ON EMBEDDABLE 1-CONVEX SPACES

VIOREL VĂJĂITU

(Received May 31, 1999)

1. Introduction

Throughout this paper all complex spaces are assumed to be reduced and with countable topology.

Let $X$ be a complex space. $X$ is said to be embeddable if it can be realized as a complex analytic subset of $\mathbb{C}^m \times \mathbb{P}^n$ for some positive integers $m$ and $n$. For instance, one checks that a complex curve of bounded Zariski dimension is embeddable.

We say that $X$ is 1-convex if $X$ is a modification at finitely many points of a Stein space $Y$, i.e., there exist a compact analytic set $S \subset X$ without isolated points and a proper holomorphic map $\pi : X \rightarrow Y$ such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ and $\pi$ induces an isomorphism between $X \setminus S$ and $Y \setminus \pi(S)$. $S$ is called the exceptional set of $X$ and $Y$ the Remmert’s reduction of $X$. See [16] for further properties of 1-convex spaces.

A criterion of Schneider [18] says that a 1-convex space $X$ of bounded Zariski dimension is embeddable if, and only if, there is a holomorphic line bundle $L$ over $X$ such that $L|_S$ is ample.

Using this, Bâncă [3] proved that a 1-convex complex surface $X$ of bounded Zariski dimension is embeddable provided that $X$ does not admit compact two dimensional irreducible components. By extending this Coltouï ([4], [5]) showed that every connected 1-convex manifold $X$ with 1-dimensional exceptional set is embeddable if $\dim(X) > 3$. This is true also for threefolds $X$ with some exceptions when the exceptional set contains a $\mathbb{P}^1$ ([5]).

In this short note we reconsider Coltouï’s example from another point of view. This is based on the following proposition which may be of independent interest.

**Proposition 1.** Let $Y \subset \mathbb{P}^n$ be a hypersurface of degree $d$ with isolated singularities, $\pi : M \rightarrow Y$ a resolution of singularities, and $H \subset \mathbb{P}^n$ a hyperplane which avoids the singular locus of $Y$ and such that $\Gamma := H \cap Y$ is smooth. Set $X := M \setminus \pi^{-1}(\Gamma)$. Then for $n \geq 4$ the following statements are equivalent:

(a) $X$ is embeddable.

(b) $X$ is Kähler.

(c) $M$ is projective.

By this and an example due to Moishezon [12] (see also [6]) we obtain:
Theorem 1. There exists a 1-convex threefold $X$ with exceptional set $\mathbb{P}^1$ such that $X$ is not Kähler; a fortiori $X$ is not embeddable.

For the proof of Proposition 1 we use several short exact sequences, Bott’s formula, Thom’s isomorphism, and some facts on pluriharmonic functions.

Also employing recent results due to Fujiki [9] we prove (see the next section for definitions):

Theorem 2. Let $\pi : X \longrightarrow Y$ be a finite holomorphic map of complex spaces with $X$ of bounded Zariski dimension. If $X$ is maximal and $Y$ is Hodge, then it holds:

(a) $Y$ compact implies $X$ projective.
(b) $Y$ is 1-convex implies $X$ is 1-convex and embeddable.

Remark 1. Note that by [23], 1-convexity is invariant under finite holomorphic surjections. However, this does not hold for embeddability.

As a consequence of Theorem 2 we improve a well-known projectivity criterion due to Grauert [10] to:

Proposition 2. Let $X$ be a compact complex space. If $X$ is Hodge and maximal, then $X$ is projective.

and the embeddability result due to Th. Peternell ([17], Theorem 2.6) to:

Proposition 3. Let $X$ be a 1-convex space of bounded Zariski dimension such that $X$ is Hodge and maximal. Then $X$ is embeddable.

2. Continuous weakly pluriharmonic functions

Let $X$ be a complex space. As usual, $\mathcal{P}_X$ denotes the sheaf of germs of pluriharmonic functions on $X$. Then the canonical map $\mathcal{O}_X \longrightarrow \mathcal{P}_X$ given by $f \mapsto \text{Re } f$ induces a short exact sequence

\begin{equation}
0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{P}_X \longrightarrow 0.
\end{equation}

Consider $\hat{\mathcal{P}}_X :=$ the sheaf of continuous weakly pluriharmonic functions, i.e., for every open subset $U$ of $X$, $\hat{\mathcal{P}}_X(U)$ consists of those $h \in C^0(U, \mathbb{R})$ which are pluriharmonic on $\text{Reg}(U)$.

