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SIMPLIFIED PROBABILISTIC APPROACH TO THE HÖRMANDER THEOREM

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1. Introduction

In this paper, we shall discuss the problem to find sufficient conditions under which the probability law of the solution to the stochastic differential equation has a smooth transition density. There are many approaches to this problem in the theory of partial differential equations. It is well known that Hörmander ([2]) showed the relation between the hypoellipticity of second order elliptic differential operators and the dimension of the Lie algebra generated by vector fields associated with coefficients of the differential operator. Malliavin ([6], [7]) introduced the new differential calculus on the Wiener space, and applied its calculus to the probabilistic proof of the Hörmander theorem. He introduced the Ornstein-Uhlenbeck operator which is an unbounded self-adjoint non-negative operator on the L^2-space over the Wiener space, and obtained the integration by parts formula on this space ([3], [9]). On the other hand, Bismut ([1]) gave the different approach from Malliavin's work. He showed the integration by parts formula by using the Girsanov transformation. For the integration by parts formula, the integrability of the inverse of so called Malliavin covariance matrix is essential. Kusuoka and Stroock ([5]) presented a key lemma for the proof of the integrability. Norris ([8]) gave a simplified proof of the key lemma. His proof of it is still considerably long and complicated.

Instead of the Kusuoka-Stroock-Norris key lemma, we shall present a new lemma that plays an important role in the Malliavin calculus for SDE's. This can be proved easily and directly only by using simple stochastic calculations, that is, the Ito formula and the Fubini type theorem for stochastic integrals. In order to show the integrability of the inverse of the Malliavin covariance matrix, it suffices to prove the exponential decay of the Laplace transform of the quadratic form of the covariance matrix. It is possible to prove the exponential decay by an iterative application of the new lemma. Therefore, by using the new key lemma, we can easily show the Hörmander theorem.

The organization of this paper is as follows: in Section 2, we give some preliminaries that need in our argument, and introduce well-known results on the integrability of the inverse of the Malliavin covariance matrix. In Section 3, our main results are stated. In the final section, the Hörmander theorem is proved.

2. Preliminaries

Let $(W_0^m, \mathcal{F}, P; \{\mathcal{F}_t\})$ be the m-dimensional Wiener space, that is, $W_0^m = C(\boldsymbol{R}_+; \boldsymbol{R}^m | w_0 = 0)$, $\mathcal{F}_t = \sigma[w_s \mid s \leq t]$, $\mathcal{F} = \bigvee_t \mathcal{F}_t$ and P is the Wiener measure on $(W_0^m, \mathcal{F})$. Let $A_j = a_j(x) \cdot \partial_x$ $(j = 0, 1, \ldots, m)$ be smooth vector fields on $\boldsymbol{R}^d$ such that derivatives of all orders of $\partial_x a_j$ are bounded.

We consider the d-dimensional SDE

$$x_t = x_0 + \int_0^t a_0(x_s)ds + \int_0^t \sum_{i=1}^m a_i(x_s) \circ dw_s^i \tag{1}$$

for $x_0 \in \boldsymbol{R}^d$. This equation is equivalent to

$$\phi(x_t) = \phi(x_0) + \int_0^t A_0\phi(x_s)ds + \int_0^t \sum_i A_i\phi(x_s) \circ dw_s^i.$$

for $\phi \in C^\infty(\boldsymbol{R}^d; \boldsymbol{R})$. From the assumption for coefficients of (1), there exists the unique solution $x_t = \Psi_t(x_0, w)$ in the pathwise sense. Moreover the mapping Ψ_t defines a stochastic flow of diffeomorphisms on $\boldsymbol{R}^d$. The Jacobi matrix $Z_t = ((\partial/\partial x_0^j)\Psi_t^i(x_0, w))_{1\leq i,j\leq d}$ of the diffeomorphism satisfies the linear SDE

