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LOGARITHMIC JETS AND HYPERBOLICITY

Jawher El GOUL

0. Introduction

In 1970 S. Kobayashi [21] posed the following problem: Is it true that the complement of a generic hypersurface of degree $d \geq e(n)$ in $\mathbb{P}^n_{\mathbb{C}}$ is hyperbolic for some number $e(n)$? Is this true for $e(n) = 2n + 1$? Later Green [15] for $n = 2$ and Zaidenberg [35] for arbitrary n proved that for $d \leq 2n$ such complements contain $\mathbb{C}^{\star}$ and so are not hyperbolic.

In this paper we study the case of complements of smooth curves in $\mathbb{P}^2_{\mathbb{C}}$ where, for $d \geq 4$ this problem is equivalent to the nonexistence of nonconstant entire curves by Brody Reparametrization Lemma [4]. When the curve has many components, this problem had been studied by many authors, see [12, 13] for a complete bibliography and the study of the case of three components (see also the recent work [2]). In the smooth case, the first example of hyperbolic complement was constructed by Azukawa and Suzuki in [1] (for even degree $d \geq 30$), then Zaidenberg in [36] showed that examples exist for all $d \geq 5$.

The first positive answer to this problem (for $n = 2$) was given in the work of Siu and Yeung [32] though the bound they obtain is quite high. Their method is rather involved and consists of an explicit construction of special second order differential operators on an associated surface in $\mathbb{P}^3_{\mathbb{C}}$ ramified over $\mathbb{P}^2_{\mathbb{C}}$. This was done by an imitation of the construction of holomorphic 1-forms on Riemann surfaces and a clever reduction of the problem to a resolution of linear systems. Those operators are such that their pullbacks by the lifting of every entire curve must vanishes identically. This follows from an Ahlfors-Schwarz type result.

In [10], after studying the compact analogue of the above conjecture and proving that a generic suface of degree $d \geq 21$ in $\mathbb{P}^3_{\mathbb{C}}$ is hyperbolic, Demailly and the author, using the same covering trick, obtained the bound 21 also for complementary of curves in $\mathbb{P}^2_{\mathbb{C}}$ (see also [25]). This is made by using the whole force of Demailly's jet bundles introduced in [8] and a result of McQuillan on holomorphic foliations [24].

In this paper, we obtain the bound 15 for the complement of a very generic curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$. Here, the terminology "very generic" refers to complements of countable unions of proper algebraic subsets of the parameter space. Our main theorem is the following

Main Theorem. *The complement of a very generic curve in $\mathbb{P}^2_{\mathbb{C}}$ is hyperbolic and hyperbolically imbedded for all degrees $d \geq 15$.*

We follow almost the same strategy as in [10] with the difference that we use an analogous logarithmic package introduced in [11], which introduces some additionnal technical complications. Dethloff and Lu's jet bundles are a compactification à la Demailly of Noguchi's logarithmic jet bundles introduced in [29]. Using Riemann-Roch and a refined study of the base loci associated to those jet bundles, we reduce the problem to the study of holomorphic foliations on log-general type surfaces. We prove that such foliations do not admit a parabolic Zariski-dense leaf. We generalize, in particular, McQuillan's result [24] on Green-Griffiths conjecture to log-general type surfaces with $\overline{c}_1^2 > \overline{c}_2$. This logarithmic point of view permits the observation that McQuillan's refined tautological inequlity is actually an easy consequence of a logarithmic tautological inequality obtained by [34] and proved in the same way as the simple non-logarithmic one (see also [7]).

The paper is organized as follows: In Section 1 we first recall the main definitions and results in [11]. Then we introduce the 2-jet threshold of a log-general type surface (X, C). We consider the case when the Picard group is $\mathbb{Z}$ and we construct, with some conditions on log-Chern classes and the 2-jet threshold, a ramified cover $\tilde{X}$ on X such that every entire curve in the complement $X \backslash C$ could be lifted as a leaf of a holomorphic foliation on $\tilde{X}$ (in this paper, we always identify a map from $\mathbb{C}$ and its image). The last part of this section is devoted to estimating the 2-jet threshold in the case of $(\mathbb{P}^2_{\mathbb{C}}, C)$ where C a generic smooth plane curve.

