DEGENERATING FAMILIES OF FINITE BRANCHED COVERINGS

MAKOTO NAMBA and MAKI TAKAI

(Received May 1, 2001)

1. Introduction

The category of finite branched coverings of a given complex projective manifold \( M \) is equivalent to the category of finite extensions \( K/\mathbb{C}(M) \) of the rational function field \( \mathbb{C}(M) \) of \( M \). Hence the study of finite branched coverings of \( M \) is nothing but a geometric study of extensions of algebraic function fields. In Namba [8], we constructed and studied the moduli space of equivalence classes of finite branched coverings of the complex projective line \( \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C}) \). If we want to compactify the moduli space, we are obliged to consider degenerations of branched coverings.

In this paper, we study degenerating families of finite branched coverings of \( \mathbb{P}^1 \) and \( \mathbb{P}^m = \mathbb{P}^m(\mathbb{C}) (m \geq 2) \) the \( m \)-dimensional complex projective space. In order to observe the degeneration, it is useful to introduce a picture which topologically represents a finite branched covering of the complex projective line. In §3, we call such a picture a Klein picture, since we can find such pictures in Klein [5]. In §5 (resp. §7), we assert that the topological type of the central fiber of a degenerating family of finite branched coverings of \( \mathbb{P}^1 \) (resp. \( \mathbb{P}^m \) (\( m \geq 2 \))) is completely determined by that of the central branch divisor and the permutation monodromy of the general fiber. In §6, we prove (Theorem 8) that the topological structure of a degenerating family of finite branched coverings of \( \mathbb{P}^1 \) can be determined by the permutation monodromy of the general fiber and the braid homomodromy of the family. Some results of this paper were announced in Namba [10].

2. Terminology

For a given connected complex manifold \( M \), a finite branched covering of \( M \) is by definition a finite proper holomorphic mapping

\[
f : X \longrightarrow M
\]

of an irreducible normal complex space \( X \) onto \( M \). A ramification point of \( f \) is a point of \( X \) such that \( f \) is not biholomorphic around the point. The image by \( f \) of a ramification point is called a branched point of \( f \). The set of all ramification points (resp. branch points) is denoted by \( R_f \) (resp. \( B_f \)). This is a hypersurface of
X (resp. M). The mapping
\[ f : X - f^{-1}(B_f) \rightarrow M - B_f \]
is a finite unbranched covering. Its mapping degree is denoted by \( \deg(f) \) and is called
the degree of \( f \). For a hypersurface \( B \) of \( M \), a finite branched covering \( f \) is said to
branch at most at \( B \) if \( B_f \) is contained in \( B \). Finite branched coverings \( f : X \rightarrow M \)
and \( f' : X' \rightarrow M \) are said to be isomorphic if there is a biholomorphic mapping \( \psi : X \rightarrow X' \)
such that \( f = f' \cdot \psi \). In this case, we denote \( f \simeq f' \). Finite branched
coverings \( f : X \rightarrow M \) and \( f' : X' \rightarrow M' \) are said to be equivalent (resp. topologically equivalent)
if there are biholomorphic mappings (resp. orientation preserving hemeomorphisms) \( \psi : X \rightarrow X' \)
and \( \varphi : M \rightarrow M' \) such that \( \varphi \cdot f = f' \cdot \psi \). In this case, we denote \( f \sim f' \)
(resp. \( f \sim f' \) (top.)).

**Theorem 1** (Grauert-Remmert [4]). Let \( B \) be a hypersurface of a connected
complex manifold \( M \) and \( f' : X' \rightarrow M - B \) be a finite unbranched covering.
Then there exists a unique (up to isomorphisms) finite covering \( f : X \rightarrow M \) which
branches at most at \( B \) and is an extension of \( f' \).

A topological version of Theorem 1 is given in Fox [3]. Theorem 1 asserts that
the correspondence \( f \leftrightarrow f' \) gives a categorical equivalence between finite unbranched
coverings of \( M - B \) and finite coverings of \( M \) branching at most at \( B \). Thus
we can apply terminology of finite unbranched coverings of \( M - B \) to finite coverings
of \( M \) branching at most at \( B \); for example, covering transformations, Galois coverings,
abelian coverings, cyclic coverings, etc.

**Corollary 1.** There is a one-to-one correspondence between the set of all iso-
morphism classes of finite coverings of \( M \) branching at most at \( B \) and the set of all
conjugacy classes of subgroups of finite index of the fundamental group \( \pi_1(M - B, q_0) \)
of \( M - B \).

3. Monodromy representations and Klein pictures

Let \( f : X \rightarrow M \) be a finite branched covering of a connected complex manifold
\( M \) of degree \( d \) branching at most at a hypersurface \( B \) of \( M \). Take a reference point
\( q_0 \) of \( M - B \) and put \( f^{-1}(q_0) = \{ p_1, \ldots, p_d \} \). The homotopy class \([\gamma]\) of a loop \( \gamma \)
in \( M - B \) starting from \( q_0 \) gives the homotopy class of the pull-back over \( f \) of \( \gamma \) starting
from every point \( p_j \), \( j = 1, \ldots, d \). Hence its end point \( p_{j'} \) is determined. Thus we
obtain a mapping
\[ \Phi_f : \pi_1(M - B, q_0) \rightarrow S_d, \]
which maps $[\gamma]$ to the permutation $j \rightarrow j'$, where $S_d$ is the $d$-th symmetric group. We define the product of pathes $\alpha$ and $\beta$ as $\alpha \beta$, where the end point of $\alpha$ is the initial point of $\beta$. We also define the product of permutations as in the following example:

\[
\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.
\]

The mapping $\Phi_f$ is then a homomorphism and is called the (permutation) monodromy representation of the covering $f$. Note that the representation class $[\Phi_f]$ of $\Phi_f$ does not depend on the choice of the arrangement of the points $p_1, \ldots, p_d$, nor the choice of the reference point $q_0$. That is, if one changes the arrangement of the points $p_1, \ldots, p_d$ or one chooses another reference point, then $\Phi_f$ is changed to $A^{-1} \Phi_f A$ for a fixed permutation $A$. Note also that the image of $\Phi_f$ is a transitive subgroup of $S_d$, for $X - f^{-1}(B)$ is connected. The image is called the monodromy group of the covering $f$. Monodromy groups of finite branched coverings correspond to Galois groups of algebraic equations. By the theorem of Grauert-Remmert and its corollary, we easily have the following 2 theorems:

**Theorem 2.** (1) Finite branched coverings $f$ and $f'$ of $M$ are isomorphic if and only if $B_f = B_{f'}$ and $[\Phi_f] = [\Phi_{f'}]$. ([\Phi_f]$ is the representation class of $\Phi_f$.)

(2) Finite branched coverings $f$ of $M$ and $f'$ of $M'$ are equivalent (resp. topologically equivalent) if and only if there is a biholomorphic mapping (resp. orientation preserving homeomorphism) $\varphi: M \rightarrow M'$ such that $\varphi(B_f) = B_{f'}$ and $[\Phi_{f'} \cdot \varphi_*] = [\Phi_f]$.

**Theorem 3.** For a given homomorphism $\Phi: \pi_1(M - B, q_0) \rightarrow S_d$ whose image is transitive, there exists a unique (up to isomorphisms) covering $f: X \rightarrow M$ of degree $d$ branching at most at $B$ such that $\Phi_f = \Phi$.

However it is a difficult problem in general to construct covering $f: X \rightarrow M$ in the theorem from a given $\Phi$ concretely (analytically or algebraically). The problem for the case $M = \mathbb{P}^1$ the complex projective line and $B = \{0, 1, \infty\}$ is studied in number theory (see Schneps [11]).

We construct branched coverings of the complex projective line $\mathbb{P}^1$ topologically for any given $\Phi$, by drawing a picture which we call a Klein picture, the idea of which comes from Klein [5]. Let $B = \{q_1, \ldots, q_n\}$ be a set of $n$ distinct points of $\mathbb{P}^1$. Let $f: X \rightarrow \mathbb{P}^1$ be a covering of degree $d$ branching at most at $B$. We draw a simple loop in $\mathbb{P}^1$ passing through all points $q_j$, $j = 1, \ldots, n$, oriented in this order which bounds a domain (the inside area) clockwise (see Fig. 1). We regard the inside area of the loop as a continent and the outside area as an ocean. We assume that the reference point $q_0$ is contained in the continent. We then pull them back over the covering $f$. Then we get a checked pattern of $d$ continents and $d$ oceans on $X$. We call such a
pattern the Klein picture of the covering $f$. The Klein picture represents the branched covering $f$ topologically. Starting from a homomorphism $\Phi : \pi_1(\mathbb{P}^1 - B, q_0) \to S_d$ such that $\text{Im } \Phi$ is transitive, we construct the branched covering $f$ in Theorem 2 topologically by drawing its Klein picture as follows: Put

$$A_j = \Phi(\gamma_j) \in S_d, \ j = 1, \ldots, n,$$

where $\gamma_j$ are lassos surrounding the points $q_j$ as in Fig. 2. Note that

$$\pi_1(\mathbb{P}^1 - B, q_0) = \langle \gamma_1, \ldots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1 \rangle,$$

$$A_n \cdots A_1 = 1 \in S_d.$$

Thus the representation $\Phi$ is determined by the permutations $A_j$. Decompose each $A_j$ into mutually prime cyclic permutations $A_{j_k}$ whose length are $e_{j_k}$. Put (by Riemann-
Hurwitz formula)

\[ g = \frac{1}{2} \left[ \sum_{j,k} (e_{jk} - 1) - 2d \right] + 1. \]

We prepare an oriented compact surface \( X \) of genus \( g \). We then draw the Klein picture, that is, a checked pattern of \( d \) continents and \( d \) oceans on \( X \) which is compatible with \( \Phi \). Here, the compatibility means that, for the point \( p_{jk} \) of \( f^{-1}(q_j) \) which corresponds to \( A_{jk}, e_{jk} \) continents and oceans are arranged alternately and counterclockwisely around \( p_{jk} \).