$$Z_t = I + \int_0^t a_0'(x_s)Z_s ds + \int_0^t \sum_i a_i'(x_s)Z_s \circ dw_s^i, \tag{2}$$

where $I = (\delta_j^i)_{1\leq i,j\leq d}$. The symbol φ' denotes the matrix $((\partial/\partial x^j)\varphi^i(x))_{1\leq i,j\leq d}$ for any $\varphi \in C^1(\boldsymbol{R}^d; \boldsymbol{R}^d)$. Let U_t be the solution to the linear SDE

$$U_t = I - \int_0^t U_s a_0'(x_s)ds - \int_0^t \sum_i U_s a_i'(x_s) \circ dw_s^i. \tag{3}$$

It is easily checked that $Z_t U_t = U_t Z_t = I$. Since coefficients of (1), (2), and (3) satisfy the linear growth condition, we have

$$E\Big[\sup_{s\leq t} (|x_s|^p + \|Z_s\|^p + \|U_s\|^p) \Big] < \infty \tag{4}$$

for all $p > 1$. For a matrix A, its transposed matrix is denoted by A^*. Define

$$Q_t = \int_0^t \sum_i U_s a_i(x_s)a_i(x_s)^* U_s^* ds, \qquad V_t = Z_t Q_t Z_t^*.$$

The matrix V_t is called the Malliavin covariance matrix. The following result is well known in the Malliavin calculus (cf. [3], [9]).

Proposition 1. *For* $T > 0$, *if* $(\det V_T)^{-1} \in \bigcap_{p>1} L^p(W_0^m, P)$, *then the probability law of* x_T *has a smooth density.*

Throughout this paper, $c.$'s denote certain positive absolute constants, which may change every lines. It is sufficient for the assumption of the above proposition that $E[\{\inf_{|z|=1}(z \cdot Q_T z)\}^{-p}]$ is bounded for all $p > 1$, because

$$\begin{aligned} E[(\det V_T)^{-p}] &= E[(\det U_T)^p \ (\det Q_T)^{-p}] \\ &\leq c.\ E\Big[\{\inf_{|z|=1}(z \cdot Q_T z)\}^{-2pd}\Big]^{1/2}. \end{aligned}$$

Lemma 1. *If the condition*

$$\sup_{|z|=1} E[(z \cdot Q_T z)^{-p}] < \infty$$

is satisfied for all $p > 1$, *then the following one is also satisfied for all* $p > 1$.

$$E\Big[\{\inf_{|z|=1}(z \cdot Q_T z)\}^{-p}\Big] < \infty.$$

Proof. For any $z, z' \in S^{d-1}$, we have $|z \cdot Q_T z - z' \cdot Q_T z'| \leq 2\ \|Q_T\|\ |z - z'|$. Hence it is possible to choose a sequence $\{z_n\}_n$ of points on S^{d-1} such that, for any $r > 0$, if $z_n \cdot Q_T z_n \geq 2r$ $(1 \leq n \leq c.(\|Q_T\|/r)^{d-1})$, then $\inf_{|z|=1}(z \cdot Q_T z) \geq r$. Therefore we have

