ON CONGRUENCES BETWEEN THE COEFFICIENTS
OF TWO L-SERIES WHICH ARE RELATED
TO A HYPERELLIPTIC CURVE OVER Q

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1. Introduction

Let \( f(x) \) be a monic irreducible polynomial with rational integer coefficients and let \( p \) be a prime integer. Reducing the coefficients of \( f(x) \) modulo \( p \), we obtain the polynomial \( f_p(x) \) with coefficients in \( \mathbb{Z}/p\mathbb{Z} \). A rule of the factorization of \( f_p(x) \) over \( \mathbb{Z}/p\mathbb{Z} \) is called a reciprocity law for \( f(x) \) (cf. Wyman [11]). For example, when \( f(x) \) is of degree 2, a reciprocity law for \( f(x) \) is given by the Legendre symbol \( (D_f/p) \) for the discriminant \( D_f \) of \( f(x) \).

In the case that \( f(x) \) is of degree 3, the minimal splitting field \( K \) of \( f(x) \) over \( \mathbb{Q} \) is the Galois extension generated by the coordinates of the two-division points of the elliptic curve \( E : y^2 = f(x) \). A reciprocity law for \( f(x) \) is given by the Legendre symbol \( (D_f/p) \) and the coefficients of the L-series of \( E \) over \( \mathbb{Q} \), which is the Mellin transform of a modular form of weight two under the Taniyama-Shimura conjecture (the Wiles theorem). Furthermore, in the case that \( f(x) \) is of degree 3 and \( D_f < 0 \), the inverse Mellin transform of the Artin L-function \( L(\pi, K/\mathbb{Q}, s) \) attached to the two-dimensional irreducible representation \( \pi \) for the Galois group of \( K \) over \( \mathbb{Q} \), is a modular form of weight one, by the Weil-Langlands theorem. Thus the Fourier coefficients of the modular form of weight one also gives a reciprocity law for \( f(x) \).

In the latter case, we can associate two modular forms with \( E \) and the Galois extension generated by the coordinates of its two-division points. Koike [3] obtained congruences between the Fourier coefficients of two modular forms. His congruences describe the relation of the above two reciprocity laws. Naito [6] gave congruences between the coefficients of the L-series of \( E \) and those of an Artin L-series attached to the Galois extension generated by the coordinates of the three-division points of \( E \).

In this paper we consider congruences modulo 2 between the coefficients of the L-series of the Jacobian variety of a hyperelliptic curve \( y^2 = f(x) \) and those of an Artin L-series which is related to the Galois extension over \( \mathbb{Q} \), generated by the coordinates of the two-division points of the same Jacobian variety.

Let \( f(x) \) be a polynomial of degree \( n \) over \( \mathbb{Q} \) with no multiple roots. Let \( C \) be a hyperelliptic curve defined by \( y^2 = f(x) \). We denote by \( g \) the genus of \( C \). We see that
either \( n = 2g + 1 \) or \( n = 2g + 2 \) holds. We assume that \( g \geq 1 \) and \( C \) has at least one \( \mathbb{Q} \)-rational point. Then we can choose its Jacobian variety \((J, \varphi)\) defined over \( \mathbb{Q} \).

Let \( K \) be the Galois extension over \( \mathbb{Q} \), generated by the coordinates of the two-division points of the Jacobian variety \( J \) and let \( G \) be its Galois group. We assume that \( n \neq 1, 2, 4 \). Then we can identify \( G \) with a suitable subgroup of the permutation group \( S_n \) of \( n \) letters (See Proposition 2.2). Let \( \pi \) be the restriction of the standard representation of \( S_n \) to \( G \). Let \( \rho_2 \) be the 2-adic representation of the absolute Galois group of \( \mathbb{Q} \) with respect to the 2-adic Tate module of \( J \).