Clearly $\mathcal{P}_X \subseteq \hat{\mathcal{P}}_X$, and if $\hat{\mathcal{O}}_X$ denotes the sheaf of continuous weakly holomorphic functions, we have a natural map $\hat{\mathcal{O}}_X \longrightarrow \hat{\mathcal{P}}_X$ given by $f \mapsto \text{Re } f$.

Here we prove:
Proposition 4. The canonical short sequence

\[ 0 \longrightarrow \mathbb{R} \longrightarrow \hat{O}_X \longrightarrow \hat{P}_X \longrightarrow 0, \]

is exact.

Proof. We check only the surjectivity of \( \hat{O}_X \longrightarrow \hat{P}_X \). We do this in two steps.

1. Suppose \( X \) is normal. Let \( \pi : M \longrightarrow X \) be a resolution of singularities. Then \( \pi_* \mathcal{P}_M = \mathcal{P}_X \) by Proposition 2.1 in [9]. Now, since on a complex manifold a continuous real-valued function \( \varphi \) is pluriharmonic if and only if \( \varphi \) and \( -\varphi \) are plurisubharmonic we obtain that \( \hat{P}_X = \mathcal{P}_X \), whence the desired surjectivity in view of (\(*\)).

2. The general case. Let \( \nu : Y \longrightarrow X \) be the normalization of \( X \). Let \( x_0 \in X \), \( U \) an open neighborhood of \( x_0 \), and \( h \in \hat{P}_X(U) \). Then, by Step 1., \( h \circ \nu \in \mathcal{P}_Y(\nu^{-1}(U)) \). By Proposition 2.3 in [9] after shrinking \( U \supseteq x_0 \), there is \( f \in \hat{O}_X(U) \) such that \( \text{Re } f = h \). Note that in loc. cit. this is done under the additional hypothesis \( h \in C^\infty(U, \mathbb{R}) \). But our case follows mutatis mutandis, whence the proposition. \( \square \)

Recall ([7], pp. 122–126) that a complex space \( Z \) is said to be maximal if \( \mathcal{O}_Z = \hat{O}_Z \) and that every complex space \( X \) admits a maximalization \( \hat{X} \), i.e., \( \hat{X} \) is maximal and there is a holomorphic homeomorphism \( \pi : \hat{X} \longrightarrow X \) which induces a biholomorphic map between \( \hat{X} \setminus \pi^{-1}(M(X)) \) and \( X \setminus M(X) \), where \( M(X) \) is the non-maximal locus of \( X \), i.e., \( M(X) = \{ x \in X : \mathcal{O}_{X,x} \neq \hat{O}_{X,x} \} \). Clearly every normal complex space is maximal. For this reason, maximal complex spaces are also called “weakly normal”.

Corollary 1. If \( X \) is maximal, then \( \mathcal{P}_X = \hat{P}_X \).

Corollary 2. If \( X \) is normal, then every pluriharmonic function \( h \) on \( \text{Reg}(X) \) extends uniquely to a pluriharmonic function on \( X \).

Proof. Since \( h \) and \( -h \) extend uniquely to plurisubharmonic functions \( \varphi \) and \( \psi \) on \( X \), we get \( \varphi = -\psi \). Hence \( \varphi \) is continuous, whence \( \varphi \) is pluriharmonic by Corollary 1. \( \square \)

By a d-closed, real (1,1)-form (in the sense of Grauert [10]) on a complex space \( X \) we mean, a d-closed, real (1,1)-form \( \omega \) on \( \text{Reg}(X) \) such that every point \( x \in X \) admits an open neighborhood \( U \) on which there is \( \varphi \in C^2(U, \mathbb{R}) \) with \( \omega = i\partial \bar{\partial} \varphi \) on \( \text{Reg}(U) \). This \( \varphi \) is called a local potential function for \( \omega \). We say that \( \omega \) is Kähler if the local potentials may be chosen strongly pluriharmonic.