**Example 1.** Put \( n = 3, d = 3 \) and

\[ A_1 = \Phi(\gamma_1) = (1 2), \quad A_2 = \Phi(\gamma_2) = (1 3), \quad A_3 = \Phi(\gamma_3) = (1 2 3). \]

The genus of \( X \) is 0. The Klein picture in this case is as in Fig. 3, in which the points \( j \) denote the points in \( f^{-1}(q_j) \) and the circled number \( \gamma_j \) denotes the \( j \)-th continent. Observe that the points 1, 2 and 3 are seaside cities (vertices) of every continents arranged colckwisely in this order, while for example the continents \( 1, 2 \) and \( 3 \) are arranged counterclockwisely in this order around the city 3, which means \( A_3 = (1 2 3) \). (Conversely, we can read the monodromy from the Klein picture.) Put

\[ f : X \longrightarrow \mathbb{P}^1, \quad (z, w) \longmapsto z, \]

where \( X \) is the Riemann surface of the algebraic function \( w = w(z) \) given by the equation \( w^3 - 3w - z = 0 \). Then \( q_1 = -2, \quad q_2 = 2, \quad q_3 = \infty \) and \( \Phi_f = \Phi \).

**Example 2.** Put \( n = 3, d = 3 \) and

\[ A_j = \Phi(\gamma_j) = (1 2 3), \quad j = 1, 2, 3. \]

The genus of \( X \) is 1. The Klein picture in this case is as in Fig. 4. Put
Fig. 4.

\[ f : X \longrightarrow \mathbb{P}^1, (z, w) \longmapsto z, \]

where \( X \) is the Riemann surface of the algebraic function \( w = w(z) \) given by the equation \( w^3 - z^3 + 1 = 0 \). Then \( f \) is a cyclic covering such that \( \Phi_f = \Phi \).

**Example 3.** Put \( n = 4, d = 3 \) and

\[
\begin{align*}
A_1 &= \Phi(\gamma_1) = (1 \ 3 \ 2), \quad A_2 = \Phi(\gamma_2) = (1 \ 3 \ 2), \\
A_3 &= \Phi(\gamma_3) = (1 \ 2 \ 3), \quad A_4 = \Phi(\gamma_4) = (1 \ 2 \ 3),
\end{align*}
\]

The genus of \( X \) is 2. The Klein picture in this case is as in Fig. 5. Put

\[ f : X \longrightarrow \mathbb{P}^1, (z, w) \longmapsto z, \]

where \( X \) is the Riemann surface of the algebraic function \( w = w(z) \) given by the equation \( w^3 - z^3(z - 1)(z - 2) = 0 \). Then \( f \) is a cyclic covering such that \( q_1 = 0, q_2 = 1, q_3 = 2, q_4 = \infty \) and \( \Phi_f = \Phi \).

4. **Families of finite branched coverings**

Let \( T \) be a connected complex manifold. A family of connected complex manifolds with the parameter space \( T \) is by definition a smooth holomorphic mapping

\[ \pi : M \longrightarrow T \]

of a connected complex manifold \( M \) onto a connected complex manifold \( T \) such that every fiber is connected. Here the smoothness means that the Jacobian matrix \( d\pi \) is of
maximal rank at every point of $M$. Every fiber $M_t = f^{-1}(t)$ of $t \in M$ is a connected complex manifold. We denote

$$M = \{M_t\}_{t \in T}.$$  

Let $M = \{M_t\}_{t \in T}$ be a family of connected complex manifolds. A family of finite branched coverings of $M = \{M_t\}_{t \in T}$ is by definition a finite branched covering

$$f : X \rightarrow M$$

such that

(i) $M_t \not\subset B_T$ for every $t \in T$,
(ii) there is a hypersurface $V$ of $T$ such that

$$f_t = f : X_t = f^{-1}(M_t) \rightarrow M_t$$

is a finite branched covering of $M_t$ for every $t \in T - V$.
(iii) For any $t$ and $t'$ in $T - V$, $f_t$ and $f_{t'}$ are topologically equivalent.

We denote $f = \{f_t\}$. In particular if $\pi : M \rightarrow T$ is a holomorphic $\mathbb{P}^m$-bundle, then we call $f : X \rightarrow M$ a family of finite branched coverings of $\mathbb{P}^m$.

**Remark.** $X$ and $X_t (t \in T - V)$ have only normal singularity, while $X_t (t \in V)$, the degenerated coverings, may not be normal. In this sense, our definition of degenerations is different from the usual one.

We are interested in $X_t$ for $t \in V$, that is, degenerated coverings. In the subse-
sequent sections, we restrict our consideration to degenerating families of finite branched coverings of $\mathbb{P}^m$ and a disc in $\mathbb{C}$.

**Example 4.** Put $Y = \{(a_0 : a_1 : a_2 : a_3), (x_0 : x_1) \in \mathbb{P}^3 \times \mathbb{P}^1 \mid a_0 x_1^3 + a_1 x_1^2 x_0 + a_2 x_1 x_0^2 + a_3 x_0^3 = 0\}$, $g: ((a_0 : a_1 : a_2 : a_3), (x_0 : x_1)) \in Y \mapsto (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3$, where $(a_0 : a_1 : a_2 : a_3)$ and $(x_0 : x_1)$ are homogeneous coordinate systems of $\mathbb{P}^3$ and $\mathbb{P}^1$, respectively. Then $Y$ is non-singular and $g$ is a branched covering of degree 3 whose branch locus is the discriminant locus

$$B_f = \{(a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3 \mid a_1^2 a_2^2 - 4a_0 a_3^3 + 18a_0 a_1 a_2 a_3 - 4a_1^3 a_3 - 27a_0^2 a_3^2 = 0\}.$$

Let $\mathbb{P}^{3*}$ be the dual projective space of $\mathbb{P}^3$ and put

$$M = \{((t_0 : t_1 : t_2 : t_3), (a_0 : a_1 : a_2 : a_3)) \in \mathbb{P}^{3*} \times \mathbb{P}^3 \mid t_0 a_0 + t_1 a_1 + t_2 a_2 + t_3 a_3 = 0\},$$

$$\pi: ((t_0 : t_1 : t_2 : t_3), (a_0 : a_1 : a_2 : a_3)) \in M \mapsto (t_0 : t_1 : t_2 : t_3) \in \mathbb{P}^{3*},$$

$$\pi': ((t_0 : t_1 : t_2 : t_3), (a_0 : a_1 : a_2 : a_3)) \in M \mapsto (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3,$$

where $(t_0 : t_1 : t_2 : t_3)$ is a homogeneous coordinate system of $\mathbb{P}^{3*}$. Then $\pi$ is a $\mathbb{P}^2$-bundle on $\mathbb{P}^{3*}$. Let $X$ be the normalization of the fiber product $M \times_{\mathbb{P}^3} Y$ of $\pi': M \mapsto \mathbb{P}^3$ and $g: Y \mapsto \mathbb{P}^3$. Let

$$f: X \mapsto M$$

be the composition of the normalization

$$X \mapsto M \times_{\mathbb{P}^3} Y$$

and the projection

$$M \times_{\mathbb{P}^3} Y \mapsto M.$$

Then $f = \{f_t\}_{t \in \mathbb{P}^3*}$ is a family of branched coverings of $\mathbb{P}^2$. ($f_t: X_t \mapsto \pi^{-1}(t)$ is the restriction to the plane $\pi^{-1}(t)$ in $\mathbb{P}^3$ of $g$.)

We explain this as follows: Let

$$C_3 = \{(1 : u : u^2 : u^3) \in \mathbb{P}^{3*} \mid u \in \mathbb{P}^1\}$$

be the rational normal curve, which is the image curve of the holomorphic imbedding

$$\Phi|_D: \mathbb{P}^1 \mapsto \mathbb{P}^{3*}$$

of the unique complete linear system $|D|$ of degree 3 ($D$ is a divisor on $\mathbb{P}^1$ of degree 3). $B_f$ is then the dual variety of $C_3$. That is, $B_f$ is the set of all planes in $\mathbb{P}^{3*}$.
which contain tangent lines to $C_3$. Every divisor in $|D|$ is the intersection of $C_3$ with a (unique) plane in $\mathbb{P}^3$. In this sense, $|D|$ is identified with $\mathbb{P}^3 = (\mathbb{P}^3)^*$. By the uniqueness of the complete linear system $|D|$ of degree 3, every automorphism of $\mathbb{P}^1$ acts on $|D| = \mathbb{P}^3$ (resp. on $\mathbb{P}^3^*$) as a projective transformation, which maps $B_f$ to $B_f$ (resp. $C_3$ to $C_3$). Let $V$ be the ruled surface in $\mathbb{P}^3^*$ consisting of tangent lines to $C_3$.