For each odd good prime \( p \) of \( J \) we put

\[
(P_p(u)) := \det(I_{n-1} - \pi(\sigma_p)u)
\]

and

\[
(Q_p(u)) := \det(I_{2g} - \rho_2(\sigma_p)u),
\]

where \( I_m \) is the unit matrix of size \( m \), \( \sigma_p \) is the Frobenius automorphism for a prime divisor \( \mathfrak{p} \) in \( \mathbb{Q} \), and \( \sigma_p \) is its restriction to \( K \). Then \( 1/P_p(p^{-s}) \) (resp. \( 1/Q_p(p^{-s}) \)) is the \( p \)-factor of Artin L-series \( L(\pi, K/\mathbb{Q}, s) \) attached to \( \pi \) (resp. the L-series \( L(J/\mathbb{Q}, s) \) of \( J \)).

**Theorem.**
(i) If \( n \) is odd and \( n \neq 1 \), the congruence \( P_p(u) \equiv Q_p(u) \mod 2 \) holds for any odd good prime \( p \) of \( J \).
(ii) If \( n \) is even and \( n \neq 2, 4 \), the congruence \( P_p(u) \equiv (1-u)Q_p(u) \mod 2 \) holds for any odd good prime \( p \) of \( J \).

In the case of \( n = 3 \), the theorem is that of Koike [3]. Thus our theorem is a generalization of Koike’s theorem.

The organization of this paper is as follows. In §2, we construct the reduction \( \rho_{2,1} \) of the 2-adic representation \( \rho_2 \) modulo 2 by matrices in \( \text{GL}(2g, \mathbb{Z}/2\mathbb{Z}) \). In §3, we construct the standard representation \( \pi^{st} \) of \( S_n \) by matrices in \( \text{GL}(n-1, \mathbb{Z}) \). By comparing two representations \( \rho_{2,1} \) and the restriction \( \pi \) of \( \pi^{st} \), we prove our theorem in §4. In §5, we give some examples of a reciprocity law for \( f(x) \) by using our theorem.

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2. The field of two-division points of the Jacobian variety of a hyperelliptic curve over \( \mathbb{Q} \)

Let \( f(x) \) be a polynomial over \( \mathbb{Q} \) of degree \( n \) with no multiple roots and let \( C \) be a hyperelliptic curve of genus \( g \) defined by \( y^2 = f(x) \). We see that either \( n = 2g + 1 \) or \( 2g + 2 \) holds. When \( n \) is even, the hyperelliptic curve \( C \) has two points \( P_\infty, P'_\infty \) at
infinity. When \( n \) is odd, the hyperelliptic curve \( C \) has one point \( P_\infty \) at infinity, which is ramified and \( \mathbb{Q} \)-rational. In the latter case we put \( P'_\infty = P_\infty \).

We assume that the hyperelliptic curve \( C \) has at least one \( \mathbb{Q} \)-rational point. Then we can assume that the Jacobian variety \( (J, \varphi) \) is defined over \( \mathbb{Q} \).

Let \( \text{Pic}^0(C) \) be the divisor class group of \( C \). The canonical mapping \( \varphi \) induces the isomorphism

\[
\varphi : \text{Pic}^0(C) \to J : \sum P \mapsto \sum \varphi(P).
\]

The point corresponding to a \( \mathbb{Q} \)-rational divisor class is \( \mathbb{Q} \)-rational.

Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be the roots of the equation \( f(x) = 0 \) and put \( P_i := (\alpha_i, 0) \in C \) for \( i = 1, 2, \ldots, n \). We see that

\[
div(x - \alpha_i) = 2P_i - P_\infty - P'_\infty \quad \text{for} \quad i = 1, 2, \ldots, n,
\]

and

\[
div(y) = \begin{cases} 
P_1 + \cdots + P_{2g+1} - (2g + 1)P_\infty & \text{if } n \text{ is odd,} \\
P_1 + \cdots + P_{2g+2} - (g + 1)(P_\infty + P'_\infty) & \text{if } n \text{ is even.}
\end{cases}
\]

Let \( J[2] \) be the group of two-division points of \( J \). By the equation (2.2) we have that

\[
(2.4) \quad \varphi(P_i - P_{2g+1}) \in J[2] \quad \text{for} \quad i = 1, 2, \ldots, 2g.
\]