In Section 2 we study foliations on log-general type surfaces as in [24]. The method we adopt is paralell to Brunella's work [6]. We obtain that these foliations do not have a Zariski dense parbolic leaf.

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1. Logarithmic Demailly jet bundles

1.1. Background material. Here we will consider a logarithmic generalization of Demailly's invariant jets introduced in [8], this is done by Dethloff and Lu in [11].

Let X be an n-dimensional complex manifold with a normal crossing divisor D. According to Iitaka [20], the logarithmic cotangent sheaf $\overline{T}^\star_X = T^\star_X(\log D)$ is defined to be the locally free sheaf locally generated by $T^\star_X$ and the logarithmic differentials ds_j/s_j, where $s_j = 0$ is a local equation for the j-th local irreducible components of D. Its dual the logarithmic tangent sheaf is the sheaf of germs of vector fields tangent to D, denoted by $\overline{T}_X = T_X(-\log D)$.

Recall from [17] that the k-jet bundle $J_k X$ is defined as the set of equivalence classes of holomorphic maps $f \colon (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if they have the same Taylor expansions of order k in some local

coordinate system of X near x. We denote the equivalence class of f by $j_k(f)$. In [29], Noguchi generalized this object to the logarithmic situation as follows. Let $\omega \in H^0(U, T_X^\star)$ be a holomorphic section over an open subset $U \subset X$. For a germ of a holomorphic map f in U we put $f^\star \omega = Z(t)\,dt$. Then we have a well defined holomorphic mapping

$$\tilde{\omega} \colon J_k X|_U \to \mathbb{C}^k \ ; \ j_k(f) \to (Z^{(j)}(0))_{0 \le j \le k-1}.$$

Now we say that a holomorphic section $s \in H^0(U, J_k X)$ is a logarithmic k-jet field if the map $\tilde{\omega} \circ s|_V \colon V \to \mathbb{C}^k$ is holomorphic for all $\omega \in H^0(V, \overline{T}_X^\star)$ and for all open subset V of U. The set of logarithmic k-jet fields over open subsets of X defines a subsheaf of $J_k X$ called the logarithmic k-jet bundle of (X, D), which we denote by $\overline{J}_k X$, and this subsheaf is the sheaf of sections of a holomorphic fibre bundle.

In [11], Dethloff and Lu constructed a more geometrically relevent $k-$jet bundles (in the same way as done in [8] for the non-logarithmic case) by considering a suitable "quotient" of this bundle $\overline{J}_k X$ by the action of the group $\mathbb{G}_k$ containing all germs of k-jets biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \qquad a_1 \in \mathbb{C}^\star, \ a_j \in \mathbb{C}, \ j \ge 2.$$

As a generalization of Demailly's directed jets to the logarithmic context, Dethloff and Lu defined a log-directed manifold to be the triple (X, D, V) where V is a holomorphic subbundle of $\overline{T}_X$ of rank r. To the log-directed manifold (X, D, V), one associates inductively a sequence of directed manifolds $(\overline{X}_k, D_k, V_k)$ as follows. Starting with $(\overline{X}_0, D_0, V_0) = (X, D, V)$, one puts inductively $\overline{X}_k = P(V_{k-1})$ with its natural projection π_k to $\overline{X}_{k-1}$ (where $P(V)$ stands for the projectivized bundle of lines in the vector bundle V), where $D_k = \pi_k^\star(D_{k-1})$ and V_k is the subbundle of $T_{\overline{X}_k}(-\log D_k)$ defined at any point $(x, [v]) \in \overline{X}_k$, $v \in V_{k-1,x}$, by

$$V_{k,(x,[v])} = \left\{ \xi \in T_{\overline{X}_k,(x,[v])}(-\log D_k) \ ; \ (\pi_k)_\star \xi \in \mathbb{C} \cdot v \right\},$$

with $\mathbb{C} \cdot v \subset V_{k-1,x} \subset T_{\overline{X}_{k-1},x}(-\log D_{k-1})$.

