} - q Y_{j_2}^{m_2} F_k) Y_{i_2}^{m_2} = Y_{j_2}^{m_2} Y_{i_2}^{m_2}.
\]

Hence we have \( Y_{i_2}^{m_2} Y_{j_2}^{m_2} = Y_{j_2}^{m_2} Y_{i_2}^{m_2} + q Y_{j_2}^{m_2} Y_{i_2}^{m_2} - q^{-1} Y_{j_2}^{m_2} Y_{i_2}^{m_2} \). The formula (Rn',2) is proved. When the condition \( (P_m^-) \) is satisfied, we can prove it similarly.
Finally we prove the formula \((R_n',3)\). Let \(l'_1, l'_2 \in |A(n')|\) satisfying \(l'_1 < l'_2\) and \((l'_1, l'_2) \neq (i_m', j_m')\) for \(1 \leq m \leq 4\). When \(l'_p = i_m' \in |A(n')|\) (resp. \(l'_p = j_m'\)), we denote \(i_m' \in |A(n)|\) (resp. \(j_m'\)) by \(l_p\) for \(p = 1, 2\). Since \(l_1 < l_2\) and \((l_1, l_2) \neq (i_m', j_m')\) for \(1 \leq m \leq 4\), we have \(Y_{l_1}Y_{l_2} = qY_{l_2}Y_{l_1}\). We have the following possibilities:

1. \(l'_1 = l_1, l'_2 = l_2, (\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = 0\),
2. \(l'_1 = l_1, (\beta_{l_1}, \alpha_k) = 0, \beta_{l'_2} = \beta_{l_2} + \alpha_k, (\beta_{l_2}, \alpha_k) = -1\),
3. \(\beta_{l'_1} = \beta_{l_1} + \alpha_k, (\beta_{l_1}, \alpha_k) = -1, l'_2 = l_2, (\beta_{l_2}, \alpha_k) = 0\),
4. \(\beta_{l'_1} = \beta_{l_1} + \alpha_k, \beta_{l'_2} = \beta_{l_2} + \alpha_k, (\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = -1\).

In the case (1) the formula \((R_n',3)\) is obvious.

In the case (2) we have \(F_k Y_{l_1} = Y_{l_1} F_k, Y_{l_2} = \text{ad}(F_k) Y_{l_2} = F_k Y_{l_2} - q Y_{l_2} F_k\). Hence we have

\[ Y_{l_1} Y_{l_2} = Y_{l_1} (F_k Y_{l_2} - q Y_{l_2} F_k) = q (F_k Y_{l_2} - q Y_{l_2} F_k) Y_{l_1} = q Y_{l_2} Y_{l'_1}. \]

In the case (3) we can prove it similarly to the case (2).

In the case (4) we have \(Y_{l'_p} = F_k Y_{l_p} - q Y_{l_p} F_k\) for \(p = 1, 2\). Since \(\beta_{l'_p} + \alpha_k = \beta_{l_p} + 2\alpha_k \notin \Delta^* \setminus \Delta_I\) and \((\beta_{l'_p}, \alpha_k) = 1\), we have \(\text{ad}(F_k) Y_{l'_p} = F_k Y_{l'_p} - q^{-1} Y_{l_p} F_k = 0\) for \(p = 1, 2\). Hence we have \(F_k Y_{l_p} Y_{l'_p} - (q + q^{-1}) F_k Y_{l_p} F_k + Y_{l_p} F_k F_k = 0, F_k Y_{l_p} F_k = (q + q^{-1})^{-1} (F_k F_k Y_{l_p} + Y_{l_p} F_k F_k)\) for \(p = 1, 2\). By these formulas we have

\[ Y_{l_1} Y_{l_2} = (F_k Y_{l_1} - q Y_{l_1} F_k)(F_k Y_{l_2} - q Y_{l_2} F_k) \]
\[ = F_k Y_{l_1} F_k Y_{l_2} - q F_k Y_{l_1} Y_{l_2} F_k - q Y_{l_1} F_k Y_{l_2} + q^2 Y_{l_1} F_k Y_{l_2} F_k \]
\[ = \frac{1}{q + q^{-1}} F_k F_k Y_{l_1} Y_{l_2} + \frac{1}{q + q^{-1}} Y_{l_1} F_k F_k Y_{l_2} - q F_k Y_{l_1} Y_{l_2} F_k - q Y_{l_1} F_k Y_{l_2} F_k \]
\[ + \frac{q^2}{q + q^{-1}} Y_{l_1} Y_{l_2} F_k F_k \]
\[ = \frac{1}{q + q^{-1}} F_k F_k Y_{l_1} Y_{l_2} - q F_k Y_{l_1} Y_{l_2} F_k + \frac{q^2}{q + q^{-1}} Y_{l_1} Y_{l_2} F_k F_k. \]