**Proposition 2.1.** \( \{\varphi(P_i - P_{2g+1})\}_{i=1}^{2g} \) is a basis of \( J[2] \).

For a divisor \( D \) on \( C \), we define the set \( L(D) \) of rational functions on \( C \) over \( \overline{\mathbb{Q}} \) by

\[
(2.5) \quad L(D) := \{ h : \text{a rational function on } C | \text{div}(h) + D \text{ is effective.} \} \cup \{0\}.
\]

\( L(D) \) is a vector space over \( \overline{\mathbb{Q}} \).

Proof. Since \( J[2] \) is a \( \mathbb{Z}/2\mathbb{Z} \)-module of rank \( 2g \), it is enough to show that \( \varphi(P_i - P_{2g+1}) \) \( (i = 1, 2, \ldots, 2g) \) are linearly independent. Suppose

\[
(2.6) \quad \sum_{i=1}^{2g} a_i \varphi(P_i - P_{2g+1}) = 0 \quad \text{for} \quad a_1, \ldots, a_{2g} \in \{0, 1\}.
\]

Then there exists a rational function \( h \) on \( C \) such that

\[
(2.7) \quad \text{div}(h) = \sum_{i=1}^{2g} a_i(P_i - P_{2g+1}).
\]
We put \(a_{2g+1} := a_1 + \cdots + a_{2g}\). For the largest integer \(m\) less than or equal to \((a_{2g+1}+1)/2\), we put \(h_1 := (x - \alpha_{2g+1})^m h\). We have

\[
\text{div}(h_1) = \sum_{i=1}^{2g} a_i P_i + (2m - a_{2g+1})P_{2g+1} - m(P_\infty + P_\infty').
\]

Since \(a_{2g+1} = \sum_{i=1}^{2g+1} a_i \leq 2g, m \leq g\). Thus \(h_1\) is contained in \(L(g(P_\infty + P_\infty'))\). By the Riemann-Roch theorem, \(h_1\) is a linear combination of \(1, x, \ldots, x^g\). Together with the fact \(P_i\) is ramified for \(i = 1, \ldots, 2g+1\), the order of \(h_1\) at \(P_i\) is even for \(i = 1, \ldots, 2g+1\). Since \(a_1, \ldots, a_{2g} = 0, 1\), we have \(a_1, \ldots, a_{2g} = 0\). Thus \(a_{2g+1} = a_1 + \cdots + a_{2g} = 0\). This completes the proof.

Let \(K\) be the Galois extension over \(\mathbb{Q}\) generated by the coordinates of the points of \(J[2]\). Since \(\varphi\) is a rational function defined over \(\mathbb{Q}\), \(\varphi(P_i)\) is defined over \(\mathbb{Q}(\alpha_i)\) for each \(i\). We note that the addition on \(\mathcal{J}\) are also defined over \(\mathbb{Q}\). Thus the point \(\tilde{\varphi}(P_i - P_{2g+1}) = \varphi(P_i) - \varphi(P_{2g+1})\) is defined over \(\mathbb{Q}(\alpha_i, \alpha_{2g+1})\). Hence \(K\) is a subfield of the minimal splitting field \(\mathbb{Q}(\alpha_1, \ldots, \alpha_n)\) of \(f\) over \(\mathbb{Q}\).

**Proposition 2.2.**

(i) If \(n \neq 1, 2, 4\), then \(K = \mathbb{Q}(f)\).

(ii) If \(n = 4\), then \(K\) is the minimal splitting field of the decomposition cubic of \(f\) over \(\mathbb{Q}\).

For the proof of Proposition 2.2, we need the following two lemmas.

**Lemma 2.3.** Assume that \(n \neq 1, 2, 4\). If \(\tilde{\varphi}(P_i - P_j) = \tilde{\varphi}(P_k - P_l)\) for \(i \neq j\) and \(k \neq l\), then \(\{P_i, P_j\} = \{P_k, P_l\}\).