We denote by $\mathcal{O}_{\overline{X}_k}(-1)$ the tautological line subbundle of $\pi_k^\star V_{k-1}$, such that

$$\mathcal{O}_{\overline{X}_k}(-1)_{(x,[v])} = \mathbb{C} \cdot v,$$

for all $(x, [v]) \in \overline{X}_k = P(V_{k-1})$. By definition, the bundle V_k fits in an exact sequence

$$0 \longrightarrow T_{\overline{X}_k/\overline{X}_{k-1}} \longrightarrow V_k \xrightarrow{\pi_{k\star}} \mathcal{O}_{\overline{X}_k}(-1) \longrightarrow 0,$$

and the Euler exact sequence of $T_{\overline{X}_k/\overline{X}_{k-1}}$ yields

$$0 \longrightarrow \mathcal{O}_{\overline{X}_k} \longrightarrow \pi_k^\star V_{k-1} \otimes \mathcal{O}_{\overline{X}_k}(1) \longrightarrow T_{\overline{X}_k/\overline{X}_{k-1}} \longrightarrow 0.$$

From these sequences, we infer

$$\text{rank } V_k = \text{rank } V_{k-1} = \cdots = \text{rank } V = r, \qquad \dim \overline{X}_k = n + k(r-1).$$

We let

$$\pi_{k,j} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : \overline{X}_k \longrightarrow \overline{X}_j,$$

be the natural projection.

The canonical injection $\mathcal{O}_{\overline{X}_k}(-1) \hookrightarrow \pi_k^\star V_{k-1}$ and the exact sequence

$$0 \longrightarrow T_{\overline{X}_{k-1}/\overline{X}_{k-2}} \longrightarrow V_{k-1} \overset{(\pi_{k-1})_\star}{\longrightarrow} \mathcal{O}_{\overline{X}_{k-1}}(-1) \longrightarrow 0$$

yield a canonical line bundle morphism

$$\mathcal{O}_{\overline{X}_k}(-1) \overset{(\pi_{k,k-2})^\star \circ (\pi_{k-1})_\star}{\hookrightarrow} \pi_k^\star \, \mathcal{O}_{\overline{X}_{k-1}}(-1)$$

which admits precisely the hyperplane section $\Gamma_k := P(T_{\overline{X}_{k-1}/\overline{X}_{k-2}}) \subset \overline{X}_k = P(V_{k-1})$ as its zero divisor. Hence we find $\mathcal{O}_{\overline{X}_k}(-1) = \pi_k^\star \, \mathcal{O}_{\overline{X}_{k-1}}(-1) \otimes \mathcal{O}(-\Gamma_k)$ and using the notation $\mathcal{O}_{\overline{X}_k}(a_1, a_2) := \pi_k^\star \mathcal{O}_{\overline{X}_{k-1}}(a_1) \otimes \mathcal{O}_{\overline{X}_k}(a_2)$,

$$\mathcal{O}_{\overline{X}_k}(-1, 1) \simeq \mathcal{O}(\Gamma_k)$$

is associated with an effective divisor in $\overline{X}_k$.