For any two points $t$ and $t'$ in $\mathbb{P}^3^* - V$, there is an automorphism $\varphi$ of $\mathbb{P}^1$ such that $\varphi(t) = t'$. In fact, there are just 3 points $p_1$, $p_2$ and $p_3$ in $C_3$ (resp. $p'_1$, $p'_2$ and $p'_3$ in $C_3$) such that the osculating plane at $p_j$ (resp. at $p'_j$) to $C_3$ passes through $t$ (resp. $t'$) for $j = 1, 2, 3$. Then $\varphi \in \text{Aut}(\mathbb{P}^1)$ such that $\varphi(p_j) = p'_j (j = 1, 2, 3)$ maps $t$ to $t'$. Thus $\text{Aut}(\mathbb{P}^1)$ acts on $\mathbb{P}^3^* - V$ transitively. (The orbits of the group action of $\text{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^3^*$ are $\mathbb{P}^3^* - V$, $V$ and $C_3$.) The projection $\pi_t$ with the center $t$ maps $C_3$ onto a rational plane cubic curve $C_t$ with a node and 3 flexes $\pi_t(p_1)$, $\pi_t(p_2)$ and $\pi_t(p_3)$. The plane projective transformation induced by $\varphi$ maps $C_t$ to $C_{t'}$. The branch locus $B_t$ (resp. $B_{t'}$) of $f_t$ (resp. $f_{t'}$) is the dual curve of $C_t$ (resp. $C_{t'}$) which is a rational plane quartic curve with 3 simple cusps. Hence the plane projective transformation induced by $\varphi$ maps $B_t$ to $B_{t'}$. Now, $\varphi$ induces an automorphism of the projective manifold $Y$ which, by the above discussion, induces an equivalence of $f_t$ and $f_{t'}$.

A similar discussion shows that if $t \in V - C_3$ (say $t = (0 : 1 : 0 : 0)$), then $B_t$ is the union of a rational plane cubic curve with 1 simple cusp and a line passing through a flex of the curve. For any points $t$ and $t'$ in $V - C_3$, $f_t$ and $f_{t'}$ are equivalent.

If $t \in C_3$ (say $t = (0 : 0 : 0 : 1)$), then $B_t$ is the union of an irreducible conic and a double tangent line to the conic. In this case, $X_t$ is not irreducible.

5. Degenerating families of finite branched coverings of $\mathbb{P}^1$

Let

$$\Delta = \Delta(0, \epsilon) = \{ t \in \mathbb{C} \mid |t| < \epsilon \}$$

be a disc and $\Delta^* = \Delta - \{0\}$ be the punctured disc. A finite branched covering

$$f : X \longrightarrow \Delta \times \mathbb{P}^1$$

is called a degenerating family of finite branched coverings of $\mathbb{P}^1$ and is denoted by $f = \{ f_t \}$, if the following three conditions are satisfied:

1. $t \times \mathbb{P}^1 \nsubseteq B_f$ for every $t \in \Delta$.
2. For every $t \in \Delta^*$, $t \times \mathbb{P}^1$ meets at $n$ points transversally with $B_f$. ($n$ is constant for $t \in \Delta^*$.)
3. For every $t \in \Delta^*$,

$$f_t = f : X_t = f^{-1}(t \times \mathbb{P}^1) \longrightarrow t \times \mathbb{P}^1$$
is a covering of $\mathbb{P}^1$ of degree $d = \deg(f)$ branching at $B_f = B_f \cap (t \times \mathbb{P}^1) = \{q_1(t), \ldots, q_n(t)\}$ (see Fig. 6).

The central fiber $X_0 = f^{-1}(0 \times \mathbb{P}^1)$ is a degeneration of a general fiber $X_t$ for $t \neq 0$. The Klein picture of $f_t$ degenerates to a picture on $X_0$, which we call the Klein picture of $f_0$. This represents $(X_0, f_0)$ topologically.

**Example 5.** Let $X_t$ be the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - 3tw - z = 0$. Put

$$f_t : X_t \longrightarrow \mathbb{P}^1, (z, w) \longmapsto z.$$  

Then $f = \{f_t\}$ is a degenerating family of branched coverings of $\mathbb{P}^1$. For a fixed non-zero $t$, the monodromy representation $\Phi_t$ and the Klein picture of $f_t$ are given as same as in Example 1. Note that $q_1(t) = -2t^{3/2}$, $q_2(t) = 2t^{3/2}$ and $q_3(t) = \infty$. As $t \longrightarrow 0$, both branch points $q_1(t)$ and $q_2(t)$ converge to $q_1(0) = q_2(0) = 0$, so the pathes connecting the points 1 and 2 in Fig. 3 converge to the point $1 = 2$, and we get the Klein picture of $f_0$ as in Fig. 7. In fact $X_0 : w^3 - z = 0$. 

Fig. 6.
Example 6. Let $X_t$ be the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^2 - z(z - t)(z - 1) = 0$. Put

$$f_t: X_t \longrightarrow \mathbb{P}^1, (z, w) \mapsto z.$$ 

Then $f = \{f_t\}$ is a degenerating family of branched double coverings of $\mathbb{P}^1$. Note that $q_1(t) = 0$, $q_2(t) = t$, $q_3(t) = 1$ and $q_4(t) = \infty$. For a fixed non-zero $t$, the monodromy representation $\Phi_t$ is given by $A_j = \Phi_t(\gamma_j) = (1 \ 2)$ for $j = 1, 2, 3, 4$. The Klein picture of $f_t$ is as in Fig. 8 in which the continent (2) is the upper backside of the torus. As $t \longrightarrow 0$, both $q_1(t)$ and $q_2(t)$ converge to $q_1(0) = q_2(0) = 0$, so the pathes connecting the points 1 and 2 in Fig. 8 converge to the point $1 = 2$, and we get the Klein picture of $f_0$ as in Fig. 9 in which the continent (2) is also the upper backside. In fact, $X_0 : w^2 - z^2(z - 1) = 0$.

Example 7. Let $X_t$ be the Riemann surface of genus 1 of the algebraic function $w = w(z)$ given by the equation $w^2 - z(z - t)(z - 1)(z - 1 - t) = 0$. Put

$$f_t: X_t \longrightarrow \mathbb{P}^1, (z, w) \mapsto z.$$
Then $f = \{f_t\}$ is a degenerating family of branched double covering of $\mathbb{P}^1$. Note that

$$q_1(t) = 0, \quad q_2(t) = t, \quad q_3(t) = 1, \quad q_4(t) = 1 + t.$$  

For fixed $t$ with $0 < |t| < 1$, the monodromy representation $\Phi_t$ is a given by $A_j = \Phi_t(\gamma_j) = (1 2)$ for $j = 1, 2, 3, 4$. The Klein picture of $f_t$ is as same as that in Fig. 8 for Example 6. As $t \to 0$, $q_2(t)$ and $q_4(t)$ converge to $q_1(0) = 0, q_3(0) = 1$, respectively, so the pathes connecting the points 1 to 2 and 3 to 4 in Fig. 8 converge to $1 = 2$ and $3 = 4$, respectively. Hence we get the Klein picture of $f_0$ as in Fig. 10 in which the continent $\mathbb{C}$ is also the upper backside. In fact

$$X_0 : w^2 - z^2(z - 1)^2 = 0,$$

which is not globally irreducible.

**Example 8.** Let $X_t$ be the Riemann surface of genus 1 of the algebraic function $w = w(z)$ given by the equation $w^2 - z(z - t)(z - 2t) = 0$. Put

$$f_t : X_t \to \mathbb{P}^1, \quad (z, w) \mapsto z.$$  

As $t \to 0$, the Klein picture of $f_t$ converges to that of $f_0$ as in Fig. 11, in which $\mathbb{C}$
are the upper backside.

$X_0$ has a cusp singularity at the point $1 = 2 = 3$. In fact

$$X_0 : w^2 - z^3 = 0.$$  

Now we assume and put

$$q_1(0) = \cdots = q_{k_1}(0) = q_1^0,$$

$$q_{k_1+1}(0) = \cdots = q_{k_1+k_2}(0) = q_2^0,$$

$$\cdots$$

$$q_{k_1+\cdots+k_r-1+1}(0) = \cdots = q_{k_1+\cdots+k_r}(0) = q_r^0$$

where $k_\rho \geq 1$ ($\rho = 1, \ldots, r$), $k_1 + \cdots + k_r = n$ and $q_1^0, q_2^0, \ldots, q_r^0$ are mutually distinct. We regard

$$B_0 = B_f \cap (0 \times \mathbb{P}^1) = \{ q_1^0, q_2^0, \ldots, q_r^0 \}$$

not as a point set but as a divisor on $\mathbb{P}^1$:

$$B_0 = k_1q_1^0 + k_2q_2^0 + \cdots + k_rq_r^0.$$  

We draw a simple loop in $\mathbb{P}^1$ passing through all points $q_1^0, \ldots, q_r^0$ oriented in this order which bounds a domain clockwisely as in Fig. 1. We call the Klein picture of $f_0$ for the checked pattern on $X_0$ which is the pull-back of the picture over $f_0$.

Now, we show that topologically, the degenerating curve $X_0 = f^{-1}(0 \times \mathbb{P}^1)$ can be described by the divisor $B_0$ and the monodromy $\Phi_t = \Phi_f$, where $t \in \Delta^*$ is a fixed point.