**Proof.** Assume that \(n = 3\). Then \(g = 1\). We have

\[
\tilde{\varphi}(P_1 - P_2) = \tilde{\varphi}(P_1 - P_3) + \tilde{\varphi}(P_2 - P_3).
\]

Since it follows from Proposition 2.1 that

\[
\tilde{\varphi}(P_1 - P_3), \tilde{\varphi}(P_2 - P_3), \tilde{\varphi}(P_1 - P_2)
\]

are distinct, our assertion follows in this case.

We assume that \(n \geq 5\). Then \(g \geq 2\). Suppose that \(\tilde{\varphi}(P_i - P_j) = \tilde{\varphi}(P_k - P_l)\). Then there exists a function \(h\) satisfying \(\text{div}(h) = P_i + P_j + P_k + P_l - 2(P_\infty + P_\infty')\). Thus \(h\) is contained in \(L(2(P_\infty + P_\infty'))\), which is spanned by \(1, x, x^2\) by the Riemann-Roch theorem, and \(h\) has zero at \(P_i\) and \(P_j\). Since \(i \neq j\), \(h\) is equal to \((x - \alpha_i)(x - \alpha_j)\) up to a constant, that is, \(\text{div}(h) = 2P_i + 2P_j - 2(P_\infty + P_\infty')\). Thus we have that \(\{P_i, P_j\} = \{P_k, P_l\}\).
Lemma 2.4. When \( n = 4 \),

\[
\phi(P_1 - P_3) = \phi(P_2 - P_4), \quad \phi(P_2 - P_3) = \phi(P_1 - P_4),
\]

and

\[
\phi(P_1 - P_3) + \phi(P_2 - P_3) = \phi(P_1 - P_2) = \phi(P_3 - P_4).
\]

Proof. These equations follow from (2.2) and (2.3).

Proof of Proposition 2.2. (i) Let \( \sigma \) be an element of the Galois group of \( \mathbb{Q}(f) \) over \( \mathbb{Q} \). Suppose that \( \sigma \) fixes all elements in \( K \). Then \( \sigma \phi(P_i - P_{2g+1}) = \phi(\sigma(P_i) - \sigma(P_{2g+1})) = \phi(P_i - P_{2g+1}) \) for \( i = 1, \ldots, 2g \). By Lemma 2.3, we have that \( \{\sigma(P_i), \sigma(P_{2g+1})\} = \{P_i, P_{2g+1}\} \) for \( i = 1, \ldots, 2g \). Thus we have \( \sigma(P_i) = P_i \), that is, \( \sigma(\alpha_i) = \alpha_i \) for \( i = 1, 2, \ldots, 2g+1 \). Hence \( \sigma \) is the identity element. Thus our assertion (i) follows.

(ii) Suppose that \( \sigma \phi(P_i - P_3) = \phi(P_i - P_3) \) for \( i = 1, 2 \). By Lemma 2.4 we have that \( \{\sigma(P_i), \sigma(P_3)\} = \{P_i, P_3\} \) or \( \{P_{2g-i}, P_4\} \) for \( i = 1, 2 \). Equivalently, \( \sigma \) fixes \( (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4), \ (\alpha_2 + \alpha_3)(\alpha_1 + \alpha_4), \) and \( (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \). Since these 3 elements are all roots of the decomposition cubic of \( f \) over \( \mathbb{Q} \), \( K \) is its minimal splitting field.

In the following we always assume \( n \neq 1, 2, 4 \). Let \( S_n \) be the permutation group of \( n \) letters \( \{1, 2, \ldots, n\} \). The group \( S_n \) acts on the set \( \{\alpha_i\}_{i=1}^{n} \) of the roots of \( f(x) = 0 \) by

\[
\sigma \alpha_i = \alpha_{\sigma(i)} \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

The group \( S_n \) acts on \( J[2] \) from the left hand side by

\[
\sigma \phi(P_i - P_{2g+1}) = \phi(P_{\sigma(i)} - P_{\sigma(2g+1)}) \quad \text{for} \quad i = 1, 2, \ldots, 2g.
\]

We take a basis \( \{w_i\}_{i=1}^{2g} \) as follows:

\[
w_i := \phi(P_i - P_{2g+1}) \quad (1 \leq i \leq 2g).
\]

For \( i = 1, 2, \ldots, n \), let \( \sigma_j := (j, 2g+1) \) be the transposition.