Alternatively, by Moishezon [14] we define a d-closed, real (1,1)-form on \( X \) as a collection \( \{(U_j, \varphi_j)\}_{j \in J} \) where \( \{U_j\}_{j} \) is an open covering of \( X \) and \( \varphi_j \in C^2(U_j, \mathbb{R}) \) are such that \( \varphi_j - \varphi_k \) is pluriharmonic. Two such collections \( \{(U_j, \varphi_j)\}_{j \in J} \) and \( \{(V_k, \psi_k)\}_{k \in K} \) define the same form if \( \varphi_j - \psi_k \) is pluriharmonic on \( U_j \cap V_k \) for all
indices $j$ and $k$.

**Corollary 3.** For a maximal complex space $X$ the above two notions of $d$-closed, real $(1, 1)$-forms coincide in an obvious sense.

Proof. This is immediate by Corollary 1. 

To every $d$-closed, real $(1, 1)$-form $\omega$ on $X$ we associate canonically an element of $H^1(X, \widehat{\mathcal{P}}_X)$, which in turn goes into its de Rham class $[\omega] \in H^2(X, \mathbb{R})$ via the cohomology sequence from Proposition 4.

We say that $\omega$ is integral if its de Rham class belongs to $\text{Im}(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{R}))$.

One has the following (see [10], proof of Satz 3)

**Lemma 1.** If $\omega$ is an integral form on a maximal space $X$, then there is a holomorphic line bundle $L \rightarrow X$ together with a class $C^2$-hermitean metric on $L$ whose Chern form is $\omega$. In particular, if $\omega$ is Kähler, then $L$ is positive.

Let $X$ be a complex space. $X$ is said to be Kähler if $X$ has a Kähler form (in the sense of Grauert). We say that $X$ is Hodge if it admits a Kähler form which is integral.

**Proposition 5.** Let $\pi : Y \rightarrow X$ be a finite holomorphic map of complex spaces such that $X$ is Hodge. Then $Y$ is Hodge. In particular, the maximalization $\widehat{X}$ and the normalization $X^*$ of $X$ are Hodge, too.

Proof. Let $\{(U_j, \psi_j)\}_j$, $U_j \subseteq X$, defines a Kähler form $\omega$ on $X$. Let $V_j \subseteq U_j$ such that $\{V_j\}_j$ is also a covering of $X$. Then by [22] for every $\delta \in C^0(X, \mathbb{R})$, $\delta > 0$, there exists $\psi \in C^\infty(Y, \mathbb{R})$, $0 < \psi < \delta$, such that $\sigma_j := \psi_j \circ \pi + \psi$ are strongly plurisubharmonic on $W_j := \pi^{-1}(V_j)$ for all $j$; hence $\{(W_j, \sigma_j)\}_j$ defines a Kähler form $\pi^*\omega$ on $Y$. Of course $\pi^*\omega$ depends on $\delta$ and $\psi$, but this is irrelevant for our discussion. Moreover, in view of a canonical commutative diagram and Proposition 4, if $\omega$ is integral, then $\pi^*\omega$ is integral too. 

Now Lemma 1 and the criteria of Grauert [10] and Schneider [18] give Theorem 2.

**Remark 2.** There is a compact, normal, two dimensional complex space $X$ with only one singularity such that $\text{Reg}(X)$ is Kähler, and $X$ is not Kähler. (This follows from [14] and [10].)
3. Proof of proposition 1

The only nontrivial implication is (b) ⇒ (c) which we now consider. First we state:

CLAIM. The restriction map $H^1(M, \mathcal{P}_M) \to H^1(X, \mathcal{P}_M)$ is surjective.

The proof of this will be done in several steps.

STEP 1. For every abelian group $G$ we have $H^1(\Gamma, G) = 0$.

Indeed, by a theorem of Siu [19], as $Y \setminus \Gamma$ is a Stein subspace of $\mathbb{P}^n \setminus \Gamma$, it admits a Stein open neighborhood $D$; thus $\mathbb{P}^n \setminus \Gamma = D \cup (\mathbb{P}^n \setminus Y)$ is a union of two Stein open subsets. On the other hand, if an $n$-dimensional complex manifold $\Omega$ is a union of $q$ Stein open subsets, then $H^i_c(\Omega, G) = 0$ for $i \leq n - q$. The assertion follows easily.