For simplicity let us consider the case $V = \overline{T}_X$ and let $f: \Delta_r \to X \setminus D$ be a non-constant trajectory tangent to V. Then f lifts to a well defined and unique trajectory $f_{[k]}: \Delta_r \to \overline{X}_k \setminus D_k$ of $\overline{X}_k$ tangent to V_k. Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]}: T_{\Delta_r} \to f_{[k]}^\star \mathcal{O}_{\overline{X}_k}(-1).$$

With any section σ of $\mathcal{O}_{\overline{X}_k}(m)$, $m \geq 0$, on any open set $\pi_{k,0}^{-1}(U)$, $U \subset X \setminus D$, we can associate a holomorphic differential operator Q of order k acting on k-jets of germs of curves $f: (\mathbb{C}, 0) \to U$ tangent to V, by putting

$$Q(f)(t) = \sigma(f_{[k]}(t)) \cdot f'_{[k-1]}(t)^{\otimes m} \in \mathbb{C}.$$

From [8], this correspondence is, in fact, bijective. To see what happen with logarithmic jets recall the following characterization of log-jet differentials in [11]:

Proposition 1.1.1 ([11]). *A holomorphic function Q on $\overline{J}_k X|_U$ on some connected open subset $U \subset X$ which satisfies*

$$(*) \qquad Q(j_k(f \circ \phi)) = \phi'(0)^m Q(j_k(f)) \quad \forall j_k(f) \in J_k X|_V \quad \text{and} \quad \forall \phi \in \mathbb{G}_k$$

over some open subset V of $U \setminus D$ defines a holomorphic section of $\mathcal{O}_{\overline{X}_k}(m)$ over U, and vice versa.

From the definition of $\overline{J}_k X$, one can deduce that the k-th derivative of the functin $\log s_j(f)$ is holomorphic on $\overline{J}_k X$ over the j-th local component of D where s_j is defined. Using this, and the fact that holomorphic functions satisfying $(*)$ for all $\phi \in \mathbb{C}^\star \subset \mathbb{G}_k$ are homogenous polynomials of degree m, we obtain

Proposition 1.1.2. *The direct image $(\pi_{k,0})_\star \mathcal{O}_{\overline{X}_k}(m)$ coincides with the sheaf $\mathcal{O}(E_{k,m}\overline{T}^\star{}_X)$ of degree m logarithmic jet differentials, that is, the locally free sheaf locally generated by all polynomial operators in the derivatives of order $1, 2, \ldots, k$ of f and of the $\log s_j(f)'s$ invariant under arbitrary reparametrization: a germ of operator $Q \in H^0(U, E_{k,m}\overline{T}^\star{}_X)$ is given by a holomorphic function Q on $\overline{J}_k X|_U$ characterized by the condition that, for every germ f in $X \setminus D$ and every germ φ of k-jet biolomorphisms of $(\mathbb{C}, 0)$,*

$$Q(f \circ \varphi) = \varphi'^m \, Q(f) \circ \varphi.$$

A basic result from [11] relying on the Ahlfors-Schwarz lemma, is the following, for the 1-jet case see [28] and [23].

Theorem 1.1.3 ([11]). *If (X, D) has a k-jet metric h_k (i.e. a singular metric in the sens of Demailly on $\mathcal{O}_{\overline{X}_k}(-1)$) with negative curvature (along V_k), then every entire curve $f : \mathbb{C} \to X \setminus D$ is such that $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$, where Σ_{h_k} denotes the singular set of h_k.*

An important case where the previous theorem applies is when there are some integers $k, m > 0$ and an ample line bundle A on X such that

$$H^0(\overline{X}_k, \mathcal{O}_{\overline{X}_k}(m) \otimes (\pi_{k,0})^\star A^{-1}) \simeq H^0(X, E_{k,m}\overline{T}^\star{}_X \otimes A^{-1})$$

has nonzero sections $\sigma_1, \ldots, \sigma_N$. Then, we can construct a k-metric of negative curvature, singular on their base locus $Z \subset \overline{X}_k$.

By definition, a line bundle L is *big* if there exists an ample divisor A on X such that $L^{\otimes m} \otimes \mathcal{O}(-A)$ admits a nontrivial global section when m is large (then there are lot of sections, namely $h^0(X, L^{\otimes m} \otimes \mathcal{O}(-A)) \gg m^n$ with $n = \dim X$).