Proposition 2.5. (i) When \( n = 2g + 1 \) and \( n \neq 1 \),

\[
\sigma_j w_i = \begin{cases} 
  w_i & \text{if either } j = 2g + 1 \text{ or } i = j, \\
  w_i + w_j & \text{if } j \neq 2g + 1 \text{ and } i \neq j.
\end{cases}
\]
Let $G$ be the Galois group of $K$ over $\mathbb{Q}$. By Proposition 2.2, for any element $\sigma \in G$, there exists the unique element $\tau$ in $S_n$ such that

$$\sigma(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_{\tau(1)}, \alpha_{\tau(2)}, \ldots, \alpha_{\tau(n)}).$$

(2.18)

We can identify $G$ with a suitable subgroup of $S_n$ through the inclusion $G \rightarrow S_n : \sigma \mapsto \tau$.

We define the representation $\rho_{2,1} : G \rightarrow \text{GL}(2g, \mathbb{Z}/2\mathbb{Z})$ by

$$\rho_{2,1}(\sigma) = \left(\begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_{2g} \end{array}\right) \text{ for } \sigma \in G.$$

The representation $\rho_{2,1}$ is the restriction to $G$ of the representation of $S_n$ defined by (2.14).

Let $T_2(J)$ be the 2-adic Tate module of $J$. $T_2(J)$ is a free $\mathbb{Z}_2$-module of rank $2g$, where $\mathbb{Z}_2$ is the 2-adic integer ring. Taking a basis $T_2(J)$, we get a representation $\rho_2$ of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$ by matrices in $\text{GL}(2g, \mathbb{Z}_2)$. We can take a basis of $T_2(J)$ satisfying

$$\rho_2(\sigma) \equiv \rho_2(\sigma') \pmod{2} \text{ for all } \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

(2.20)

where $\sigma'$ is the restriction of $\sigma$ to $K$. We call the representation $\rho_2$ is the 2-adic representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with respect to $T_2(J)$ and we call the representation $\rho_{2,1}$ the reduction modulo 2 of $\rho_2$.

### 3. Standard representation of $S_n$

Let $S_n$ be the permutation group of $n$ letters $\{1, 2, \ldots, n\}$. Let $V^\text{pr}$ be an $n$-dimensional vector space over $\mathbb{Q}$ with basis $\{\varepsilon_i\}_{i=1}^n$. The group $S_n$ acts on the vector space $V^\text{pr}$ from the left hand side by

$$\sigma \varepsilon_i := \varepsilon_{\sigma(i)} \text{ for } i = 1, 2, \ldots, n, \text{ and } \sigma \in S_n.$$

(3.1)

The vector space $V^\text{pr}$ is called the permutation representation of $S_n$. The permutation representation $V^\text{pr}$ of $S_n$ is decomposed into the direct sum of two irreducible representations of $S_n$. Namely, the 1-dimensional subspace $V^\text{tr}$ spanned by $\varepsilon_1 + \cdots + \varepsilon_n$ and the $(n-1)$-dimensional subspace $V^\text{st}$ with basis $\{\varepsilon_i - \varepsilon_n\}_{i=1}^{n-1}$. The representations $V^\text{tr}$ and $V^\text{st}$ are irreducible.
and $V^{st}$ are called the *trivial representation* and the *standard representation*, respectively.

In this section, we investigate the standard representation $V^{st}$ of $S_n$. As a matter of convenience, we denote by $g$ the largest integer less than or equal to $(n - 1)/2$. Then either $n = 2g + 1$ or $n = 2g + 2$ holds.