$$\begin{aligned} E\Big[\{\inf_{|z|=1}(z \cdot Q_T z)\}^{-p}\Big] &= \Gamma(p)^{-1} \int_0^\infty \lambda^{p-1} E\Big[\exp\{-\lambda \inf_{|z|=1}(z \cdot Q_T z)\}\Big] d\lambda \\ &\leq c. \int_0^\infty \lambda^{p-1} \Big\{ E\Big[\exp\{-\lambda \inf_{|z|=1}(z \cdot Q_T z)\}\ ;\ \|Q_T\| \leq \lambda\Big] + P[\|Q_T\| > \lambda] \Big\} d\lambda \\ &= c. \int_0^\infty \lambda^{p-1} \Big\{ \int_0^\infty \lambda e^{-\lambda r} P\Big[\inf_{|z|=1}(z \cdot Q_T z) \leq r,\ \ \|Q_T\| \leq \lambda\Big]\ dr \\ &\qquad + P[\|Q_T\| > \lambda] \Big\} d\lambda \\ &\leq c. \int_0^\infty \lambda^{p-1} \Big\{ \int_0^\infty \lambda e^{-\lambda r} \sum_{n=1}^{c.(\lambda/r)^{d-1}} P[z_n \cdot Q_T z_n \leq 2r]\ dr \\ &\qquad + \Big(\frac{E[\|Q_T\|^{p+1}]}{\lambda^{p+1}}\Big) \wedge 1 \Big\} d\lambda \\ &\leq c. \int_0^\infty \lambda^{2d+p-2} \Big\{ \int_0^\infty e^{-\lambda r} \sum_{n \geq 1} \frac{r^{2-2d}}{n^2} P[z_n \cdot Q_T z_n \leq 2r]\ dr \Big\} d\lambda + c. \\ &= c. \sum_{n \geq 1} \frac{1}{n^2} \int_0^\infty r^{3-4d-p}\ \ P[z_n \cdot Q_T z_n \leq r] dr + c. \end{aligned}$$

$$
\begin{aligned}
&= c. \sum_{n \geq 1} \frac{1}{n^2} E[(z_n \cdot Q_T z_n)^{4-4d-p}] + c. \\
&\leq c. \sup_{|z|=1} E[(z \cdot Q_T z)^{4-4d-p}] + c. \\
&< \infty.
\end{aligned}
$$

The proof is complete. □

From Lemma 1 and the equality

$$
E[(z \cdot Q_T z)^{-p}] = \Gamma(p)^{-1} \int_0^\infty \lambda^{p-1} E[\exp\{-\lambda(z \cdot Q_T z)\}] d\lambda,
$$

our main purpose is to find a sufficient condition under which

$$
\sup_z E[\exp\{-\lambda(z \cdot Q_T z)\}] = o(\lambda^{-p}) \quad (\lambda \to \infty) \tag{5}
$$

is satisfied for all $p > 1$.

3. Lower bounds of functionals of semimartingales

Let $g(t)$ be a continuous semimartingale. Then

$$
\begin{aligned}
\int_0^T g(t)^2 dt &\geq \left\{ \int_\delta^T - \int_0^{T-\delta} \right\} g(t)^2 dt \\
&= \int_\delta^T dt \int_{t-\delta}^t dg(s)^2 = \int_0^T \delta_T(s) dg(s)^2,
\end{aligned} \tag{6}
$$

where $\delta_T(s) = (s + \delta) \wedge T - s \vee \delta$ for $\delta > 0$. This inequality is very simple, but it will turn out that the new key lemma is a consequence of this inequality.

For the sake of simlicity, set

$$
\boldsymbol{M}_T[(\zeta_i)] = \exp\left\{ - \int_0^T \sum_{i=1}^m \zeta_i(t) dw_t^i - \frac{1}{2} \int_0^T \sum_{i=1}^m \zeta_i(t)^2 dt \right\}
$$

for an $\boldsymbol{R}^m$-valued continuous semimartingale $\zeta(t) = \{\zeta_i(t)\}_{i=1}^m$. Let the symbol $\|\xi\|$ denote the value $\sup_t |\xi(t)|$ for a given process $\xi(t)$.

Consider semimartingales

$$
dF(t) = f_0(t)dt + \sum_{i=1}^m f_i(t) dw_t^i, \qquad df_0(t) = f_{00}(t)dt + \sum_{i=1}^m f_{0i}(t) dw_t^i,
$$

and define

$$\Gamma(\lambda) = \Big\{ w \in W_0^m \mid \sum_{j=0}^m \|f_j^2\| + \sum_{j=0}^m \|f_{0j}^2\| > \lambda^{1/4} \Big\}.$$

Theorem 1. *Let $\lambda > 1$. There exist positive constants C_0, C_1, C_2 independent of $F(\cdot)$ and λ, and a positive random variable $M_T^{(\lambda)}$ with $E[M_T^{(\lambda)}] \le 1$ such that the inequality*

$$C_0 \int_0^T \lambda^4 F(t)^2 dt + \lambda^{-1/8} \log M_T^{(\lambda)} \ge C_1 \int_0^T \lambda^{1/4} \sum_{j=0}^m f_j(t)^2 dt - C_2 \tag{7}$$

holds on the complement of the set $\Gamma(\lambda)$ for sufficiently large λ.