STEP 2. $H^2(Y, \mathcal{O}_Y) = 0$.

For this, we let $\mathcal{I}_Y$ be the coherent ideal sheaf of $Y$ in $\mathbb{P}^n$. Then $\mathcal{I}_Y \simeq \mathcal{O}(\mathcal{Y})$, where $\mathcal{Y}$ denotes the canonical line bundle associated to the divisor defined by $Y$.

Now Bott’s formula gives the vanishing of $H^i(\mathbb{P}^n, \mathcal{O}(k))$ for integers $i, k$ with $1 \leq i < n$, and by the long exact cohomology sequence associated to the short exact sequence $0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0$, the assertion of Step 2 results immediately.

STEP 3. The maps $H^1(M, \mathcal{O}) \to H^1(X, \mathcal{O})$ and $H^2(M, \mathcal{O}) \to H^2(X, \mathcal{O})$ are surjective and injective respectively.

Let $V$ be an arbitrary open neighborhood of $\Gamma$ in $Y$. Since $Y \setminus \Gamma$ is Stein, the Mayer-Vietoris sequence for $Y = (Y \setminus \Gamma) \cup V$ and Step 2 give that the maps $H^1(V, \mathcal{O}) \to H^1(V \setminus \Gamma, \mathcal{O})$ and $H^2(V, \mathcal{O}) \to H^2(V \setminus \Gamma, \mathcal{O})$ are surjective and injective respectively.

Assume now $V \subset \text{Reg}(Y)$; hence $\pi^{-1}(V)$ is biholomorphic to $V$ via $\pi$. This and the above discussion plus the Mayer-Vietoris sequence for $M = X \cup \pi^{-1}(V)$ completes the proof of Step 3.

STEP 4. $H^2(M, G) \to H^2(X, G)$ is surjective for every abelian group $G$.

We view $\Gamma$ as a smooth complex hypersurface in $M$. The inclusion $X \subset M$ gives rise to an exact cohomology sequence (coefficients in any abelian group $G$)

$$\cdots \to H^i(M, X : G) \to H^i(M ; G) \to H^i(X ; G) \to H^{i+1}(M, X : G) \to \cdots$$

On the other hand since $\Gamma$ is a non-singular complex hypersurface, a tubular neighborhood of $\Gamma$ is diffeomorphic to a neighborhood of the 0-section of the normal bundle of $\Gamma$ in $M$. This bundle being holomorphic is naturally oriented. We thus have, see [2], a Thom isomorphism:

$$H^i(M, X ; G) \cong H^{i-2}(\Gamma ; G),$$

whence the assertion of Step 4 using Step 1.

(●) The proof of the claim follows by diagram chasing using Steps 3 and 4 and
the next commutative diagram with exact rows:

\[
\begin{array}{cccccc}
H^1(M, \mathcal{O}) & \longrightarrow & H^1(M, \mathcal{P}) & \longrightarrow & H^2(M, \mathbb{R}) & \longrightarrow & H^2(M, \mathcal{O}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(X, \mathcal{O}) & \longrightarrow & H^1(X, \mathcal{P}) & \longrightarrow & H^2(X, \mathbb{R}) & \longrightarrow & H^2(X, \mathcal{O}),
\end{array}
\]

(•) For the proof of the proposition we let \( K^{1,1}_M \) be the sheaf of germs of real smooth \((1,1)\)-forms on \( M \) which are \( d \)-closed. As usual, \( \mathcal{E}_M \) represents the sheaf of germs of smooth real functions on \( M \). The short exact sequence on \( M \),

\[
0 \longrightarrow \mathcal{P}_M \longrightarrow \mathcal{E}_M \longrightarrow K^{1,1}_M \longrightarrow 0,
\]

where the last non trivial map is given by \( \varphi \mapsto \sqrt{-1} \partial \bar{\partial} \varphi \), induces a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
H^0(M, \mathcal{E}_M) & \longrightarrow & H^0(M, K^{1,1}_M) & \longrightarrow & H^1(M, \mathcal{P}_M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(X, \mathcal{E}_M) & \longrightarrow & H^0(X, K^{1,1}_M) & \longrightarrow & H^1(X, \mathcal{P}_M) & \longrightarrow & 0,
\end{array}
\]