We take a basis $\{v_i\}_{i=1}^{n-1}$ of $V^{st}$ as follows:

When $n = 2g + 1$,

$$v_i := \varepsilon_i - \varepsilon_{2g+1} \text{ if } 1 \leq i \leq 2g;$$  

When $n = 2g + 2$,

$$v_i := \begin{cases} 
\varepsilon_i - \varepsilon_{2g+1} & \text{if } 1 \leq i \leq 2g, \\
\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \cdots + \varepsilon_{2g+1} - \varepsilon_{2g+2} & \text{if } i = 2g + 1.
\end{cases}$$

We define the matrix representation $\pi^{st}$ of $S_n$ by

$$\sigma(v_1, v_2, \ldots, v_{n-1}) = (v_1, v_2, \ldots, v_{n-1})\pi^{st}(\sigma).$$

For $j = 1, 2, \ldots, n$, let $\sigma_j := (j, 2g + 1)$ be the transposition in $S_n$.

**Proposition 3.1.**

(i) When $n = 2g + 1$, we have

$$\sigma_j v_i = \begin{cases} 
v_i & \text{if } j = 2g + 1, \\
-v_i & \text{if } i = j \text{ and } j \neq 2g + 1, \\
v_i - v_j & \text{if } i \neq j \text{ and } j \neq 2g + 1.
\end{cases}$$

(ii) When $n = 2g + 2$, we have

$$\sigma_j v_i = \begin{cases} 
v_i & \text{if } j = 2g + 1, \\
-v_i & \text{if } i \neq 2g + 1, j \neq 2g + 1, 2g + 2 \text{ and } i = j, \\
v_i - v_j & \text{if } i \neq 2g + 1, j \neq 2g + 1, 2g + 2 \text{ and } i \neq j, \\
\sum_{m=1}^{2g} (-1)^m v_m + v_i + v_{2g+1} & \text{if } i \neq 2g + 1 \text{ and } j = 2g + 2, \\
v_{2g+1} & \text{if } i = 2g + 1 \text{ and } j \text{ is odd,} \\
v_{2g+1} + 2v_j & \text{if } i = 2g + 1 \text{ and } j \neq 2g + 2 \text{ is even,} \\
-v_{2g+1} + 2\sum_{m=1}^{2g} (-1)^{m-1} v_m & \text{if } i = 2g + 1 \text{ and } j = 2g + 2.
\end{cases}$$
Since $\sigma_j$'s generate $S_n$, it follows from Proposition 3.1 that $\pi^{st}(\sigma)$ is a matrix in $\text{GL}(n-1,\mathbb{Z})$. Thus we can consider the reduction of the representation $\pi^{st}$ modulo 2.

**Proposition 3.2.** (i) When $n = 2g + 1$, we have

\[
\sigma_j v_i \equiv \begin{cases} 
v_i & \text{mod } 2 \text{ if either } j = 2g + 1 \text{ or } i = j, \\
v_i + v_j & \text{mod } 2 \text{ if } i \neq j \text{ and } j \neq 2g + 1. \end{cases}
\]

(ii) When $n = 2g + 2$,

\[
\sigma_j v_i \equiv \begin{cases} 
v_i & \text{mod } 2 \text{ if } i \neq 2g + 1, j = 2g + 1, \\
v_i + v_j & \text{mod } 2 \text{ if } i \neq 2g + 1, j \neq 2g + 1, 2g + 2, \\
\sum_{m=1}^{2g} v_m + v_i + v_{2g+1} & \text{mod } 2 \text{ if } i \neq 2g + 1 \text{ and } j = 2g + 2, \\
v_{2g+1} & \text{mod } 2 \text{ if } i = 2g + 1. \end{cases}
\]

Proof. Proposition 3.2 follows from (3.5) and (3.6).

The conjugate classes of $S_n$ correspond to partitions of $n$ bijectively. We call an element $\sigma$ in $S_n$ of type $(n_1, n_2, \ldots, n_r)$ if $\sigma$ belongs to the conjugacy class corresponding to the partition $(n_1, n_2, \ldots, n_r)$. The following is well-known.