Proof. STEP 1. By using inequality (6) and the Ito formula, we have

$$\begin{aligned}
\int_0^T \lambda^4 F^2 dt &\ge \int_0^T \lambda^4 \delta_T dF^2 \\
&= \int_0^T \lambda^4 \delta_T \Big\{ \Big(\sum_i f_i^2 + 2F f_0 \Big) dt + 2F \sum_i f_i dw^i \Big\} \\
&= -\frac{1}{N_1} \log \widetilde{M}_T^{(1)} + \int_0^T \lambda^4 \delta_T \sum_i f_i^2 dt + \int_0^T 2\lambda^4 \delta_T F f_0 dt \\
&\quad - \int_0^T 2N_1 \lambda^8 \delta_T^2 F^2 \sum_i f_i^2 dt \\
&= -\frac{1}{N_1} \log \widetilde{M}_T^{(1)} + I_1 + I_2 + I_3,
\end{aligned}$$

where $3\lambda^{1/8} < N_1 \le 3\lambda^{7/4}$ and $\widetilde{M}_T^{(1)} = \boldsymbol{M}_T[(2N_1\lambda^4\delta_T F f_i)]$. We consider lower estimates of I_1, I_2, I_3 on $\Gamma(\lambda)^c$. Set $\delta = \lambda^{-3}$. Since $\lambda^3 \delta_T = 1$ on $[\delta, T - \delta]$,

$$I_1 = \int_0^T \lambda \sum_i f_i^2 dt - \int_0^T \lambda(1 - \lambda^3 \delta_T) \sum_i f_i^2 dt \ge \int_0^T \lambda \sum_i f_i^2 dt - 2\lambda^{-7/4}.$$

Since $0 \le \delta_T \le \delta = \lambda^{-3}$, we see that

$$\begin{aligned}
I_2 &\ge -\int_0^T \lambda^4 F^2 dt - \int_0^T \lambda^{-2} f_0^2 dt \ge -\int_0^T \lambda^4 F^2 dt - T\lambda^{-7/4}, \\
I_3 &\ge -2N_1 \lambda^2 \int_0^T F^2 \sum_i f_i^2 dt \ge -2N_1 \lambda^{-7/4} \int_0^T \lambda^4 F^2 dt.
\end{aligned}$$

Therefore we obtain the inequality

$$c.\int_0^T \lambda^4 F(t)^2 dt + \frac{1}{N_1}\log \widetilde{M}_T^{(1)} \geq \int_0^T \lambda \sum_{i=1}^m f_i(t)^2 dt - c. \tag{8}$$

STEP 2. For the sake of simplicity, set $\varepsilon = \lambda^{-1}$ and $\mu = \lambda^{1/4}$. From the Fubini type theorem for stochastic integrals, we obtain the equality