Similarly we have

\[ Y_{l'_2} Y_{l'_1} = \frac{1}{q + q^{-1}} F_k F_k Y_{l'_2} Y_{l'_1} - q F_k Y_{l'_2} Y_{l'_1} F_k + \frac{q^2}{q + q^{-1}} Y_{l'_2} Y_{l'_1} F_k F_k. \]

Since \(Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1}\), we have \(Y_{l'_1} Y_{l'_2} = q Y_{l'_2} Y_{l'_1}\).

By [4] and Proposition 2.3 we obtain the following:

**Theorem 2.4.** The formulas \((Q6)\) give fundamental relations for the generator system \([Y_i]_{i \in \Lambda}\) of the algebra \(A_q = U_q(n^-)\).

We shall construct a quantum deformation of the lowest degree part \(J_{C_0}^0\) of the defining ideal \(J_{C_0}\) and we shall give canonical generators of a quantum analogue of
Set
\[ \psi_n = Y_i^* Y_j^* - q Y_i^* Y_j^* + q^2 Y_i^* Y_j^* - q^3 Y_i^* Y_j^*. \]
for \( 1 \leq n \leq 10 \). Recall that \( A(n) = (i_n^*, i_n^*, i_n^*, i_n^*, j_n^*, j_n^*, j_n^*, j_n^*, j_n^*, j_n^*) \). Using the formulas \((Rn,1),(Rn,2)\), we can write \( \psi_n = Y_i^* Y_j^* - q^{-1} Y_i^* Y_j^* + q^{-2} Y_i^* Y_j^* - q^{-3} Y_i^* Y_j^*\).

\textbf{Lemma 2.5.} We have
\[
\text{ad}(F_k)\psi_n = \begin{cases} 
\psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n + \alpha_k = \delta_{n'}, \\
0 & \text{otherwise,}
\end{cases}
\]
\[
\text{ad}(E_k)\psi_n = \begin{cases} 
\psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n - \alpha_k = \delta_{n'}, \\
0 & \text{otherwise}
\end{cases}
\]
for \( k \in I \), and
\[
\text{ad}(K_k)\psi_n = q^{-(\delta_n,\alpha_k)}\psi_n
\]
for \( k \in I_0 \).

\textbf{Proof.} Let \((k,n,n') \in \mathcal{A}\). We shall show \( \text{ad}(F_k)\psi_n = \psi_{n'} \). If the condition \((P^*_m)\) is satisfied, then we have \( \text{ad}(F_k)Y_{i_n^*} = 0 \), \( Y_{i_n^*} = Y_{i_n^*} \), \( \text{ad}(K_k)Y_{i_n^*} = Y_{i_n^*} \), \( \text{ad}(F_k)Y_{j_n^*} = Y_{j_n^*} \).

Hence
\[
\text{ad}(F_k)(Y_{i_n^*} Y_{j_n^*}) = \text{ad}(F_k)Y_{i_n^*} Y_{j_n^*} + \text{ad}(K_k)Y_{i_n^*} \text{ad}(F_k)Y_{j_n^*} = Y_{i_n^*} Y_{j_n^*}
\]
If the condition \((P_m)\) is satisfied, then we have \( \text{ad}(F_k)Y_{i_n^*} = Y_{i_n^*} \), \( \text{ad}(F_k)Y_{j_n^*} = 0 \). Hence \( \text{ad}(F_k)(Y_{i_n^*} Y_{j_n^*}) = Y_{i_n^*} Y_{j_n^*} \) similarly. Therefore we have \( \text{ad}(F_k)\psi_n = \psi_{n'} \).

Next we prove \( \text{ad}(E_k)\psi_{n'} = \psi_n \). We have \( \text{ad}(E_k)Y_{i_n^*} = 0 \), \( \text{ad}(E_k)Y_{j_n^*} = Y_{j_n^*} \) if the condition \((P^*_m)\) is satisfied, and we have \( \text{ad}(E_k)Y_{i_n^*} = Y_{i_n^*} \), \( \text{ad}(K_k^{-1})Y_{j_n^*} = Y_{j_n^*} \), \( j_m^* = j_n^* \), \( \text{ad}(E_k)Y_{j_n^*} = 0 \) if the condition \((P_m)\) is satisfied. Hence we have
\[
\text{ad}(E_k)(Y_{i_n^*} Y_{j_m^*}) = \text{ad}(E_k)Y_{i_n^*} \text{ad}(K_k^{-1})Y_{j_m^*} + Y_{i_n^*} \text{ad}(E_k)Y_{j_n^*} = Y_{i_n^*} Y_{j_m^*}
\]
for \( 1 \leq m \leq 4 \). Therefore we have \( \text{ad}(E_k)\psi_{n'} = \psi_n \).