As a consequence, Theorem 1.1.3 can be applied when $\mathcal{O}_{\overline{X}_k}(1)$ is big or its restriction on some subvariety is big (see Theorem 4.3 in [11]).

In view of studying the degeneration of entire curve drawn on a variety of log-general type and of Theorem 1.1.3, it is especially interesting to compute the stable

base loci of the global sections of log-jet differentials, that is, the intersections

$$\overline{B}_k := \bigcap_{m>0} \overline{B}_{k,m} \subset \overline{X}_k$$

of the base loci $\overline{B}_{k,m}$ of all line bundles $\mathcal{O}_{\overline{X}_k}(m) \otimes \pi_{k,0}^{\star}\mathcal{O}(-A)$, where the intersections are in fact independent of the ample divisor A over X. We call B_k the k-th tautological stable base locus.

REMARK 1.1.4. The 1-jet case was studied in [23]. Using Riemann-Roch [19], we prove that if (X, D) is a nonsingular surface of log-general type with logarithmic Chern classes $\overline{c}_1^2 > \overline{c}_2$ then there is a lot of log-symmetric differentials, i.e. sections in $E_{1,m}\overline{T}^{\star} = S^m\overline{T}_X^{\star}$, and the base locus $\overline{B}_1$ is 2-dimentionnal. Unfortunately, the "order 1" techniques are insufficient to deal with complement of smooth curve C of degree d in $\mathbb{P}^2$, because in this case

$$\overline{c}_1^2 = (d-3)^2 < \overline{c}_2 = (d^2 - 3d + 3).$$

Lemma 1.4.1 below shows in fact that $H^0(X, S^m\overline{T}_X^{\star}) = 0$ for all $m > 0$.

1.2. Base locus of logarithmic 2-jet differentials. From now on, we suppose that (X, D) is a nonsingular minimal surface of log-general type (i.e. with $\overline{K}_X := K_X \otimes \mathcal{O}(D)$ big and nef) and we study the base locus $\overline{B}_2$ in $\overline{X}_2$. As in the non-logarithmic case, the bundle of log-jet differentials of order 2 has the following filtration

$$\mathrm{Gr}^\bullet E_{2,m}\overline{T}_X^{\star} = \bigoplus_{0 \le j \le m/3} S^{m-3j}\overline{T}_X^{\star} \otimes \overline{K}_X^{\otimes j}.$$

This filtration consists of writing an invariant polynomial log-differential operator outside D as

$$Q(f) = \sum_{0 \le j \le m/3} \ \sum_{\alpha \in \mathbb{N}^2, \, |\alpha|=m-3j} a_{\alpha,j}(f)\,(f')^{\alpha}(f' \wedge f'')^j$$

where

$$f = (f_1, f_2), \qquad (f')^{\alpha} = (f_1')^{\alpha_1}(f_2')^{\alpha_2}, \qquad f' \wedge f'' = f_1' f_2'' - f_1'' f_2'.$$

On a component of D, say D_1, given in local coordinate by $z_1 = 0$ we replace only f_1 by $\log f_1$ in this expression. A calculation based on the above filtration of $E_{2,m}\overline{T}_X^{\star}$ and Riemann-Roch yields

$$\chi\left(X, E_{2,m}\overline{T}_X^{\star}\right) = \frac{m^4}{648}(13\overline{c}_1^2 - 9\overline{c}_2) + O(m^3).$$

On the other hand,

$$H^2(X, E_{2,m}\overline{T}_X^\star \otimes \mathcal{O}(-A)) = H^0(X, K_X \otimes E_{2,m}\overline{T}_X \otimes \mathcal{O}(A))$$

by Serre duality. From the filtration above, $K_X \otimes (E_{2,m}\overline{T}_X) \otimes \mathcal{O}(A)$ admits a filtration with graded pieces

$$S^{m-3j}\overline{T}_X \otimes \overline{K}_X^{\otimes -j} \otimes K_X \otimes \mathcal{O}(A).$$