$$\begin{aligned}
\mu^9 \int_\varepsilon^T \Big(\int_{s-\varepsilon}^s f_0(u)du\Big)^2 ds &= 2\mu^9 \int_\varepsilon^T ds \int_{s-\varepsilon}^s \Big(\int_u^s f_0(v)dv\Big) f_0(u)du \\
&= 2\mu^9 \int_\varepsilon^T ds \int_{s-\varepsilon}^s \Big\{F(s) - F(u) - \int_u^s \sum_i f_i(v)dw_v^i\Big\} f_0(u)du \\
&= 2\mu^9 \int_\varepsilon^T ds \int_{s-\varepsilon}^s \{F(s) - F(u)\} f_0(u)du - \mu^4 \int_0^T h_\lambda(v) \sum_i f_i(v)dw_v^i \\
&= \frac{1}{N_2}\log \widetilde{M}_T^{(2)} + 2\mu^9 \int_\varepsilon^T ds \int_{s-\varepsilon}^s \{F(s) - F(u)\} f_0(u)du \\
&\quad + \frac{N_2}{2}\int_0^T \mu^8 h_\lambda^2 \sum_i f_i^2 dv \\
&= \frac{1}{N_2}\log \widetilde{M}_T^{(2)} + I_4 + I_5,
\end{aligned}$$

where $3\mu^{1/2} < N_2 \leq 3\mu$,

$$h_\lambda(v) = 2\mu^5 \int_{v\vee\varepsilon}^{(v+\varepsilon)\wedge T} ds \int_{s-\varepsilon}^v f_0(u)du, \quad \widetilde{M}_T^{(2)} = \boldsymbol{M}_T[(N_2\mu^4 h_\lambda f_i)].$$

We see that

$$|h_\lambda(v)| \leq 2\mu^5 \|f_0\| \int_v^{v+\varepsilon} (v - s + \varepsilon)ds \leq \mu^{-5/2}$$

on $\Gamma(\lambda)^c$. To estimate of I_4, I_5 on $\Gamma(\lambda)^c$ are routine works.

$$\begin{aligned}
I_4 &\leq \int_\varepsilon^T ds \int_{s-\varepsilon}^s 2\mu\|f_0\|\mu^8 |F(s) - F(u)|du \\
&\leq \mu^2 \|f_0^2\| \varepsilon(T - \varepsilon) + \int_\varepsilon^T ds \int_{s-\varepsilon}^s \mu^{16}|F(s) - F(u)|^2 du \\
&\leq T\mu^{-2}\|f_0^2\| + \int_\varepsilon^T ds \int_{s-\varepsilon}^s 2\mu^{16}\{F(s)^2 + F(u)^2\}du \\
&\leq T\mu^{-1} + 4\int_0^T \mu^{12} F^2 ds,
\end{aligned}$$

$$I_5 \le \frac{N_2}{2} \int_0^T \sum_i \mu^3 f_i^2 dv.$$

Therefore we obtain the estimate

$$\begin{aligned} (9) \qquad & \mu^9 \int_\varepsilon^T \Big(\int_{s-\varepsilon}^s f_0(u)du \Big)^2 ds - \frac{1}{N_2} \log \widetilde{M}_T^{(2)} \\ & \le c. \int_0^T \mu^{16} F^2 dt + c. \int_0^T \mu^4 \sum_i f_i^2 dt + c. \end{aligned}$$

STEP 3. Next we consider the lower estimate of the left-hand side of (9). Define a new process

$$\xi_\lambda(t;s) = (s-t) f_0(t) + \int_{s-\varepsilon}^t f_0(u)du = \varepsilon f_0(s-\varepsilon) + \int_{s-\varepsilon}^t (s-u) df_0(u).$$

Then $|\xi_\lambda(t;s)| \le \varepsilon \| f_0 \|$ for $s - \varepsilon \le t \le s$. By using the Ito formula, we have

$$\begin{aligned} & \xi_\lambda(s;s)^2 \\ & = \varepsilon^2 f_0(s-\varepsilon)^2 + \int_{s-\varepsilon}^s 2(s-u)\xi_\lambda(u;s) df_0(u) + \int_{s-\varepsilon}^s (s-u)^2 \sum_i f_{0i}(u)^2 du \\ & \ge \varepsilon^2 f_0(s-\varepsilon)^2 + \int_{s-\varepsilon}^s 2(s-u)\xi_\lambda(u;s) df_0(u). \end{aligned}$$