In other 50 cases, where \( \delta_n + \alpha_k \notin \{ \delta_l \mid 1 \leq l \leq 10 \} \), we can check \( \text{ad}(F_k)\psi_n = 0 \) by a case-by-case consideration as follows.

In the 10 cases where there exists \( n' \) satisfying \( \text{ad}(F_k)\psi_{n'} = \psi_n \), \( (k,n) = (6,2), (5,3), (3,4), (2,5), (4,6), (2,7), (4,7), (3,8), (5,9), (6,10) \), we have \( \text{ad}(F_k)Y_{i_m^*} = \text{ad}(F_k)Y_{j_m^*} = 0 \) for \( 1 \leq m \leq 4 \), and hence the assertion is obvious.

In the 8 cases \( (k,n) = (5,1), (6,3), (6,4), (6,5), (6,6), (6,7), (6,8), (5,10) \), we have \( \text{ad}(F_k)Y_{i_m^*} = \text{ad}(F_k)Y_{j_m^*} = 0 \) for \( m = 3,4 \), \( \text{ad}(F_k)Y_{i_2^*} = Y_{i_2^*} \), \( \text{ad}(F_k)Y_{j_2^*} = 0 \),
ad(F_k)Y_{j_1}^r = Y_{j_2}^r, \ ad(F_k)Y_{j_2}^r = 0, and hence ad(F_k)(Y_{j_1}^r Y_{j_2}^r) = Y_{j_1}^r Y_{j_2}^r. \ Thus we have ad(F_k)\psi_n = q^2(Y_{j_1}^r Y_{j_2}^r - q Y_{j_2}^r Y_{j_1}^r) = 0 by Proposition 2.3.

In the remaining 32 cases there exists $m' \in \{2, 3, 5\}$ such that $\text{ad}(F_k)Y_{m'}^r = 0 (m \neq m')$, $\text{ad}(F_k)Y_{0}^r = 0 \ (m \neq m' - 1)$, $\text{ad}(F_k)Y_{m}^r = Y_{m-1}^r$, $\text{ad}(F_k)(Y_{m'}^r Y_{m}^r) = q^{-1}Y_{m'}^r Y_{m}^r$. Then we have $\text{ad}(F_k)(Y_{m'}^r Y_{m-1}^r) = Y_{m'}^r Y_{m-1}^r$, $\text{ad}(F_k)(Y_{m}^r Y_{m-1}^r) = q^{-1}Y_{m}^r Y_{m-1}^r$, $\text{ad}(F_k)\psi_n = q^{4-m'}(1 - q^{-1})Y_{m'}^r Y_{m}^r = 0$.

The weight $\beta_{m'} + \beta_{m}^r$ does not depend on $m$. Hence we have $\text{ad}(K_k)\psi_n = q^{-(\delta_n, \alpha_k)}\psi_n$ where $\delta_n = \beta_{m'} + \beta_{m}^r$.

Finally we show $\text{ad}(E_k)\psi_n = 0$ if $\delta_n - \alpha_k \notin \{\delta_l | 1 \leq l \leq 10\}$. We can check $\text{ad}(E_k)\psi_1 = 0$ for any $k = 2, 3, \ldots, 6$ directly. It follows that $\sum_{n=1}^{10} C(q)\psi_n = U_q(F_1)\psi_1$ and hence $\sum_{n=1}^{10} C(q)\psi_n$ is an $\text{ad}(U_q(F_1))$-stable subspace with weights in $\{-\delta_l | 1 \leq l \leq 10\}$. Therefore we have $\text{ad}(E_k)\psi_n = 0$ if $\delta_n - \alpha_k \notin \{\delta_l | 1 \leq l \leq 10\}$. \hfill \Box

**Proposition 2.6.** $\sum_{n=1}^{10} C(q)\psi_n$ is an irreducible highest weight $U_q(F_1)$-module with highest weight vector $\psi_1$.