Let $\gamma_j(t)$ ($1 \leq j \leq n$) be the lasso around $q_j(t)$ as in Fig. 2 and put

$$A_1 = \Phi_t(\gamma_1), \ldots, A_n = \Phi_t(\gamma_n).$$  

Let $H_\rho$ ($1 \leq \rho \leq r$) be the subgroup of $S_d$ generated by

$$A_{k_1+\cdots+k_{\rho-1}+1}, \ldots, A_{k_1+\cdots+k_{\rho-1}+k_\rho}.$$
$H_\rho$ may not be a transitive subgroup of $S_d$. We denote

$$\mathcal{A}_1^\rho, \ldots, \mathcal{A}_{v_\rho}^\rho$$

the orbits of $H_\rho$ on \{1, 2, \ldots, d\}. $v_\rho$ is the number of orbits. Put

$$A_1^0 = A_{k_1}A_{k_1-1}\cdots A_1,$$
$$A_2^0 = A_{k_1+k_2}A_{k_1+k_2-1}\cdots A_{k_1+1},$$
$$\ldots$$
$$A_r^0 = A_{k_1+\cdots+k_r}A_{k_1+\cdots+k_r-1}\cdots A_{k_1+\cdots+k_{r-1}+1}$$

and

$$K = \langle A_1^0, \ldots, A_r^0 \rangle$$

the subgroup of $S_d$ generated by $A_1^0, \ldots, A_r^0$. Let

$$\gamma_1^0, \ldots, \gamma_r^0$$

be lassos around the points

$$q_1^0, \ldots, q_r^0,$$

respectively in $0 \times \mathbb{P}^1$ as in Fig. 2. Put

$$\Phi_0(\gamma_\rho^0) = A_\rho^0 \quad (1 \leq \rho \leq r).$$

Then

$$\Phi_0: \pi_1(0 \times \mathbb{P}^1 - \{q_1^0, \ldots, q_r^0\}, q_0) \longrightarrow S_d$$

is a homomorphism.

**Definition 1.** For a permutation $A \in S_d$, if $A$ is written as $A = A_1 \cdots A_m$, the product of mutually prime cyclic permutations, then we call the number $w = w(A)$ the weight of $A$. ($w(A)$ depends also on $d$. For example, if $d = 4$ and $A = (1 2 3)$, then $w(A) = w((1 2 3)(4)) = 2$.)

Let $\chi(X_t)$ denote the Euler characteristic of $X_t$.

**Theorem 4.** Let $t \neq 0$. Then the following (1)–(4) hold:

1. $\chi(X_t) = 2 - 2g = 2d - nd + \sum_{j=1}^d w(A_j)$.
2. $\chi(X_0) = 2d - \{nd - \sum_{\rho=1}^r (k_\rho - 1)d\} + \sum_{\rho=1}^r v_\rho$.
3. $\chi(X_0) - \chi(X_t) = \sum_{\rho=1}^r (k_\rho - 1)d + \sum_{\rho=1}^r v_\rho - \sum_{j=1}^d w(A_j)$. 


(4) $\chi(X_0) \geq \chi(X_f)$.

Proof. (1) The Klein picture of the covering $f_i: X_i \rightarrow \mathbb{P}^1$ gives a cellular decomposition of $X_f$. The number of vertices is $\sum_{j=1}^n w(A_j)$, the number of sides is $nd$ and the number of faces is $2d$. Hence

$$\chi(X_f) = 2d - nd + \sum_{j=1}^n w(A_j).$$

(2) Let $\hat{G}$ be the (oriented) graph on $X_t$ of the pull-back by $f_t$ of the cycle

$$q_1(t) \rightarrow q_2(t) \rightarrow \cdots \rightarrow q_r(t) \rightarrow q_1(t).$$

Then $\hat{G}$ is the graph whose points and lines are vertices and sides, respectively, of the Klein picture of $f_t$. Every point of $\hat{G}$ has been numbered as a vertix of the Klein picture. We put the circled number ($j$) on every sides of $j$-th continent. Thus we get a graph $\hat{G}$ with numbered points and circle numbered lines.

Let $G_\rho (1 \leq \rho \leq r)$ be the (oriented) graph on $X_t$ of the pull-back by $f_t$ of the tree

$$q_{k_1+\cdots+k_{\rho-1}+1}(t) \rightarrow q_{k_1+\cdots+k_{\rho-1}+2}(t) \rightarrow \cdots \rightarrow q_{k_1+\cdots+k_{\rho-1}+k_\rho}(t).$$

Then every $G_\rho$ is a subgraph of $\hat{G}$. Let

$$k_1 + \cdots + k_{\rho-1} + 1 \leq i < i + 1 \leq k_1 + \cdots + k_{\rho-1} + k_\rho.$$

If the permutation $A_i$ and $A_{i+1}$ are written as, say,

$$A_i = \begin{pmatrix} \cdots & a & \cdots \end{pmatrix}, \quad A_{i+1} = \begin{pmatrix} \cdots & a & \cdots \end{pmatrix},$$

then there are lines $\overline{a}$ and $\overline{b}$ in $G_\rho$ which have the starting point $i$ (a point of $G_\rho \cap f_t^{-1}(q_i)$), and so are connected at the point $i$. Moreover there is a line $\overline{c}$ in $G_\rho$ such that the lines $\overline{a}$ and $\overline{c}$ have the same end point $i + 1$, and so are connected at the point $i + 1$. Hence the lines $\overline{a}$, $\overline{b}$ and $\overline{c}$ are connected in $G_\rho$.

Now the Klein picture of $f_0$ gives a cellular decomposition of $X_0$ which can be obtained from that of $X_f$ by converging every connected component of the graphs $G_\rho (1 \leq \rho \leq r)$ to a point. Hence the number of vertices is $\sum_{\rho=1}^r v_\rho$, the number of sides is $nd - \sum_{\rho=1}^r (k_\rho - 1)d$ and the number of faces is $2d$. Hence

$$\chi(X_0) = 2d - \left\{ nd - \sum_{\rho=1}^r (k_\rho - 1)d \right\} + \sum_{\rho=1}^r v_\rho.$$

(3) This follows from (1) and (2).
(4) For a graph $G$, the following inequality holds:

$$b \geq a - c,$$

where $a$ is the number of points of $G$, $b$ is the number of lines of $G$ and $c$ is the number of connected components of $G$. Here the equality holds if and only if every component is a tree, that is, a graph without cycles.

We apply this to every graph $G_{\rho}$. Then

$$a = a_{\rho} = w(A_{k_{1}} + \cdots + k_{\rho - 1} + 1),$$
$$b = b_{\rho} = (k_{\rho} - 1)d,$$
$$c = c_{\rho} = v_{\rho}.$$

Note that

$$\sum_{\rho=1}^{r} a_{\rho} = \sum_{j=1}^{n} w(A_{j}).$$

Hence by (3)

$$\chi(X_{0}) - \chi(X_{1}) = \sum_{\rho=1}^{r} b_{\rho} + \sum_{\rho=1}^{r} c_{\rho} - \sum_{\rho=1}^{r} a_{\rho} = \sum_{\rho=1}^{r} (b_{\rho} + c_{\rho} - a_{\rho}) \geq 0. \tag*{\square}$$

**Theorem 5.** (1) $f_{0}^{-1}(q_{0})$ consists of $v_{\rho}$ points, which can be identified with $\mathfrak{A}_{1}^{0}, \ldots, \mathfrak{A}_{\rho}^{0}$.

(2) Every $A_{\rho}^{0}$ ($1 \leq \rho \leq r$) induces a permutation $A_{\rho}^{0} : \mathfrak{A}_{j}^{0} \rightarrow \mathfrak{A}_{j}^{0}$. $X_{0}$ has local $w(A_{\rho}^{0})$ irreducible components at the point corresponding to $\mathfrak{A}_{j}^{0}$.

(3) There is a natural one-to-one correspondence between the set of global irreducible components of $X_{0}$ and the set of orbits of $K = \langle A_{1}^{0}, \ldots, A_{r}^{0} \rangle$ on $\{1, \ldots, d\}$. $\Phi_{0}$, regarded as the representation to permutations on an orbit of $K$ on $\{1, \ldots, d\}$, gives the monodromy representation of the branched covering

$$f_{0} : \eta : \tilde{X}_{0} \longrightarrow 0 \times \mathbb{P}^{1},$$

where $\eta : \tilde{X}_{0} \longrightarrow X_{0}$ is the normalization of the global irreducible component $X_{0}'$ of $X_{0}$ corresponding to the orbit of $K$.

**Proof.** (1) follows from the proof (2) of Theorem 4. For a sufficiently small $|t|$, $q_{k_{1} + \cdots + k_{\rho - 1} + 1}(t), \ldots, q_{k_{1} + \cdots + k_{\rho - 1} + k_{\rho}}(t)$ are in a small neighborhood of $q_{\rho}^{0}$. Hence the lasso $\gamma_{\rho}^{0}$ is homotopic to the product

$$\gamma_{k_{1} + \cdots + k_{\rho - 1} + k_{\rho}}(t) \cdots \gamma_{k_{1} + \cdots + k_{\rho - 1} + 1}(t).$$
Let

\[ i : N = 0 \times \mathbb{P}^1 \longrightarrow M = \Delta \times \mathbb{P}^1 \]

be the inclusion mapping. Then the fiber product \( N \times_M X \) can be identified with \( X_0 \). Now, (2) and (3) of Theorem 5 follow from the following lemma, whose proof is straightforward and is omitted.

**Lemma 1.** Let \( M \) and \( N \) be connected complex manifolds and \( h : N \longrightarrow M \) be a holomorphic mapping. Let \( f : X \longrightarrow M \) be a finite unbranched covering of \( M \) of degree \( d \) and \( \Phi_f \) be its monodromy representation. Let \( f' : N \times_M X \longrightarrow N \) be the projection of the fiber product \( N \times_M X \) onto \( N \). Then the followings hold:

1. There is a one-to-one correspondence between the set of orbits of \( \Phi_f \cdot h_\ast(\pi_1(N, p_0)) \) on \( \{1, \ldots, d\} \) and the set of connected components of \( N \times_M X \).
2. For a connected component \( Y \) of \( N \times_M X \), \( f' : Y \longrightarrow N \) is a finite unbranched covering of \( N \) whose monodromy representation is equal to \( \Phi_f \cdot h_\ast \) regarded as the representation to permutations on the orbit of \( \Phi_f \cdot h_\ast(\pi_1(N, p_0)) \) on \( \{1, \ldots, d\} \) corresponding to \( Y \).