Hence

$$\begin{aligned} & \mu^9 \int_\varepsilon^T \Big(\int_{s-\varepsilon}^s f_0(u)du \Big)^2 ds \\ & \ge \mu \int_\varepsilon^T f_0(s-\varepsilon)^2 ds + 2\mu^9 \int_\varepsilon^T ds \int_{s-\varepsilon}^s (s-u)\xi_\lambda(u;s) df_0(u) \\ & = \mu \int_0^{T-\varepsilon} f_0^2 ds + \int_0^T \eta_\lambda df_0, \end{aligned}$$

where

$$\eta_\lambda(u) = 2\mu^9 \int_{u \vee \varepsilon}^{(u+\varepsilon)\wedge T} (s-u)\xi_\lambda(u;s) ds.$$

Since it holds that

$$|\eta_\lambda(u)| \le 2\mu^9 \int_u^{u+\varepsilon} (s-u) \, |\xi_\lambda(u;s)| ds \le \mu^{-5/2}$$

on $\Gamma(\lambda)^c$, we have

$$\begin{aligned}
\mu^9 \int_\varepsilon^T \Big(\int_{s-\varepsilon}^s f_0(u)du \Big)^2 ds &\geq \int_0^T \mu f_0^2 ds - \mu^{-2} + \int_0^T \eta_\lambda df_0 \\
&= \int_0^T \mu f_0^2 ds - \mu^{-2} + \int_0^T \eta_\lambda f_{00} du + \int_0^T \eta_\lambda \sum_i f_{0i} dw^i \\
&\geq \int_0^T \mu f_0^2 ds - c.\mu^{-2} - \frac{1}{N_3} \log \widetilde{M}_T^{(3)} - \frac{N_3}{2} \int_0^T |\eta_\lambda|^2 \sum_i f_{0i}^2 du \\
&\geq \int_0^T \mu f_0^2 ds - c.\mu^{-2} - \frac{1}{N_3} \log \widetilde{M}_T^{(3)} - c.N_3 \mu^{-4},
\end{aligned}$$

where $3\mu^{1/2} < N_3 \leq 3\mu^4$ and $\widetilde{M}_T^{(3)} = \boldsymbol{M}_T[(N_3 \eta_\lambda f_{0i})]$. Therefore we obtain the estimate

$$\mu^9 \int_\varepsilon^T \Big(\int_{s-\varepsilon}^s f_0(u)du \Big)^2 ds \geq -\frac{1}{N_3} \log \widetilde{M}_T^{(3)} + \int_0^T \mu f_0^2 ds - c. \tag{10}$$

on $\Gamma(\lambda)^c$. Set $M_T^{(\lambda)} = (\widetilde{M}_T^{(1)})^{\sqrt{\mu}/N_1} (\widetilde{M}_T^{(2)})^{\sqrt{\mu}/N_2} (\widetilde{M}_T^{(3)})^{\sqrt{\mu}/N_3}$. Then

$$E[M_T^{(\lambda)}] \leq E[\widetilde{M}_T^{(1)}]^{\sqrt{\mu}/N_1} E[\widetilde{M}_T^{(2)}]^{\sqrt{\mu}/N_2} E[\widetilde{M}_T^{(3)}]^{\sqrt{\mu}/N_3} \leq 1.$$

From (8), (9) and (10), we obtain inequality (7). □

4. Existence of the smooth density

In this section, we prove the Hörmander theorem by using Theorem 1. For $\Phi = \Phi_x = \phi(x) \cdot \partial_x$ ($\phi \in \boldsymbol{C}^\infty(\boldsymbol{R}^d; \boldsymbol{R}^d)$), let $[A_j, \Phi] = A_j \Phi - \Phi A_j$, and set

$$\varrho[A_0, \Phi] = [A_0, \Phi] + \frac{1}{2} \sum_{i=1}^m [A_i, [A_i, \Phi]], \quad \varrho[A_j, \Phi] = [A_j, \Phi] \ \ (j = 1, \dots, m).$$