**Proof.** By Lemma 2.5 $\sum_{n=1}^{10} C(q)\psi_n$ is a finite dimensional $U_q(F_1)$-submodule generated by a highest weight vector $\psi_1$ with highest weight $-\delta_1$. Thus it is irreducible. \hfill \Box

By [4] and Proposition 2.6 we obtain the following:

**Theorem 2.7.** A quantum analogue of the defining ideal $J_{C_0}$ of the closure of the non-trivial non-open orbit $C_0$ is given by the two-sided ideal of $A_q$ generated by $\{\psi_n | 1 \leq n \leq 10\}$.

### 3. Case of type $E_7$

Let $g$ be a simple Lie algebra of type $E_7$. We shall use the labelling of the vertices of the Dynkin diagram 1. Hence we have $I_0 = \{1, 2, 3, 4, 5, 6, 7\}$. Set $I = \{2, 3, 4, 5, 6, 7\}$. In this case we have $\eta_i^r \neq \{0\}, [\eta_i^r, \eta_j^r] = \{0\}$. Then $F_1$ is isomorphic to $C \oplus g_{E_6}$, where $g_{E_6}$ is a Lie algebra of type $E_6$ over $\mathbb{C}$, and $\eta_i^r$ is a 27-dimensional irreducible prehomogeneous vector space. There are four $L_{F_1}$-orbits $\{0\}, C_1, C_2, O$ on $\eta_i^r$ satisfying $\{0\} \subset C_1 \subset C_2 \subset O$. Let $J_{C_1} \subset \mathbb{C}[\eta_i^r]$ be the defining ideal of the closure of $C_1$, and let $J_{C_1}^0$ denote the subspace of $J_{C_1}$ consisting of the polynomials in $J_{C_1}$ with homogeneous degree 2. Then $J_{C_1}^0$ is a 27-dimensional irreducible $L_{F_1}$-module, and it generates the ideal $J_{C_1}$. Let $J_{C_2} \subset \mathbb{C}[\eta_i^r]$ be the defining ideal of the closure of $C_2$, and let $J_{C_2}^0$ denote the subspace of $J_{C_2}$ consisting of the polynomials in $J_{C_2}$ with homogeneous degree 3. Then $J_{C_2}^0$ is a one-dimensional irreducible $L_{F_1}$-module generated by the irreducible relative invariant, and it generates the ideal $J_{C_2}$.
We fix a reduced expression of $W/W_{Q}$ and define the elements $Y_i$ ($i \in \Lambda = \{1, 2, \ldots , 27\}$) as in Section 1.

Set $I'_0 = \{1, 2, 3, 4, 5, 6\}$, $I' = \{2, 3, 4, 5, 6\}$, $\Lambda' = \{1, 2, \ldots , 10\}$. Then $\{\alpha_i\}_{i \in I'_0}$ is a set of simple roots of type $D_6$. Let $g'$ be the simple subalgebra of $Q$ corresponding to $I'_0$. We choose a reduced expression $w_{I'}w'_{I_0} = s_1s_2s_3s_4s_5s_6s_7s_8s_9s_{10}$ of $w_{I'}w'_{I_0}$. The elements $Y_i$ ($i \in \Lambda'$) can be computed inside $U_q(g')$.

Let $m = \sum_{i \in I} m_i \alpha_i$ and set $m_j = (m_1, \ldots , m_{27})$ for $j \in \Lambda$. Then we have

\[
\begin{align*}
    m^1 &= (1, 0, 0, 0, 0, 0, 0), & m^2 &= (1, 1, 0, 0, 0, 0, 0), & m^3 &= (1, 1, 1, 0, 0, 0, 0), \\
    m^4 &= (1, 1, 1, 1, 0, 0), & m^5 &= (1, 1, 1, 1, 1, 0), & m^6 &= (1, 1, 1, 2, 1, 1), \\
    m^7 &= (1, 1, 1, 1, 1, 0), & m^8 &= (1, 1, 1, 1, 1, 1), & m^9 &= (1, 1, 1, 1, 1, 1), \\
    m^{10} &= (1, 1, 1, 1, 1, 0), & m^{11} &= (1, 1, 1, 1, 1, 1), & m^{12} &= (1, 1, 1, 1, 1, 1), \\
    m^{13} &= (1, 1, 1, 1, 1, 0), & m^{14} &= (1, 1, 1, 1, 1, 1), & m^{15} &= (1, 1, 1, 1, 1, 1), \\
    m^{16} &= (1, 1, 1, 1, 1, 0), & m^{17} &= (1, 1, 1, 1, 1, 1), & m^{18} &= (1, 1, 1, 1, 1, 1), \\
    m^{19} &= (1, 1, 1, 1, 1, 0), & m^{20} &= (1, 1, 1, 1, 1, 1), & m^{21} &= (1, 1, 1, 1, 1, 1), \\
    m^{22} &= (1, 1, 1, 1, 1, 0), & m^{23} &= (1, 1, 1, 1, 1, 1), & m^{24} &= (1, 1, 1, 1, 1, 1), \\
    m^{25} &= (1, 1, 1, 1, 1, 0), & m^{26} &= (1, 1, 1, 1, 1, 1), & m^{27} &= (1, 1, 1, 1, 1, 1).
\end{align*}
\]