**Proposition 3.3.** Let $\sigma$ be an element in $S_n$ of type $(n_1, n_2, \ldots, n_r)$. Then the characteristic polynomial of $\sigma$ in $S_n$ for $\pi^{st}$ is given by

\[
\det(I_{n-1} - \pi^{st}(\sigma)u) = \frac{1}{1-u} \prod_{i=1}^{r} (1-u^{n_i}),
\]

where $I_{n-1}$ is the unit matrix of size $n-1$

Proof. We note that $V^{pr} = V^{st} \oplus V^{tr}$. Our assertion follows from direct computations.

4. **Proof of Theorem**

Let the notation be the same as in §1. We note that any odd good prime is unramified in $K$.

Let $\rho_{2,1} : G \to \text{GL}(2g, \mathbb{Z}/2\mathbb{Z})$ be the representation defined by (2.19). It follows
from (2.20) that

\[(4.1) \quad Q_p(u) = \det(I_{2g} - \rho_2(\sigma_p)u) \equiv \det(I_{2g} - \rho_{2,1}(\sigma_p)u) \mod 2.\]

We can take \(\pi^*: S_n \to \text{GL}(n-1, \mathbb{Z})\) defined by (3.4) in §4 as the standard representation of \(S_n\). Compared with (2.16), (2.17) and (3.7), (3.8), we have

\[(4.2) \quad \pi(\sigma) \equiv \rho_{2,1}(\sigma) \left( \begin{array}{cc} \rho_{2,1}(\sigma) & 0 \\ \ast & 1 \end{array} \right) \mod 2 \quad \text{for all } \sigma \in G,
\]

if \(n\) is odd and \(n \neq 1\) (resp. if \(n\) is even and \(n \neq 2, 4\)). Thus we have

\[(4.3) \quad P_p(u) \equiv Q_p(u) \ (\text{resp. } P_p(u) \equiv (1 - u)Q_p(u)) \mod 2.\]

5. Numerical examples

Let the notation be the same as in §1. We assume that \(f(x)\) is a monic polynomial with rational integer coefficients. We denote by \(f_p(x)\) the reduction of \(f(x)\) modulo \(p\). The type of the factorization of \(f_p(x)\) corresponds to that of the conjugate class of the Frobenius automorphism \(\sigma_p\). By Proposition 3.3 and by our Theorem, we have:

**Proposition 5.1.** (i) If \(f_p(x) = g_1(x)g_2(x) \cdots g_r(x)\) in \(\mathbb{Z}/p\mathbb{Z}[x]\) for some irreducible polynomials \(g_i(x)\) of degree \(n_i\), then

\[(5.1) \quad Q_p(u) \equiv \frac{1}{(1 - u)^\varepsilon} \prod_{i=1}^{r} (1 - u^{n_i}) \mod 2,
\]

where \(\varepsilon = 1\) (resp. \(\varepsilon = 2\)) if \(n\) is odd and \(n \neq 1\) (resp. if \(n\) is even and \(n \neq 2, 4\)).

(ii) The signature of \(\sigma_p\) in \(S_n\) is equal to the Legendre symbol \((D_f/p)\).

In the following we give three examples, which describe the law of decomposition of primes in terms of \(Q_p(u) \mod 2\) and \((D_f/p)\), in the case of \(g = 2\). We note that an odd prime integer \(q\) is a good prime of \(J\) if \(q\) is prime to the discriminant \(D_f\) of \(f(x)\).