**Theorem 6.** The following four conditions are mutually equivalent:

1. \( X_0 \) is homeomorphic to \( X_t \) for \( t \neq 0 \).
2. \( \chi(X_0) = \chi(X_t) \) for \( t \neq 0 \).
3. \( \sum_{\rho=1}^r (k_\rho - 1)d = \sum_{j=1}^n w(A_j) - \sum_{\rho=1}^r v_\rho \).
4. \( \sum_{\rho=1}^r (k_\rho - 1)d = \sum_{j=1}^n w(A_j) - \sum_{\rho=1}^r w(A_\rho^0) \).

Proof. If \( X_0 \) is homeomorphic to \( X_t \), then \( \chi(X_0) = \chi(X_t) \). If \( \chi(X_0) = \chi(X_t) \), then every connected component of the graphs \( G_\rho \) (1 \( \leq \rho \leq r \)) is a tree as is shown in the proof of Theorem 4. When \( t \) converges to 0, every connected component of the graphs \( G_\rho \) (1 \( \leq \rho \leq r \)) converges to a point. This means that \( X_0 \) is homeomorphic to \( X_t \). Next, note that

\[ A_\rho^0 = A_{\rho 1}^0 \cdots A_{\rho v_\rho}^0, \]

where \( A_{\rho j}^0 \) is the permutation on the orbit \( \mathcal{A}_j^0 \) induced by \( A_\rho^0 \). Hence

\[ w(A_\rho^0) = w(A_{\rho 1}^0) + \cdots + w(A_{\rho v_\rho}^0). \]

In particular

\[ w(A_\rho^0) \geq v_\rho. \]

Here the equality holds if and only if every \( A_{\rho j}^0 \) is a cyclic permutation. Hence, by (2) of Theorem 5, the equality holds if and only if \( X_0 \) is locally irreducible at ev-
very point $\mathcal{X}^\rho_j (1 \leq j \leq \nu_\rho)$. Now, by (4) of Theorem 4, the following inequality holds:

$$\sum_{\rho=1}^{r}(k_\rho - 1)d \geq \sum_{j=1}^{n}w(A_j) - \sum_{\rho=1}^{r}v_\rho \geq \sum_{j=1}^{n}w(A_j) - \sum_{\rho=1}^{r}w(A^0_\rho),$$

If

$$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{k}w(A_j) - \sum_{\rho=1}^{r}w(A^0_\rho),$$

then

$$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{n}w(A_j) - \sum_{\rho=1}^{r}v_\rho.$$ 

Hence $\chi(X_0) = \chi(X_t)$ by Theorem 4.

Conversely, if $\chi(X_0) = \chi(X_t)$, then $X_0$ is homeomorphic to $X_t$. In particular, $X_0$ is locally irreducible at every point $\mathcal{X}^\rho_j (1 \leq j \leq \nu_\rho, 1 \leq \rho \leq r)$. Thus

$$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{n}w(A_j) - \sum_{\rho=1}^{r}v_\rho = \sum_{j=1}^{n}w(A_j) - \sum_{\rho=1}^{r}w(A^0_\rho).$$

\[\square\]

**Remark.** If one of the conditions of Theorem 6 is satisfied, then $X_0$ is non-singular. In fact, if one of the conditions of Theorem 6 is satisfied, then every connected component of every graph $G_\rho (1 \leq \rho \leq r)$ is a tree. $X_0$ is obtained from $X_t$ converging every tree to a point. Hence $X_0$ is still a manifold. (The total space $X$ is also non-singular.)

This can be also shown in the following way: The arithmetic genus is constant. In particular, the arithmetic genus of $X_0$ is equal to the geometric genus of $X_t (t \neq 0)$. If $X_0$ is singular, then the geometric genus of $X_0$ is less than the arithmetic genus, a contradiction to the assumption $\chi(X_0) = \chi(X_t)$. On the other hand, if $\chi(X_0) > \chi(X_t)$, then the graph $G$ contains a cycle. As $t \to 0$, such a cycle $\Gamma'$ converges to a point $p$, while, for a connected open neighborhood $U$ of $\Gamma$, $U - \Gamma$, which has two connected components, moves homeomorphically. Hence $X_0$ is locally a cone with the vertex $p$. Thus $X_0$ can not be a manifold, so $X_0$ is singular.

**6. Topological equivalence of families**

In this section we show that the topological structure of the degenerating family $f = \{ f_t \}$ of finite branched coverings of $\mathbb{P}^1$ is not determined by $\Phi_t$ alone, but depends also on the braid monodromy $\theta(\delta)$. Here

$$\delta: u \mapsto t = t_0 e^{iu}, \ (0 \leq u \leq 2\pi)$$
is the loop around $t = 0$. ($t_0 \in \Delta^*$ is a fixed point.) In this section we assume for simplicity that $q_j(t) \neq \infty$ for every $t \in \Delta$ and $1 \leq j \leq n$. Then

$$\{q_1(t_0 e^{i \theta}), \ldots, q_n(t_0 e^{i \theta})\}_{0 \leq \theta \leq 2 \pi}$$

gives an (Artin) braid of strings, which is called the braid monodromy of the curve $B_f$ around $t = 0$ and is denoted by $\theta(\delta)$. The braid $\theta = \theta(\delta)$ can not be arbitrary. It is given by a complex analytic curve $B_f$. So such a braid we call a complex analytic braid. We fix a reference point $t_0 \in \Delta^*$ and put

$$q_j = q_j(t_0) \quad \text{for} \quad 1 \leq j \leq n.$$

Then the Artin braid group $B_n$ naturally acts on the fundamental group $\pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0)$ as follows:

$$\sigma_i(\gamma_i) = \gamma_i^{-1} \gamma_{i+1} \gamma_i,$$

$$\sigma_i(\gamma_{i+1}) = \gamma_i,$$

$$\sigma_i(\gamma_j) = \gamma_j \quad (j \neq i, i + 1),$$

where $\gamma_j$ ($j = 1, \ldots, n$) are the lassos as in Fig. 2 and $\sigma_i$ ($i = 1, \ldots, n - 1$) are the generators of $B_n$ defined as in Fig. 12.

A theorem of Zariski-van Kampen (see e.g. Dimca [2]) asserts

**Theorem 7** (Zariski-van Kampen).

$$\pi_1(\Delta \times \mathbb{P}^1 - B_f, q_0) = \langle \gamma_1, \ldots, \gamma_n \mid \gamma_n \cdots \gamma_1 \gamma_1 = 1, \theta(\delta) \gamma_j = \gamma_j, (1 \leq j \leq n) \rangle,$$

where $\gamma_j$ are lassos as in Fig. 2 for $f_{t_0}: X_{t_0} \longrightarrow \mathbb{P}^1$.

The monodromy representation $\Phi_f$ of $f: X \longrightarrow \Delta \times \mathbb{P}^1$ is equal to $\Phi_{t_0} = \Phi_{f_{t_0}}$. 
By Theorem 7, $\Phi_{f_0}$ satisfies

$$\Phi_{f_0} \cdot \theta(\delta) = \Phi_{f_0}.$$ 

**Definition 2.** $f = \{f_t\}$ and $f' = \{f'_t\}$ are said to be topologically equivalent if there are orientation preserving homeomorphisms $\psi, \varphi$ and $\eta$ which make the following diagram commutative:

$$
\begin{array}{c}
X \xrightarrow{\psi} X' \\
\downarrow f \quad \quad \downarrow f' \\
\Delta \times \mathbb{P}^1 \xrightarrow{\varphi} \Delta' \times \mathbb{P}^1 \\
\downarrow \quad \quad \downarrow \\
\Delta \xrightarrow{\eta} \Delta'
\end{array}
$$

Using fundamental results in the theory of fiber bundles (see Steenrod [12]), we get the following theorem, which can be regarded as a branched covering version of a theorem in Matsumoto-Montesinos [6]:

**Theorem 8.** There exists a one to one correspondence between $\{\text{topological equivalence class of } f = \{f_t\}, \text{ where } f_{t_0} (t_0 \neq 0) \text{ has the degree } d \text{ and } n \text{ branched points } q_1, \ldots, q_n\}$ and $\{(\Phi), \theta) | \{\Phi\} \text{ is the representation class of } \Phi: \pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0) \rightarrow S_d \text{ such that } \text{Im } \Phi \text{ is transitive, and } \theta \in B_n \text{ is a complex analytic braid such that } \Phi \cdot \theta = \Phi)/B_n. \text{ Here } \sigma \in B_n \text{ acts on } ((\Phi), \theta) \text{ as follows:}

$$
\sigma([\Phi], \theta) = ([\Phi \cdot \sigma^{-1}], \sigma \theta \sigma^{-1}).
$$

**Proof.** For two families

$$(1) \quad f = \{f_t\}: X \rightarrow \Delta \times \mathbb{P}^1, \quad f' = \{f'_t\}: X' \rightarrow \Delta' \times \mathbb{P}^1,$$

with the assumption

$$(\Delta \times \{\infty\}) \cap B_f = \emptyset, \quad (\Delta' \times \{\infty\}) \cap B_{f'} = \emptyset,$$

we may assume that there are $q_0 \in \mathbb{C}$ and $q'_0 \in \mathbb{C}$ such that

$$(\Delta \times \{q_0\}) \cap B_f = \emptyset, \quad (\Delta' \times \{q'_0\}) \cap B_{f'} = \emptyset.$$

(For example, take $q_0$ and $q'_0$ such that $|q_0|$ and $|q'_0|$ are sufficiently large.)
Take reference points \( t_0 \in \Delta^* \) and \( t'_0 \in \Delta'^* \). Put
\[
(t_0 \times \mathbb{C}) \cap \mathcal{B}_f = \{ q_1 = q_1(t_0), \ldots, q_n = q_n(t_0) \},
\]
\[
(t'_0 \times \mathbb{C}) \cap \mathcal{B}_{f'} = \{ q'_1 = q'_1(t'_0), \ldots, q'_n = q'_n(t'_0) \}.
\]

There is an orientation preserving homeomorphism
\[
(2) \quad \xi: t_0 \times \mathbb{C} = \mathbb{C} \longrightarrow t'_0 \times \mathbb{C} = \mathbb{C}
\]
such that
\[
\xi(q_j) = q'_j \quad (j = 0, 1, \ldots, n).
\]

We identify \( q'_j \) with \( q_j \) \( (j = 0, 1, \ldots, n) \) through \( \xi \).