If $(\beta_j, \alpha_k) = -1$ for $j \in \Lambda$ and $k \in I$, then $s_k(\beta_j) = \beta_j + \alpha_k \in \Delta^+ \setminus \Delta_I$ and there exists $l \in \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. Conversely if $\beta_j, \beta_l \in \Delta^+ \setminus \Delta_I$ satisfying $\beta_j - \beta_l = \alpha_k$ ($k \in I$), then we have $(\beta_j, \alpha_k) = -1, s_k(\beta_j) = \beta_l$.

For $k \in I, j \in \Lambda$, we have $\beta_j - 2\alpha_k, \beta_j + 2\alpha_k \not\in \Delta^+ \setminus \Delta_I$.

Set

$$\mathcal{B} = \{(k, j, l) \in I \times \Lambda \times \Lambda \mid \beta_j + \alpha_k = \beta_l\}.\]$$

We have

$$\mathcal{B} = \{(2, 1, 2), (3, 2, 3), (4, 3, 4), (5, 4, 5), (6, 4, 6), (6, 5, 7), (5, 6, 7), (4, 7, 8), (3, 8, 9), (2, 9, 10), (7, 6, 11), (7, 7, 12), (7, 8, 13), (7, 9, 14), (7, 10, 19), (5, 11, 12), (4, 12, 13), (3, 13, 14), (6, 13, 15), (6, 14, 16), (3, 15, 16), (4, 16, 17), (5, 17, 18), (2, 14, 19), (2, 16, 20), (2, 17, 21), (2, 18, 22), (6, 19, 20), (4, 20, 21), (5, 21, 22), (3, 21, 23), (3, 22, 24), (5, 23, 24), (4, 24, 25), (6, 25, 26), (7, 26, 27)\}.$$ In particular, we have $|\mathcal{B}| = 36$.

**Lemma 3.1.** Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ ($k \in I$). Then we can choose a reduced expression $w_{I}w_0 = s_1s_2\cdots s_{12}$, and $p \in \Lambda$ satisfying

$$\begin{align*}
    \beta &= s_1s_2\cdots s_{p-1}(\alpha_{i_p}), & \beta' &= s_1s_2\cdots s_{p-1}s_p(\alpha_{i_{p+1}}), & (\alpha_{i_p}, \alpha_{i_{p+1}}) &= -1, \\
    \alpha_k &= s_1s_2\cdots s_{p-1}(\alpha_{i_{p+1}}).
\end{align*}$$
Proof. The 21 triplets \((k, j, l)\) in \(B\) satisfy \(l = j + 1\), \((\alpha\_i, \alpha\_j) = -1\). Therefore it is sufficient to deal with the remaining 15 cases. In the cases \((k, j, l) = (6, 4, 6), (6, 5, 7), (6, 13, 15), (6, 14, 16), (3, 21, 23), (3, 22, 24)\), we can take
\[
w_j w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} s_{15} s_{16} s_{17} s_{18} s_{19} s_{20} s_{21}
\]
with \(p = 4, 6, 13, 15, 21, 23\), and in the cases \((k, j, l) = (7, 6, 11), (7, 7, 12), (7, 8, 13), (7, 9, 14), (7, 10, 19)\), we can take
\[
w_j w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} s_{15} s_{16} s_{17} s_{18} s_{19} s_{20} s_{21}
\]
with \(p = 6, 8, 10, 12, 14\), and in the cases \((k, j, l) = (2, 14, 19), (2, 16, 20), (2, 17, 21), (2, 18, 22)\), we can take
\[
w_j w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} s_{15} s_{16} s_{17} s_{18} s_{19} s_{20} s_{21}
\]
with \(p = 15, 17, 19, 21\).

We can show the following similarly to the case \(E\_6\). We omit the details.