By using the Ito formula,

$$d(U_t \Phi_{x_t}) = U_t \ \varrho[A_0, \Phi]_{x_t} dt + \sum_i U_t \ \varrho[A_i, \Phi]_{x_t} dw_t^i.$$

We shall introduce sets of vector fields:

$$\begin{aligned}
\mathcal{A}_0 &= \{A_1, \dots, A_m\}, \\
\mathcal{A}_k &= \{\varrho[A_j, \Phi] \mid \ \Phi \in \mathcal{A}_{k-1}, \ j = 0, 1, \dots, m\} \quad (k \geq 1).
\end{aligned}$$

Theorem 2. *Assume that Hörmander's condition is satisfied, that is, the linear space generated by* $\bigcup_k \mathcal{A}_k$ *at* x_0 *is* $\boldsymbol{R}^d$. *Then for* $T > 0$, *the probability law of* x_T *has a smooth density.*

Proof. By the Hörmander condition, there exist an integer $n \geq 0$ and constants $\gamma,\ \eta > 0$ such that

$$\sum_{\Phi \in \bigcup_{k=0}^{n} \mathcal{A}_k} (z \cdot \phi(y))^2 \geq \eta \tag{11}$$

for all $z \in S^{d-1}$ and y with $|y - x_0| < \gamma$. Consider the following stopping time:

$$S = \inf\Big\{ t > 0 \mid \sup_{s \leq t} |x_s - x_0| \geq \gamma \text{ or } \sup_{s \leq t} \|U_s - I\| \geq 1/2 \Big\} \wedge T.$$

From the Chebyshev inequality, the Burkholder inequality and (4), we see that

$$\begin{aligned} P[S < \lambda^{-\beta}] &\leq P\Big[\sup_{t \leq \lambda^{-\beta}} |x_t - x_0| \geq \gamma \Big] + P\Big[\sup_{t \leq \lambda^{-\beta}} \|U_t - I\| \geq \frac{1}{2} \Big] \\ &\leq c.\ E\Big[\sup_{t \leq \lambda^{-\beta}} |x_t - x_0|^q \Big] + c.\ E\Big[\sup_{t \leq \lambda^{-\beta}} \|U_t - I\|^q \Big] \\ &= o\big(\lambda^{-q\beta/2}\big) \end{aligned} \tag{12}$$

for all $q \geq 4/\beta$, where $\beta = 2^{-1-4n}$. Remark that $|z^* U_t| \geq 1 - \|I - U_t\| \geq 1/2$ under $t \leq S$. Set $\alpha_k = 2^{-1-4k}$ $(k = 0, 1, \ldots, n)$. Consider a functional $\mathcal{Q}_T(\lambda^{\alpha_k} z, \mathcal{A}_k)$ defined by

$$\mathcal{Q}_T(\lambda^{\alpha_k} z, \mathcal{A}_k) = \int_0^T \sum_{\phi \cdot \partial_x \in \mathcal{A}_k} (\lambda^{\alpha_k} z \cdot U_t \phi(x_t))^2 dt$$

for $z \in S^{d-1}$. For a given vector field $\Phi = \phi(x) \cdot \partial_x$, let $\|\Phi\|$ denote the value $\sup_{0 \leq t \leq T} |U_t \phi(x_t)|$. Define subsets $\Gamma_k(\lambda)$ of W_0^m by $\Gamma_0(\lambda) = \emptyset$ and

$$\Gamma_k(\lambda) = \bigcup_{\Phi \in \mathcal{A}_{k-1}} \Big\{ w \in W_0^m \mid \sum_{j=0}^{m} \|\varrho[A_j, \Phi]\|^2 + \sum_{j=0}^{m} \|\varrho[A_j, \varrho[A_0, \Phi]]\|^2 > \lambda^{2\alpha_k} \Big\}$$