Recall now that $\overline{T}_X$ is semi-stable (see [22] and [33]) so by Bogomolov's vanishing theorem [3], we have $h^0\big(X, S^p\overline{T}_X \otimes \overline{K}_X^{\otimes q}\big) = 0$, $p - 2q > 0$. This implies that

$$h^2\big(X, E_{2,m}\overline{T}_X^\star \otimes \mathcal{O}(-A)\big) = 0$$

for m large. Consequently we get the following

Theorem 1.2.1. *If (X, D) is an algebraic surface of general type and A an ample line bundle over X, then*

$$h^0(X, E_{2,m}\overline{T}_X^\star \otimes \mathcal{O}(-A)) \geq \frac{m^4}{648}(13\,\overline{c}_1^2 - 9\,\overline{c}_2) - O(m^3).$$

In particular: If $13\,\overline{c}_1^2 - 9\,\overline{c}_2 > 0$, then $\overline{B}_2 \neq \overline{X}_2$.

In the special case when $X = \mathbb{P}^2$ and $D = C$ is a smooth plane curve of degree d, we take $A = \mathcal{O}_{\mathbb{P}^2}(1)$. Then we have $\overline{c}_1 = (3 - d)h$ and $\overline{c}_2 = (d^2 - 3d + 3)h^2$ where $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$, $h^2 = 1$, thus

$$\chi\big(E_{2,m}\overline{T}_X^\star \otimes \mathcal{O}(-A)\big) = (4d^2 - 51d + 90)\frac{m^4}{648} + O(m^3).$$

A straightforward computation shows that the leading coefficient $4\,d^2 - 51\,d + 90$ is positive if $d \geq 11$. Thus, we obtain

Corollary 1.2.2. *For every smooth curve of degree $d \geq 11$ the associated log-surface $(\mathbb{P}^2, C)$ has its 2-jets base locus $\overline{B}_2 \neq \overline{X}_2$.*

Remark 1.2.3. As a consequence of the calculus above and with the previous condition on Chern classes, every holomorphic entire curve f into $X \setminus D$ could be lifted in $\overline{X}_2$ and its image is contained in an irreducible component Z of $\overline{B}_2$. We have to distinguish three cases
(a) $\pi_{2,1}(Z) = \overline{X}_1$, then Z is three dimensional and called horizontal, in this case the 2-jet lifting of f is a leaf of a foliation by curves on Z. In fact the lifting of f is tangent to $T_Z \cap V_2$ which defines a distribution of lines on a Zariski open subset of Z which is obviously integrable.

(b) $\pi_{2,0}(Z) = X$ and we are not in case (a), then our curve f could be lifted to $\overline{X}_1$ as a leaf of foliation by curves on the surface $Y := \pi_{2,1}(Z)$ (defined by the distribution $T_Y \cap V_1$)

(c) The image of f is algebraically degenerate.

The difficulty, in the case (a), is to study (singular) foliation by curves on a variety of dimension bigger than two. Actually, we have no reasonable model with reducible foliated singularities in this case currently (see however [26]). So the next step is to show that with a slightly stronger condition on Chern classes, in the case of $(\mathbb{P}^2, C)$, we only have to consider foliations on surfaces.

1.3. Existence of the multi-foliation. Our aim now is to study the restriction of the tautological line bundle on 2-jets on a 3-dimentionnal horizontal component Z of $\overline{B}_2$. Let us first make the following useful definition (as in [10]).