**Lemma 3.2.** For \(k \in I, j \in \Lambda\), we have
\[
ad(F_k)Y_j = \begin{cases} Y_i & \text{if there exists } (k, j, l) \in B, \\ 0 & \text{otherwise,} \end{cases}
\]
\[
ad(E_k)Y_j = \begin{cases} Y_l & \text{if there exists } (k, l, j) \in B, \\ 0 & \text{otherwise.} \end{cases}
\]

The \(U_q(\mathfrak{l}_1)\)-module \(\bigoplus_{j \in \Lambda} C(q)Y_j\) is an irreducible highest weight module with highest weight vector \(Y_1\) and lowest weight vector \(Y_{27}\). Hence, for any \(1 \leq m \leq 26\), there exists a sequence \(((k_1, n'_1, n_1), \ldots, (k_s, n'_s, n_s))\) of \(B\) satisfying \(n_1 = 27, n'_1 = m, n'_j = n_{j+1} (1 \leq j \leq s - 1)\).

Next we shall consider relations among the elements \(Y_i\). We can write
\[
Y_i Y_j = \sum_{\beta, \gamma, \delta} a^{i, j}_{\beta, \gamma, \delta} Y_\beta Y_\gamma \quad (a^{i, j}_{\beta, \gamma, \delta} \in C(q))
\]
for \(i > j\) (see [4]). Hence if \(\beta + \beta'\) does not have another decomposition \(\beta + \beta' (\beta, \beta' \in \Delta^* \setminus \Delta_I, \beta_1 + \beta_2 = \beta + \beta')\) then we have \(Y_i Y_j = a_{i, j} Y_j Y_i\) for some \(a_{i, j} \in C(q)\). Set \(\delta = 2\sigma_1 = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7\), where \(\sigma_1\) is the fundamental weight corresponding to \(\alpha_1\). We denote a set of weights of the 27-dimensional irreducible highest weight \(U_q(\mathfrak{l}_I)\)-module \(J_{\mathfrak{c}_1}^0\) with highest weight \(-\beta_1 - \beta_{10}\) by \(\Gamma\). Set \(\gamma_n = \delta - \beta_n (n \in \Lambda)\), and we have \(\Gamma = \{-\gamma_n \mid n \in \Lambda\}\). For \(\beta, \beta' \in \Delta^* \setminus \Delta_I\), a weight \(\beta + \beta'\) has another decomposition if and only if we have \(-(\beta + \beta') \in \Gamma\). For each \(n \in \Lambda\) there
exist exactly five pairs \((i, j) \in \Lambda^2\) such that \(i < j\), \(\beta_i + \beta_j = \gamma_n\). We denote them by \((i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n), (i_5^n, j_5^n)\) \(\in \Lambda^2\) where \(i_2^n < i_4^n < i_3^n < i_5^n\), \(j_1^n < j_2^n < j_3^n < j_5^n\), and \(i_1^n, j_1^n\) satisfy the following condition \((P_i^n)\) or \((P_j^n)\). Set \(B(n) = (i_1^n, i_2^n, i_3^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n) \in \Lambda^{10}\) \((n \in \Lambda)\). Then we have

\[
\begin{align*}
B(1) &= (10, 19, 20, 21, 22, 24, 25, 26, 27), \\
B(2) &= (9, 14, 16, 17, 23, 18, 24, 25, 26, 27), \\
B(3) &= (8, 13, 15, 17, 21, 18, 22, 24, 25, 26, 27), \\
B(4) &= (7, 12, 15, 16, 20, 18, 22, 24, 25, 26, 27), \\
B(5) &= (6, 11, 15, 16, 20, 17, 21, 23, 26, 27), \\
B(6) &= (5, 12, 13, 14, 19, 18, 22, 24, 25, 27), \\
B(7) &= (4, 11, 13, 14, 19, 17, 21, 23, 25, 27), \\
B(8) &= (3, 11, 12, 14, 19, 16, 20, 23, 24, 27), \\
B(9) &= (2, 11, 12, 13, 19, 15, 20, 21, 22, 27), \\
B(10) &= (1, 11, 12, 13, 14, 15, 16, 17, 18, 27), \\
B(11) &= (5, 7, 8, 9, 10, 18, 22, 24, 25, 26), \\
B(12) &= (4, 6, 8, 9, 10, 17, 21, 23, 25, 26), \\
B(13) &= (3, 6, 7, 9, 10, 16, 20, 23, 24, 26), \\
B(14) &= (2, 6, 7, 8, 10, 15, 20, 21, 22, 26), \\
B(15) &= (3, 4, 5, 9, 10, 14, 19, 23, 24, 25), \\
B(16) &= (2, 4, 5, 8, 10, 13, 19, 21, 22, 25), \\
B(17) &= (2, 3, 5, 7, 10, 12, 19, 20, 22, 24), \\
B(18) &= (2, 3, 4, 6, 10, 11, 19, 20, 21, 23), \\
B(19) &= (1, 6, 7, 8, 9, 15, 16, 17, 18, 26), \\
B(20) &= (1, 4, 5, 8, 9, 13, 14, 17, 18, 25), \\
B(21) &= (1, 3, 5, 7, 9, 12, 14, 16, 18, 24), \\
B(22) &= (1, 3, 4, 6, 9, 11, 14, 16, 17, 23), \\
B(23) &= (1, 2, 5, 7, 8, 12, 13, 15, 18, 22), \\
B(24) &= (1, 2, 4, 6, 8, 11, 13, 15, 17, 21), \\
B(25) &= (1, 2, 3, 6, 7, 11, 12, 15, 16, 20), \\
B(26) &= (1, 2, 3, 4, 5, 11, 12, 13, 14, 19), \\
B(27) &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10).
\end{align*}
\]