Now \( \Delta^* \times \mathbb{C} - \mathcal{B}_f \) is a topological fiber bundle with the base space \( \Delta^* \) and the standard fiber \( \mathbb{C} - \{ n \text{ points} \} \) (see Dimca [2] and Matsuno [7]). Put
\[
G = \{ \alpha: \mathbb{C} \longrightarrow \mathbb{C} \mid \alpha \text{ is an orientation preserving homeomorphism such that} \}
\]
\[
\alpha(q_0) = q_0, \quad \alpha(\{ q_1, \ldots, q_n \}) = \{ q_1, \ldots, q_n \}
\].

\( G \) is then a topological group with compact-open topology. Let \( G_e \) be its connected component of the identity. Put
\[
\pi_0(G) = G/G_e.
\]

Then \( \pi_0(G) \) can be naturally identified with the Artin braid group \( B_n \) of \( n \) strings (see Birman [1, p. 165]).

Now assume that the above two families
\[
f = \{ f_t \}: X \longrightarrow \Delta \times \mathbb{P}^1,
\]
\[
f' = \{ f'_t \}: X' \longrightarrow \Delta' \times \mathbb{P}^1
\]
are topologically equivalent. We may assume that
\[
\eta(t_0) = t'_0,
\]
\[
\varphi: t_0 \times \mathbb{C} = \mathbb{C} \longrightarrow t'_0 \times \mathbb{C} = \mathbb{C},
\]
\[
\varphi(q_0) = q'_0.
\]

Let
\[
\chi: \pi_1(S^1) \longrightarrow \pi_0(G)
\]
(resp. \( \chi': \pi_1(S^1) \longrightarrow \pi_0(G) \))
be the characteristic homomorphism of the bundle
\[ \Delta^* \times \mathbb{C} - B_f \rightarrow \Delta^* \]
(resp. \( \Delta'^* \times \mathbb{C} - B_f \rightarrow \Delta'^* \))
(see Steenrod [12, p. 96]). Let \( \delta \) (resp. \( \delta' \)) be the loop around \( t = 0 \) as before.

Two bundles
\[ \Delta^* \times \mathbb{C} - B_f \quad \text{and} \quad \Delta'^* \times \mathbb{C} - B_f \]
over the base space \( \Delta^* \) (which is homeomorphic to \( (0, 1) \times S^1 \)) and \( \Delta'^* \) are weakly equivalent in the sense of Steenrod [12, p. 99]. Hence by Steenrod [12, p. 100], the characteristic \( \chi(\delta) \) and \( \chi'(\delta') \) of these bundles satisfy either
\[ \chi(\delta) = \chi'(\delta') \quad \text{or} \quad \chi(\delta) = \chi'(\delta')^{-1} \]
in \( \pi_0(G) \). The equality here is up to conjugacy in \( \pi_0(G) \). But the last equality does not occur by Steenrod [12, p. 100], for \( \eta \) is orientation preserving. Hence
\[ \chi(\delta) = \chi'(\delta') \quad \text{(up to conjugacy)}. \]

But \( \pi_0(G) \) can be identified with \( B_\eta \) as noted above. Under the identification, \( \chi(\delta) \) (resp. \( \chi'(\delta') \)) is equal to \( \theta(\delta) \) (resp. \( \theta'(\delta') \)), the braid monodromy. Hence by (3), there is \( \sigma \in B_\eta \) such that
\[ \theta'(\delta') = \sigma \theta(\delta) \sigma^{-1}. \]

Now the restriction
\[ \varphi: t_0 \times \mathbb{C} = \mathbb{C} \rightarrow t'_0 \times \mathbb{C} = \mathbb{C} \]
of \( \varphi \) is an orientation preserving homeomorphism. By the assumption of topological equivalence,
\[ [\Phi_{f_0} \cdot \varphi^{-1}_*] = [\Phi_{f'_0}]. \]

Consider an isotopy \( \varphi_t \) \((0 \leq t \leq 1)\) on \( \mathbb{C} \) such that \( \varphi_0 = \) the identity and \( \varphi_1 = \varphi \). This gives a braid \( \sigma \). We may write
\[ \varphi = \sigma. \]

Then by (5)
\[ [\Phi_{f_0} \cdot \sigma^{-1}] = [\Phi_{f'_0}]. \]
Now the braid \( \sigma \) in (4) and \( \sigma \) in (6) are the same. In fact, the braid \( \sigma \) in the relation
\[
\theta (\delta') = \sigma \theta (\delta) \sigma^{-1}
\]
is nothing but
\[
\sigma = \varphi : t_0 \times C \longrightarrow t_0' \times C
\]
if we regard \( \theta (\delta') \) and \( \theta (\delta) \) as elements of \( \pi_0 (G) \) (see Steenrod [12, p. 97–p. 98, p. 9–p. 12]). On the other hand, \( \sigma \) in (6) is also
\[
\sigma = \varphi : t_0 \times C \longrightarrow t_0' \times C.
\]
Hence the braid \( \sigma \) in (4) and \( \sigma \) in (6) are the same. Thus there is \( \sigma \in B_n \) such that
\[
([\Phi_f], \, \theta (\delta')) = ([\Phi_f \cdot \sigma^{-1}], \, \sigma \theta (\delta) \sigma^{-1}).
\]
Conversely, for two families in (1), we identify \( q_j' \) with \( q_j \) \((j = 0, 1, \ldots, n)\) through \( \xi \) in (2) and suppose that there is \( \sigma \in B_n \) such that
\[
([\Phi_f'], \, \theta (\delta')) = ([\Phi_f \cdot \sigma^{-1}], \, \sigma \theta (\delta) \sigma^{-1}).
\]
Since \( \theta (\delta') = \sigma \theta (\delta) \sigma^{-1} \), the above discussion shows that two bundles
\[
\Delta^* \times C - B_f \quad \text{and} \quad \Delta'^* \times C - B_{f'}
\]
over \( \Delta^* \) and \( \Delta'^* \) respectively are weakly equivalent. That is, there are orientation preserving homeomorphism \( \varphi \) and \( \eta \) such that (i) the following diagram commutes:
\[
\begin{array}{ccc}
\Delta^* \times C - B_f & \xrightarrow{\varphi} & \Delta'^* \times C - B_{f'} \\
\downarrow & & \downarrow \\
\Delta^* & \xrightarrow{\eta} & \Delta^*
\end{array}
\]
(ii) \( \eta(t_0) = t_0' \) and
(iii) \( \varphi = \sigma : t_0 \times C = C \longrightarrow t_0' \times C = C. \)
Now the fiber bundle structures on \( \Delta^* \times C - B_f \) and \( \Delta'^* \times C - B_{f'} \) can be naturally extended to those on \( \Delta^* \times \mathbb{C} \) and \( \Delta'^* \times \mathbb{C} \) respectively (see Lemma 2 in Matsuno [7]). Hence \( \varphi \) can be extended to an orientation preserving homeomorphism
\[
\varphi : \Delta^* \times \mathbb{P}^1 \longrightarrow \Delta'^* \times \mathbb{P}^1
\]
such that the following diagram commutes:

\[
\begin{array}{ccc}
\Delta^* \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta'^* \times \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\Delta^* & \xrightarrow{\eta} & \Delta'^*
\end{array}
\]

We show that \( \varphi \) and \( \eta \) can be extended so that the following diagram commutes:

\[
\begin{array}{ccc}
\Delta \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta' \times \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{\eta} & \Delta'
\end{array}
\]

We assume and put as in §5

\[
q_1(0) = \cdots = q_{k_1}(0) = q_1^0, \\
(\text{resp. } q'_1(0) = \cdots = q'_{k_1}(0) = q'_1),
\]

\[
q_{k_1+1}(0) = \cdots = q_{k_1+k_2}(0) = q_2^0, \\
(\text{resp. } q'_{k_1+1}(0) = \cdots = q'_{k_1+k_2}(0) = q'_2),
\]

\[
\ldots \\
q_{k_1+\cdots+k_{r-1}+1}(0) = \cdots = q_{k_1+\cdots+k_{r-1}+k_r}(0) = q_r^0, \\
(\text{resp. } q'_{k_1+\cdots+k_{r-1}+1}(0) = \cdots = q'_{k_1+\cdots+k_{r-1}+k_r}(0) = q'_r),
\]

where \( k_\nu \geq 1 \) (\( \nu = 1, \ldots, r \)), \( k_1 + \cdots + k_r = n \) and \( q_1^0, \ldots, q_r^0 \) (resp. \( q'_1, \ldots, q'_r \)) are mutually distinct.