DEFINITION 1.3.1. Let (X, D) be a nonsingular projective variety of log-general type. We define the k-jet log-threshold $\overline{\theta}_k$ of (X, D) to be the infimum

$$\overline{\theta}_k = \inf_{m>0} \overline{\theta}_{k,m} \in \mathbb{R},$$

where $\overline{\theta}_{k,m}$ is the smallest rational number t/m such that there is a non zero section in $H^0(X, E_{k,m}\overline{T}_X^\star \otimes \mathcal{O}(t\,\overline{K}_X))$ (assuming that $t\,\overline{K}_X$ is an integral divisor, $t \in \mathbb{Q}$).

In the case when the Picard group equals $\mathbb{Z}$ we have a more clear idea about the jet log-threshold. Recall that Γ_2 is the effective divisor in $\overline{X}_2$ associated to $\mathcal{O}_{\overline{X}_2}(-1, 1)$. Then we have

Lemma 1.3.2. *Let (X, D) be a nonsingular surface of log-general type with* Pic$(X) = \mathbb{Z}$. *Suppose that $\overline{\theta}_1 \geq 0$ and $\overline{\theta}_2 < 0$. Then*

$$\overline{B}_2 \subset Z_\sigma = Z \cup \Gamma_2,$$

where Z is an irreducible component and Z_σ is the set of zeros of a section σ in $H^0(\overline{X}_2, \mathcal{O}_{\overline{X}_2}(m_0) \otimes \mathcal{O}(t_0\,\overline{K}_X))$, with $t_0 < 0$ in $\mathbb{Z}$ Moreover, in the case $\overline{B}_2 = Z_\sigma$, we have $\overline{\theta}_2 = t_0/m_0$.

Proof.   As $\overline{\theta}_2 < 0$ we have a nontrivial section

$$s \in H^0(X, E_{2,m}\overline{T}_X^\star \otimes \mathcal{O}(t\,\overline{K}_X)), \qquad m > 0, \quad t \in \mathbb{Q} \quad \text{and} \quad t < 0.$$

Let $u_1 = \pi_{2,1}^\star \mathcal{O}_{\overline{X}_1}(1)$ and $u_2 = \mathcal{O}_{\overline{X}_2}(1)$, then its zero divisor

$$Z_s = m\,u_2 + t\,\pi_{2,0}^\star K_X \qquad \text{in Pic}(X_2),$$

Let $Z_s = \sum p_j Z_j$ be the decomposition of Z_s in irreducible components. From the equality $\mathrm{Pic}(\overline{X}_2) = \mathrm{Pic}(X) \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$ and the assumption $\mathrm{Pic}(X) \simeq \mathbb{Z}$, we find

$$Z_j \sim a_{1,j}u_1 + a_{2,j}u_2 + t_j\, \pi_{2,0}^\star \overline{K}_X,$$

for suitable integers $a_{1,j},\ a_{2,j} \in \mathbb{Z}$ and rational numbers $t_j \in \mathbb{Q}$. We can prove that (see Lemma 3.3 in [10]), as Z_j is effective, we must have one of the following three disjoint cases:

- $(a_{1,j}, a_{2,j}) = (0,0)$ and $Z_j \in \pi_{2,0}^\star \mathrm{Pic}(X)$, $t_j > 0$;
- $(a_{1,j}, a_{2,j}) = (-1, 1)$, then Z_j contains Γ_2, so $Z_j = \Gamma_2$ and $t_j = 0$;
- $a_{1,j} \geq 2a_{2,j} \geq 0$ and $m_j := a_{1,j} + a_{2,j} > 0$.

We can suppose $t_0/m_0 = \min t_j/m_j$ then t_0 is clearly negative. Now we have $a_{2,0} \neq 0$ because $\theta_1 \geq 0$ and then Z_0 gives a section

$$\sigma \in H^0\big(\overline{X}_2, \mathcal{O}_{\overline{X}_2}(m_0) \otimes \pi_{2,0}^\star \mathcal{O}(t_0 \overline{K}_X)\big)$$