For \(n \in \Lambda\) we denote the set \([i_1^n, i_2^n, i_3^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n]\) by \([B(n)]\). For any \(i, j \in \Lambda\) there exists \(n \in \Lambda\) satisfying \(i, j \in [B(n)]\).

For \((k, n', n) \in B\) and \(m \in \{1, 2, 3, 4, 5\}\), we have either

\[
\begin{align*}
(P_{i,n}^k) &\quad (\beta_{i,n}^k, \alpha_k) = 0, \quad \beta_{i,n}^k = i_m^n, \quad (\beta_{j,n}^k, \alpha_k) = -1, \quad \beta_{j,n}^k = j_m^n + \alpha_k \\
(P_{j,n}^k) &\quad (\beta_{j,n}^k, \alpha_k) = -1, \quad \beta_{i,n}^k = i_m^n + \alpha_k, \quad (\beta_{j,n}^k, \alpha_k) = 0, \quad j_m^n = j_m^n.
\end{align*}
\]

Proposition 3.3. For any \(i, j \in \Lambda\) satisfying \(i < j\), we have

\[
Y_i Y_j = \begin{cases} 
Y_j Y_i & \text{if there exists } n \in \Lambda \text{ such that } \{i, j\} = \{i_1^n, j_1^n\}, \\
Y_j Y_i + (q - q^{-1})Y_i Y_j & \text{if there exists } n \in \Lambda \text{ such that } i = i_2^n, j = j_2^n, \\
Y_j Y_i + q Y_{j,m-1} Y_{i,m-1} - q^{-1} Y_{i,m-1} Y_{j,m-1} & \text{if there exist } n \in \Lambda, m \in \{3, 4, 5\} \text{ such that } i = i_m^n, j = j_m^n,
\end{cases}
\]

otherwise.

Proof. Since there exists \(n \in \Lambda\) satisfying \(i, j \in [B(n)]\) for any \(i, j \in \Lambda\), it is
sufficient to show
\[
\begin{align*}
Y_{i_1} Y_{j_1} &= Y_{i_1} Y_{j_1}, \quad \text{(Rn, 1)} \\
Y_{i_2} Y_{j_2} &= Y_{i_2} Y_{j_2} + q Y_{i_2} Y_{j_2} - q^{-1} Y_{i_2} Y_{j_2} \quad \text{(2 \leq m \leq 5)} \tag{Rn, 2} \\
Y_{i} Y_{j} &= q Y_{i} Y_{j} \quad \text{for } (1 \leq m \leq 5) \tag{Rn, 3}
\end{align*}
\]

for \( n \in \Lambda \) and \( 1 \leq m \leq 5 \).

When \( n = 27 \), the elements \( Y_{i} \) (\( 1 \leq i \leq 10 \)) satisfy the same relations as those for type \( D_6 \), and hence relations (R27) hold.

Since there exists a sequence \( ((k, n, n'), \ldots, (k, n, n)) \) of \( B \) satisfying \( n_1 = 27, n_1' = m, n_j' = n_{j+1} \) (\( 1 \leq j < s - 1 \)) for any \( 1 \leq m \leq 26 \), it is sufficient to show (R\( n' \)) for \( (k, n', n) \in B \) assuming (R\( n \)). This is proved similarly to Proposition 2.3. Details are omitted.

By [4] and Proposition 3.3 we obtain the following:

**Theorem 3.4.** The formulas (Q7) give fundamental relations for the generator system \( \{Y_{i}\}_{i \in \Lambda} \) of the algebra \( A_{q} = U_{q}(\mathfrak{n}_{\gamma}) \).