**Example 1.** We put \(f(x) := x^5 - x - 1\). Then \(D_f = 2869 = 19 \cdot 151\) and \(G = S_5\) (cf. [4], p. 121). For any \(p \neq 2, 19, 151\) we have the following:
\[ \frac{2869}{p} \]

<table>
<thead>
<tr>
<th>Degrees of irreducible factors of ( f_p )</th>
<th>Example of ( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1, 1, 1, 1, 1 )</td>
<td>1973, 3769, 5101</td>
</tr>
<tr>
<td>( 1, 2, 2 )</td>
<td>67, 71</td>
</tr>
<tr>
<td>( 1, 1, 1, 2 )</td>
<td>163, 193, 227</td>
</tr>
<tr>
<td>( 1, 4 )</td>
<td>23, 29, 31, 61, 97</td>
</tr>
<tr>
<td>( 1, 3 )</td>
<td>7, 13, 37, 59, 73, 83</td>
</tr>
<tr>
<td>( 2, 3 )</td>
<td>3, 5, 11, 79, 89</td>
</tr>
</tbody>
</table>

**Example 2.** We put \( f(x) := x^6 - 4x^5 - 12x^4 + 2x^3 + 8x^2 + 8x - 7 \). Then \( D_f = 2^{12}29^5 \) and the hyperelliptic curve \( C \) is the modular curve \( X_0(29) \) (cf. [5]). We can check that the endomorphism algebra of \( J \) is \( \mathbb{Q}(\sqrt{2}) \). By choosing suitable indices of roots of \( f \), \( G = \langle (1, 2, 3)(4, 5, 6), (1, 2)(4, 5), (1, 4)(2, 5)(3, 6) \rangle \), which is isomorphic to the dihedral group of order 12 (cf. [8]). For any \( p \neq 2, 29 \) we have the following:

<table>
<thead>
<tr>
<th>Degrees of irreducible factors of ( f_p )</th>
<th>Example of ( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1, 1, 1, 1, 1 )</td>
<td>173, 197, 277</td>
</tr>
<tr>
<td>( 1, 1, 2, 2 )</td>
<td>7, 23, 59, 67, 71, 83</td>
</tr>
<tr>
<td>( 2, 2, 2 )</td>
<td>17, 19, 37, 41, 61, 73, 89, 97</td>
</tr>
<tr>
<td>( 3, 3 )</td>
<td>5, 13, 53</td>
</tr>
<tr>
<td>( 6 )</td>
<td>3, 11, 31, 43, 47, 79</td>
</tr>
</tbody>
</table>

In this example, by using the fact that \( K \) contains \( \mathbb{Q}(\sqrt{-1}) \), we can distinguish the first row and the second row by the Legendre symbol \((-1/p)\). And also the fourth row and the fifth row.

**Example 3.** We put \( f(x) := x^6 - 4x^5 + 6x^4 - 6x^3 + 9x^2 - 14x + 9 \). Then \( D_f = 2^{12}67^2 \) and the hyperelliptic curve \( C \) is the modular curve \( X_1^2(67) \) (cf. [5]). Then we can checked that the endomorphism algebra of \( J \) is \( \mathbb{Q}(\sqrt{5}) \). By choosing suitable indices of roots of \( f \), \( G = \langle (1, 2, 6)(3, 5, 4), (1, 2, 3, 4, 5), (2, 5)(3, 4) \rangle \), which is isomorphic to the alternative group of degree 5 (cf. [8]). For any \( p \neq 2, 67 \) we have the following:

<table>
<thead>
<tr>
<th>Degrees of irreducible factors of ( f_p )</th>
<th>Example of ( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1, 1, 1, 1, 1 )</td>
<td>311, 1163, 1453</td>
</tr>
<tr>
<td>( 1, 1, 2, 2 )</td>
<td>17, 59, 73</td>
</tr>
<tr>
<td>( 3, 3 )</td>
<td>5, 11, 23</td>
</tr>
<tr>
<td>( 1, 5 )</td>
<td>3, 7, 13</td>
</tr>
</tbody>
</table>

In Example 2 and in Example 3, there exist modular forms \( h_1, h_2 \) of weight two with respect to a congruence subgroup such that \( L(J/\mathbb{Q}, s) \) and the product \( L(h_1, s)L(h_2, s) \) of their Mellin transforms are essentially same as in Shimura's sense (cf. [7]). Thus by our theorem, we can consider congruences between the coefficients.
of the Artin L-series $L(\tau, K/Q, s)$ and the Fourier coefficients of the modular forms $h_1, h_2$ of weight two in those examples.

References


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