We may assume that there is a continuous function \( \rho(|t|) \) of \( |t| \) such that

(i) \( \rho(|t|) \) > 0 for \( |t| > 0 \),

(ii) \( \rho(0) = 0 \),

(iii) \( \Delta(q_\nu^0, \rho(|t|)) \) (resp. \( \Delta(q'_\nu^0, \rho(|t|)) \)) (\( \nu = 1, \ldots, r \)) are mutually disjoint,

(iv) each \( \Delta(q_\nu^0, \rho(|t|)) \) (resp. \( \Delta(q'_\nu^0, \rho(|t|)) \)) (\( \nu = 1, \ldots, r \)) contains

\[
q_{k_1+\cdots+k_{r-1}+1}(t), \ldots, q_{k_1+\cdots+k_{r-1}+k_r}(t) \\
(\text{resp. } q'_{k_1+\cdots+k_{r-1}+1}(t), \ldots, q'_{k_1+\cdots+k_{r-1}+k_r}(t)).
\]

Now Lemma 2 in Matsuo [7] implies that the bundle structure on \( \Delta^* \times \mathbb{C} - B_f \) coincides with that of the product bundle \( \Delta^* \times \mathbb{C} \) outside

\[
T = \bigcup_{0 < |t| < T} \bigcup_{\nu = 1}^r \Delta(q_\nu^0, \rho(|t|)).
\]

Similar assertion holds for the bundle structure on \( \Delta'^* \times \mathbb{C} - B_{f'} \). Hence we may assume that \( \varphi \) does not depend on \( t \) outside \( T \). Thus \( \varphi \) can be extended to an orientation
preserving homeomorphism

\[ \varphi: \Delta \times \mathbb{C} - \{q^0_1, \ldots, q^0_r\} \longrightarrow \Delta' \times \mathbb{C} - \{q'^0_1, \ldots, q'^0_r\}. \]

Moreover if we define

\[ \varphi(q^0_\nu) = q'^0_\nu \quad (\nu = 1, \ldots, r), \]

then \( \varphi \) is extended to an orientation preserving homeomorphism

\[ \varphi: \Delta \times \mathbb{P}^1 \longrightarrow \Delta' \times \mathbb{P}^1. \]

Put also \( \eta(0) = 0 \). Then \( \eta \) is extended to an orientation preserving homeomorphism

\[ \eta: \Delta \longrightarrow \Delta' \]

and the following diagram commutes:

\[
\begin{array}{ccc}
\Delta \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta' \times \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{\eta} & \Delta'
\end{array}
\]

Next, note that

\[ \varphi(B_f) = B_{f'}, \]

\[ \varphi = \sigma: t_0 \times \mathbb{C} \longrightarrow t'_0 \times \mathbb{C}. \]

Note also that

\[ \varphi \cdot f: X \longrightarrow \Delta' \times \mathbb{P}^1 \]

is unbranched on \( \Delta' \times \mathbb{P}^1 - B_{f'} \). By Theorem 1, \( \varphi \cdot f \) can be extended to a branched covering

\[ f'': X'' \longrightarrow \Delta' \times \mathbb{P}^1. \]

\( \varphi \cdot f \) and \( f'' \) coincides on \( \Delta' \times \mathbb{C} - B_{f'} \) and both are Fox completions of the same unbranched coverings of \( \Delta' \times \mathbb{C} - B_{f'}. \) Hence by the uniqueness of the Fox completion (see Fox [3]), there is a homeomorphism

\[ \psi': X \longrightarrow X'' \]
such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\psi'} & X'' \\
\varphi \cdot f & \downarrow & \downarrow f'' \\
\Delta' \times \mathbb{P}^1 & \xrightarrow{id} & \Delta' \times \mathbb{P}^1
\end{array}
\]

Note that \(\psi'\) is orientation preserving. Now, the representation class of the monodromy of \(f''\) is equal to that of \(\varphi \cdot f\), which clearly equal to \([\Phi_f \cdot \varphi^{-1}]\). By the assumption \([\Phi_f \cdot \varphi^{-1}] = [\Phi_f \cdot \sigma^{-1}] = [\Phi_f]\), we have

\([\Phi_{f'}] = [\Phi_f]\).

Hence there is a biholomorphic mapping

\(\psi'' : X'' \to X'\)

which makes the following diagram commutative:

\[
\begin{array}{ccc}
X'' & \xrightarrow{\psi''} & X' \\
\downarrow f'' & & \downarrow f' \\
\Delta' \times \mathbb{P}^1 & \xrightarrow{id} & \Delta' \times \mathbb{P}^1
\end{array}
\]

Now put \(\psi = \psi'' \cdot \psi'\). Then

\(\psi : X \to X'\)

is an orientation preserving homeomorphism which makes the following diagram commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{\psi} & X' \\
\downarrow f & & \downarrow f' \\
\Delta \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta' \times \mathbb{P}^1
\end{array}
\]

Hence \(f = \{f_t\}\) and \(f' = \{f'_t\}\) are topologically equivalent. \(\square\)

Considering a trivial family, we get the following corollary, which can be also derived directly from Theorem 2.
Corollary 2 (cf. Wajnryb [13]). There exists a one to one correspondence between \( \{ \text{topological equivalence class of } f : X \to \mathbb{P}^1 \text{ of degree } d \text{ with } n \text{ branched points } q_1, \ldots, q_n \} \) and \( \{ [\Phi] \mid [\Phi] \text{ is the representation class of } \Phi : \pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0) \to S_d \text{ such that } \text{Im } \Phi \text{ is transitive} \} / B_n \).

Remark. As one can see in the proof of Theorem 8, we do not need to mention the points \( \{ q_1, \ldots, q_n \} \) in the statement of Theorem 8 and its corollary, if we replace \( \pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0) \) by the abstract group

\[ \langle \gamma_1, \ldots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1 \rangle. \]

7. Degenerating families of finite branched coverings of \( \mathbb{P}^m \)

Let \( \Delta = \Delta(0, \epsilon) \) be a disc and

\[ f : X \to \Delta \times \mathbb{P}^m \]

be a finite branched covering. As in the case of \( \mathbb{P}^1 \), \( f \) is called a degenerating family of finite branched covering of \( \mathbb{P}^m \) and is denoted by \( f = \{ f_t \} \) if the following 4 conditions are satisfied

1. \( t \times \mathbb{P}^m \not\subset B_f \) for every \( t \in \Delta \).
2. For every \( t \in \Delta^* \), \( t \times \mathbb{P}^m \) meets transversally with \( B_f \) and putting \( (t \times \mathbb{P}^m) \cap B_f = t \times B_f \), \( B_f \) is a hypersurface of \( \mathbb{P}^m \) of degree \( n. (n \text{ is constant for } t \in \Delta^*.) \)
3. For every \( t \in \Delta^* \),

\[ f_t = f : X_t = f^{-1}(t \times \mathbb{P}^m) \to t \times \mathbb{P}^m \]

is a covering of \( \mathbb{P}^m \) of degree \( d = \deg(f) \) branching at \( B_t \).

4. For any points \( t \) and \( t' \) in \( \Delta^* \), \( f_t \) and \( f_{t'} \) are topologically equivalent.

The central fiber \( X_0 = f^{-1}(0 \times \mathbb{P}^m) \) is a degeneration of a general fiber \( X_t \) for \( t \neq 0 \).

We show that, topologically, the central fiber \( X_0 \) can be described by the central branch divisor \( B_0 \), where \( 0 \times B_0 = (0 \times \mathbb{P}^m) \cap B_f \) and by the monodromy \( \Phi_t = \Phi_{f_t} \), where \( t \in \Delta^* \) is a fixed point. We explain this as follows:

Let \( L \) be a general line in \( \mathbb{P}^m \). We may assume that \( L \) meets transversally with every \( B_t \) for \( t \in \Delta \). Consider the restriction

\[ f^L : X^L = f^{-1}(\Delta \times L) \to \Delta \times L \]

of \( f \) to \( X^L = f^{-1}(\Delta \times L) \). Then

Lemma 2. (1) Every point of \( (X - f^{-1}(B_f)) \cap X^L \) is a non-singular point of \( X^L \).
(2) For \( t \neq 0 \), every point of \( f^{-1}(\text{Reg}(B_f) \cap (t \times L)) \) is non-singular point of \( X^L \). 
(\text{Reg}(B_f) \) is the set of non-singular points of the branch locus \( B_f \).)

(3) For \( t \neq 0 \), the restriction

\[
f_t^L : X_t^L = f^{-1}(t \times L) \longrightarrow t \times L
\]

of \( f^L \) is a branched covering of degree \( d = \deg(f) \).

Proof. (1) Let \( p \in (X - f^{-1}(B_f)) \cap X^L \). Then there are local coordinate systems

\((t, x_1, \ldots, x_m) \) and \((t, y_1, \ldots, y_m) \) around \( p \) in \( X \) and \( q = f(p) \) in \( \Delta \times \mathbb{P}^m \) such that

(i) \( t \) is a local coordinate system in \( \Delta \) and \((y_1, \ldots, y_m) \) is in \( \mathbb{P}^m \),
(ii) \( L \) is locally given by the equation \( y_2 = \cdots = y_m = 0 \) and
(iii) \( f \) is locally given by

\[
f : (t, x_1, \ldots, x_m) \longmapsto (t, y_1, \ldots, y_m) = (t, x_1, \ldots, x_m).
\]

Then \( f^L \) is locally given by

\[
f^L : (t, x_1) \longmapsto (t, y_1) = (t, x_1),
\]

In particular, \( p \) is a non-singular point of \( X^L \).