We shall construct a quantum deformation of the lowest degree part \( J_{C_1}^0 \) of the defining ideal \( J_{C_1} \), and we shall give canonical generators of a quantum deformation of \( J_{C_1} \).

Set
\[
\psi_n = Y_{i_2} Y_{j_2} - q Y_{i_2} Y_{j_2} + q^2 Y_{i_2} Y_{j_2} - q^3 Y_{i_2} Y_{j_2} + q^4 Y_{i_2} Y_{j_2},
\]
for \( n \in \Lambda \), where \( B(n) = (i_2^n, i_2^n, i_2^n, i_2^n, i_2^n, j_2^n, j_2^n, j_2^n, j_2^n, j_2^n) \). Using the formulas (R\( n,1 \), (R\( n,2 \)), we can write
\[
\psi_n = Y_{i_2} Y_{j_2} - q Y_{i_2} Y_{j_2} + q^2 Y_{i_2} Y_{j_2} - q^3 Y_{i_2} Y_{j_2} + q^4 Y_{i_2} Y_{j_2}.
\]

Similarly to Lemma 2.5 and Proposition 2.6 we can show the following:

**Lemma 3.5.** We have
\[
\text{ad}(F_k) \psi_n = \begin{cases} 
\psi_{n'} & \text{if there exists } (k, n', n) \in B, \\
0 & \text{otherwise},
\end{cases}
\]
\[
\text{ad}(E_k) \psi_n = \begin{cases} 
\psi_{n'} & \text{if there exists } (k, n, n') \in B, \\
0 & \text{otherwise}.
\end{cases}
\]
for $k \in I$, and

$$\text{ad}(K_k)\psi_n = q^{-\langle \gamma_n, \alpha_k \rangle} \psi_n$$

for $k \in I_0$.

**Proposition 3.6.** $\sum_{n \in \Lambda} \mathbb{C}(q)\psi_n$ is an irreducible highest weight $U_q(sl_1)$-module with highest weight vector $\psi_27$.

By [4] and Proposition 3.6 we obtain the following:

**Theorem 3.7.** A quantum deformation of the defining ideal $J_{C_1}$ of the closure of the non-open orbit $C_1$ is given by the two-sided ideal of $A_q$ generated by $\{\psi_n \mid n \in \Lambda\}$.

Set

$$\varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n| - 1} Y_n \psi_n,$$

where $|\beta| = \sum_{i \in I_0} m_i$ ($\beta = \sum_{i \in I_0} m_i \alpha_i$).

**Proposition 3.8.** $\mathbb{C}(q)\varphi$ is a one-dimensional $U_q(sl_1)$-module.

Proof. By Proposition 3.3 we can check that the coefficient $a_{1,10,27}$ of $Y_1 Y_{10} Y_{27}$ in $\varphi = \sum_{i,j,k} a_{ijk} Y_i Y_j Y_k$ is $1 + q^8 + q^{16}$. Therefore we have $\varphi \neq 0$.

Let $(k, n, n') \in B$. Then we have $|\beta_n'| = |\beta_n| + 1$, $\text{ad}(F_k)Y_n = Y_{n'}$, $\text{ad}(F_k)Y_{n'} = 0$, $\text{ad}(F_k)\psi_{n'} = \psi_n$, $\text{ad}(F_k)\psi_n = 0$, $(\beta_{n'}, \alpha_k) = 1$. Hence $\text{ad}(F_k)(Y_n \psi_n - q Y_{n'} \psi_{n'}) = Y_{n'} \psi_n - q q^{-1} Y_n \psi_n = 0$. Therefore we have $\text{ad}(F_k)\varphi = 0$ for any $k \in I$, and similarly we have $\text{ad}(E_k)\varphi = 0$ for any $k \in I$. Since $\gamma_n + \beta_n = \delta$ for any $n \in \Lambda$, we have $\text{ad}(K_k)\varphi = q^{-(\delta, \alpha_k)}\varphi$ for any $k \in I_0$. In particular, we have $\text{ad}(K_k)\varphi = \varphi$ for any $k \in I$, and $\text{ad}(K_1)\varphi = q^{-2} \varphi$.

The element $\varphi$ is a quantum deformation of the irreducible relative invariant on the prehomogeneous vector space.

**Theorem 3.9.** A quantum deformation of the defining ideal $J_{C_2}$ of the closure of the non-open orbit $C_2$ is given by the two-sided ideal of $A_q$ generated by $\varphi$. 
References


Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima, 739-8526,
Japan
e-mail: morita@math.sci.hiroshima-u.ac.jp