for $k \geq 1$. From (4), $P[\Gamma_k(\lambda)] = o(\lambda^{-p})$ as $\lambda \to \infty$ for $p > 1$. For $k \geq 1$, by Theorem 1, there exists a positive random variable $M_T^{(k,\lambda)}$ with $E[M_T^{(k,\lambda)}] \leq 1$ such that the inequality

$$c.\mathcal{Q}_T(\lambda^{\alpha_{k-1}} z, \mathcal{A}_{k-1}) \geq -\lambda^{-\alpha_k} \log M_T^{(k,\lambda)} + c.\mathcal{Q}_T(\lambda^{\alpha_k} z, \mathcal{A}_k) - c. \tag{13}$$

holds on $\Gamma_k(\lambda)^c$. Set $\mathcal{A}'_n = \bigcup_{k=0}^{n} \mathcal{A}_k$. By the iterative application of (13),

$$\begin{aligned} c.\mathcal{Q}_T(\lambda^{\alpha_0} z, \mathcal{A}_0) &\geq -\lambda^{-\alpha_1} \log M_T^{(1,\lambda)} + c.\mathcal{Q}_T(\lambda^{\alpha_1} z, \mathcal{A}'_1) - c. \\ &\geq \cdots\cdots \end{aligned} \tag{14}$$

$$\geq -\sum_{k=1}^{n} \lambda^{-\alpha_k} \log M_T^{(k,\lambda)} + c.\mathcal{Q}_T(\lambda^{\alpha_n} z, \mathcal{A}'_n) - c.$$

on $\bigcap_{k=0}^{n} \Gamma_k(\lambda)^c$. By the Jensen inequality, we see

$$E\Big[\prod_{k=1}^{n} \big(M_T^{(k,\lambda)}\big)^{2\lambda^{-\alpha_k}}\Big] \leq \prod_{k=1}^{n} E\Big[\big(M_T^{(k,\lambda)}\big)^{2n\lambda^{-\alpha_k}}\Big]^{1/n} \leq 1$$

for sufficiently large λ. From (11), (12) and (14), we have

$$\begin{aligned}
&E\big[\ \exp\{-\lambda(z\cdot Q_T z)\}\ \big] \\
&\leq E\big[\ \exp\{-\lambda(z\cdot Q_T z)\} I_{\{S\geq\lambda^{-\beta}\}}\ \big] + P\big[S<\lambda^{-\beta}\big] \\
&\leq c.E\Big[\prod_{k=1}^{n}\big(M_T^{(k,\lambda)}\big)^{\lambda^{-\alpha_k}} \exp\Big\{-c.\mathcal{Q}_T(\lambda^{\alpha_n} z, \mathcal{A}'_n)\Big\} I_{\{S\geq\lambda^{-\beta}\}}\Big] \\
&\quad + P\Big[\bigcup_{k=0}^{n}\Gamma_k(\lambda)\Big] + P\big[S<\lambda^{-\beta}\big] \\
&\leq c.E\Big[\exp\Big\{-c.\mathcal{Q}_{\lambda^{-\beta}}\big(\lambda^{\alpha_n} z, \mathcal{A}'_n\big)\Big\} I_{\{S\geq\lambda^{-\beta}\}}\Big]^{1/2} + o\big(\lambda^{-p}\big) \\
&\leq c.\exp\Big\{-c.\lambda^{2\alpha_n-\beta} \inf_{|z|=1}\ \inf_{|y-x_0|<\gamma} \sum_{\Phi\in\mathcal{A}'_n} (z\cdot\phi(y))^2\Big\} + o\big(\lambda^{-p}\big) \\
&= o\big(\lambda^{-p}\big).
\end{aligned}$$

Since property (5) is satisfied, the transition density of x_T is smooth. □

REMARK. The new key lemma can be extended to the general semimartingale with the jump term. Hence we can discuss the regularity on the transition density associated with the jump type SDE. This will be discussed in the forthcoming paper ([4]).

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