(2) Let \( l_0 \neq 0 \) and \( p \in f^{-1}(\text{Reg}(B_f) \cap (l_0 \times L)) \). Then there are local coordinate systems

\((t, x_1, \ldots, x_m) \) and \((t, y_1, \ldots, y_m) \) around \( p \) in \( X \) and \( q = f(p) \) in \( \Delta \times \mathbb{P}^m \) such that

(i) \( t \) is a local coordinate system in \( \Delta \) around \( l_0 \) and \((y_1, \ldots, y_m) \) is in \( \mathbb{P}^m \),
(ii) \( L \) is locally given by the equation \( y_2 = \cdots = y_m = 0 \) and
(iii) \( f \) is locally given by

\[
f : (t, x_1, x_2, \ldots, x_m) \longmapsto (t, y_1, y_2, \ldots, y_m) = (t, x_1^e, x_2, \ldots, x_m).
\]

Then \( f^L \) is locally given by

\[
f^L : (t, x_1) \longmapsto (t, y_1) = (t, x_1^e).
\]

In particular, \( p \) is a non-singular point of \( X^L \). Moreover \( p \) is a ramification point of
\( f_{l_0}^L \) with the ramification index \( e \).

(3) For \( t \neq 0 \), the branched covering

\[
f_t : X_t \longrightarrow \mathbb{P}^m
\]

gives a linear system on \( X_t \). By Bertini’s theorem, \( X_t^L \) is non-singular and globally
irreducible. Hence, by the proof of (2),

\[
f_t^L : X_t^L \longrightarrow t \times L
\]
is a branched covering of degree \( d = \text{deg}(f) \).

This lemma shows that the singular locus \( \text{Sing}(X^L) \) of \( X^L \) is contained in \( f^{-1}(0 \times (B_0 \cap L)) \), which is a finite set. \( X^L \) is globally irreducible. Let

\[
\mu : \tilde{X}^L \rightarrow X^L
\]

be the normalization of \( X^L \). Since \( \text{Sing}(X^L) \) is a finite set, \( \mu \) is a bijective holomorphic mapping. In fact, suppose that there are distinct points \( p_1 \) and \( p_2 \) in \( \tilde{X}^L \) such that

\[
p = \mu(p_1) = \mu(p_2) \in f^{-1}(0 \times (B_0 \cap L)).
\]

Then there are disjoint connected open neighborhoods \( W_1 \) and \( W_2 \) of \( p_1 \) and \( p_2 \) respectively such that

\[
\mu(W_1) = \mu(W_2) = W
\]

and \( W \) is connected open neighborhood of \( p \) in \( X^L \). Since \( f^{-1}(0 \times (B_0 \cap L)) \) is a finite set, we may assume that

\[
f^{-1}(0 \times (B_0 \cap L)) \cap W = \{ p \}.
\]

We may assume that \( X^L_t \cap W \) is connected for non-zero \( t \) with \( |t| \) sufficiently small. Hence \( W - \{ p \} \) is a connected 2-dimensional complex manifold. Since \( \mu \) is the normalization of \( X^L \),

\[
W_1 - \{ p_1 \} = W_2 - \{ p_2 \}
\]

and

\[
\mu : W_1 - \{ p_1 \} = W_2 - \{ p_2 \} \rightarrow W - \{ p \}
\]

is biholomorphic, a contradiction. Thus \( \mu \) is bijective. The composition

\[
f^L \cdot \mu : \tilde{X}^L \rightarrow \Delta \times L
\]

is a degenerating family of finite branched coverings of \( L = \mathbb{P}^1 \), which we denote

\[
f^L \cdot \mu = \{ f^L_t \}
\]

by abuse of notation.

**Lemma 3.** (1) Let \( X_0 = X_{01} \cup \cdots \cup X_{0t} \) be the global irreducible decomposition of \( X_0 \). Then

\[
X^L_0 = (X_{01} \cap X^L_0) \cup \cdots \cup (X_{0t} \cap X^L_0)
\]
is the global irreducible decomposition of $X^L_0$.

(2) Let $\text{Sing}_m(X_0)$ be the union of global irreducible components of $\text{Sing}(X_0)$ which are hypersurfaces of $X_0$. Then (i) $\text{Sing}_m(X_0) \subset f_0^{-1}(B_0)$ and (ii) $\text{Sing}_m(X_0) \cap X^L_0 = \text{Sing}(X^L_0)$.

(3) For a point $p \in \text{Sing}(X^L_0)$, let

$$(X_0)_p = Z_1 \cup \cdots \cup Z_u$$

be the local irreducible decomposition of $X_0$ at $p$. Then the local irreducible decomposition of $X^L_0$ at $p$ is given by

$$(X^L_0)_p = (Z_1 \cap X^L_0) \cup \cdots \cup (Z_u \cap X^L_0).$$

Proof. (1) Let

$$\mu_j : \hat{X}_{0j} \rightarrow X_{0j}$$

$(1 \leq j \leq u)$ be the normalization of $X_{0j}$. By the proof of (1) of Lemma 2,

$$f_{0j} : \mu_j : \hat{X}_{0j} \rightarrow 0 \times \mathbb{P}^m \quad (f_{0j} = f_0 \mid X_{0j})$$

is a finite branched covering. By Bertini’s theorem, $(f_{0j} \cdot \mu_j)^{-1}(0 \times L)$ is a non-singular connected curve of $\hat{X}_{0j}$. Hence $f_{0j}^{-1}(0 \times L)$ is a global irreducible component of $X^L_0$ and

$$X^L_0 = f_0^{-1}(0 \times L) = \bigcup_{j=1}^u f_{0j}^{-1}(0 \times L) = \bigcup_{j=1}^u (X_{0j} \cap X^L_0)$$

is the irreducible decomposition of $X^L_0$.

(2) By (2) of Lemma 2, every component of $\text{Sing}_m(X_0)$ is a global irreducible component $R_0$ of $f_0^{-1}(B_{01})$, where $B_{01}$ is a global irreducible component of $B_0$. Let $p$ be a point of $X^L_0 \cap R_0$. Then $p$ is clearly a singular point of $X^L_0$. Conversely, if $p$ is a singular point of $X^L_0$, then $f_0(p) = q$ is on $L \cap B_{01}$ for an irreducible component $B_{01}$ of $B_0$. Since $L$ is a general line, every point on a global irreducible component $R_0$ with $p \in R_0$ of $f_0^{-1}(B_{01})$ is a singular point of $X_0$. Hence $R_0$ is a component of $\text{Sing}_m(X_0)$. This shows (i) and (ii) of (2).

(3) We use the same notation as in the proof of (1). Every $Z_k$ is an open set of some $X_{0j}$. Hence

$$\mu_{jk} : \hat{Z}_k = \mu_j^{-1}(Z_k) \rightarrow Z_k \quad (\mu_{jk} = \mu_j \mid \hat{Z}_k)$$

is the normalization of $Z_k$. $(f_{0jk} \mu_{jk})^{-1}(0 \times L)$ is a non-singular connected curve of $\hat{Z}_k$, where $f_{0jk} = f_{0j} \mid Z_k$. Hence $f_{0jk}^{-1}(0 \times L) = Z_k \cap X^L_0$ is a local irreducible component
of $X^L_0$ at $p$ and

$$(X^L_0)_p = (Z_1 \cap X^L_0) \cup \cdots \cup (Z_n \cap X^L_0)$$

is the local irreducible decomposition of $X^L_0$ at $p$. □

Now we refer to a theorem of Zariski-van Kampen. Let $B$ be a hypersurface of degree $n$ in $\mathbb{P}^m$. Take a general point $q_0$ in $\mathbb{P}^m - B$ and let

$$\pi : \mathbb{P}^m - \{q_0\} \longrightarrow \mathbb{P}^{m-1}$$

be the projection with the center $q_0$. Put

$$\hat{\pi} = \pi \mid B : B \longrightarrow \mathbb{P}^{m-1}$$

be the restriction. Let $D$ be the branch locus of $\hat{\pi}$. A theorem of Zariski-van Kampen in this case can be described as follows (cf. Matsuno [7]).

**Theorem 9** (Zariski-van Kampen).

$$\pi_1(\mathbb{P}^m - B, q_0) = \langle \gamma_1, \ldots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1, \theta(\delta_k)\gamma_j = \gamma_j (1 \leq j \leq n, 1 \leq k \leq s) \rangle,$$

where $\gamma_j$ are lassos as in Fig. 2 on $\pi^{-1}(q_0)$, the line deleted the point $\{q_0\}$, $\delta_k$ are the generators of $\pi_1(\mathbb{P}^{m-1} - D, r_0)$ for a reference point $r_0 \in \mathbb{P}^{m-1} - D$, and $\theta(\delta_k)$ are the braid monodromy along $\delta_k$.

This theorem shows in particular that the monodromy $\Phi_{f_t}$ of $f_t$ is equal to the monodromy $\Phi^L_{f_t}$ of $f^L_t$ for a general line $L$ passing through $q_0$. Hence, we conclude by Theorems 4, 5, 6 and Lemma 3 that topologically, the central fiber $X_0$ can be determined by the central branched divisor $B_0$ and by the monodromy $\Phi_t = \Phi_{f_t}$, where $t \in \Delta^*$ is a fixed point.

**Remark.** If $\deg B_0 = \deg B_t \ (t \neq 0)$, then there is a surjective homomorphism

$$\pi_1(\mathbb{P}^m - B_0, q_0) \longrightarrow \pi_1(\mathbb{P}^m - B_t, q_0) \longrightarrow 0$$

(see Zariski [14]). In this case, $X_0$ is irreducible and

$$\dim \operatorname{Sing}(X_0) \leq m - 2.$$

Hence degenerations such that

$$\dim \operatorname{Sing}(X_0) = m - 1$$
happen only if \( \deg B_0 < \deg B_t \) \((t \neq 0)\), that is, only if \( B_0 \) has a multiple component as a divisor.

References