By diagram chasing and the above claim if \( \omega \) is the Kähler form of \( X \), then there are: a smooth, \( d \)-closed, real \((1,1)\)-form \( \alpha \) on \( M \) and a smooth real-valued function \( \varphi \) on \( X \) such that

\[
\alpha|_X - \omega = \sqrt{-1} \partial \bar{\partial} \varphi.
\]

Now, select \( \chi \in C^\infty(X, \mathbb{R}) \) which vanishes on a neighborhood \( \Omega \) of \( \pi^{-1}(\text{Sing}(Y)) \) and equals 1 outside a compact subset of \( X \). By (•), the smooth \((1,1)\)-form \( \omega + \sqrt{-1} \partial \bar{\partial} \chi \varphi \) on \( X \) extends trivially to a smooth, real, and \( d \)-closed \((1,1)\)-form \( \tilde{\omega} \) on \( M \).

Let \( \beta \) be the canonical Kähler form on \( \mathbb{P}^n \). For every \( c > 0 \) define a \( d \)-closed \((1,1)\)-form \( \tilde{\omega}_c \) on \( M \) by setting:

\[
\tilde{\omega}_c := \tilde{\omega} + c \pi^*(\beta).
\]

Clearly \( \tilde{\omega}_c \) restricted to \( \Omega \) is positive definite for every \( c > 0 \). On the other hand, there is \( c > 0 \) sufficiently large such that \( \tilde{\omega}_c \) is positive definite near the compact set \( M \setminus \Omega \). Thus \( M \) is Kähler. Since \( M \) is Moishezon, by [13] \( M \) is projective.

**Remark 3.** In [20] a similar version to our Proposition 1, without any smoothness assumption on \( H \cap Y \) and with the additional assumption that \( H^2(X, \mathcal{O}_X) = 0 \), is stated.
Unfortunately, the “given proof” is wrong. See Colțoiu’s pertinent comments [5] for this and many, many other fatal errors, which, to our unpleasant surprise, are used again in [21].

4. Proof of theorem 1

Let \( Y \subset \mathbb{P}^4 \) be a hypersurface of degree \( d > 2 \) having a nondegenerate quadratic point \( y_o \) as its only singularity [12]. Let \( \sigma : V \rightarrow \mathbb{P}^4 \) be the quadratic transform with center \( y_o \). Set \( \Sigma := \sigma^{-1}(y_o) \), \( W := \) the proper transform of \( Y \) (\( W \) is a nonsingular hypersurface in \( V \)), and \( T := \Sigma \cap W \cong \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( S \) be one of the two factors and \( \rho : T \rightarrow S \) the corresponding projection.

If \( N \) denotes the normal bundle of \( T \) in \( W \), the restriction of \( N \) to each of the fibres of \( \rho \) is the negative of the hyperplane bundle, so the criterion of Nakano and Fujiki applies ([8], [15]).

In other words \( W \) is obtained by blowing-up a non singular \( M \) along a rational non singular curve \( S \). One obtains easily a holomorphic map \( \pi : M \rightarrow Y \) which resolve the singularity \( y_o \) of \( Y \) and \( S = \pi^{-1}(y_o) \cong \mathbb{P}^1 \).

On the other hand, by [6], \( M \) is not Kähler if \( d > 2 \). Therefore, if we choose a linear hyperplane \( H \) in \( \mathbb{P}^1 \), \( H \not\subset y_o \), such that \( H \cap Y \) is smooth, then by Proposition 1, \( X := M \setminus \pi^{-1}(Y \cap H) \) is the desired example.

\( \square \)

Remark 4. As a counterexample for embeddability this example is due to Colțoiu [5] where by a different method he obtained that \( H^1(X, \mathcal{O}_X) = 0 \) under the additional hypothesis that \( H \) intersects \( Y \) transversally.

Here we emphasize the non-Kähler property of the example.

References