(we use the identity $\mathcal{O}_{\overline{X}_2}(a_1, a_2) = \mathcal{O}_{\overline{X}_2}(a_1 + a_2) \otimes \mathcal{O}_{\overline{X}_2}(-a_1 \Gamma_2)$). Then, by definition, we obtain $\overline{B}_2 \subset Z_\sigma$ and we have equality if and only if Z is the unique irreducible section with $t < 0$. As $t_j/m_j \leq t/m$ we conclude, in this case that $\overline{\theta}_2 = t_0/m_0$. $\qquad\square$

As a generalization to the log-case of the main theorem of [10] we have the following

Theorem 1.3.3. *Let (X, D) be a nonsingular surface of log-general type with $\mathrm{Pic}(X) = \mathbb{Z}$. Suppose that $\overline{\theta}_2 < 0$ and that the log-Chern numbers of X satisfy*

$$(13 + 12\,\overline{\theta}_2)\overline{c}_1^2 > 9\,\overline{c}_2.$$

Then every Zariski-dense holomorphic map $f : \mathbb{C} \to X \setminus D$ is a leaf of an algebraic multi-foliation on X.

Proof. If $\overline{\theta}_1 < 0$, then $\overline{B}_1 \neq \overline{X}_1$ and we conclude by a direct application of Theorem 1.1.3. (where the foliation is defined by the intersection of V_1 and the tangent space of the irreducible component of $\overline{B}_1$ which contains the lifting of f to 1-jets) so we suppose that $\overline{\theta}_1 \geq 0$. As $\overline{\theta}_2 < 0$, we have $\overline{B}_2 \neq \overline{X}_2$ and the discussion made in Remark 1.2.3. shows that we have to consider only the case when the lifting of f to 2-jets is contained in a horizontal irreducible divisor Z in $\overline{X}_2$. By Lemma 1.3.2 we have

$$Z \sim a_1 u_1 + a_2 u_2 + t_0 \pi_{2,0}^\star \overline{K}_X \qquad \text{in } \mathrm{Pic}(\overline{X}_2), \quad t_0 \in \mathbb{Q}, \quad t_0 < 0, \qquad a_1 + a_2 = m_0,$$

where $t_0/m_0 = \overline{\theta}_2$ Our aim now is to prove that the restriction of the tautological line

bundle to Z is big. First, we have the following intersection equalities

$$u_1^4 = 0, \quad u_1^3 u_2 = \overline{c}_1^2 - \overline{c}_2, \quad u_1^2 u_2^2 = \overline{c}_2, \quad u_1 u_2^3 = \overline{c}_1^2 - 3\overline{c}_2, \quad u_2^4 = 5\overline{c}_2 - \overline{c}_1^2,$$
$$u_1^3 \cdot F = 0, \quad u_1^2 u_2 \cdot F = -\overline{c}_1 \cdot F, \quad u_1 u_2^2 \cdot F = 0, \quad u_2^3 \cdot F = 0,$$

where F is any divisor in $\mathrm{Pic}(X)$.

Using this table, we obtain easily

$$(2u_1 + u_2)^3 \cdot Z = m_0(13\,\overline{c}_1^2 - 9\,\overline{c}_2) + 12\,t_0\,\overline{c}_1^2,$$

moreover, we have $t_0/m_0 = \overline{\theta}_2$, and hence

$$(2u_1 + u_2)^3 \cdot Z = m_0((13 + 12\,\overline{\theta}_2)\,\overline{c}_1^2 - 9\,\overline{c}_2) > 0.$$

As in [10] Proposition 3.4., we conclude that the restriction $\mathcal{O}_{\overline{X}_2}(1)_{|Z}$ is big. Consequently, by Theorem 1.1.3, every nonconstant entire curve $f \colon \mathbb{C} \to X$ is such that $f_{[2]}(\mathbb{C})$ is contained in the base locus of $\mathcal{O}_{\overline{X}_2}(l) \otimes \pi_{2,0}^\star \mathcal{O}(-A)_{|Z}$ for l large. This base locus is at most 2-dimensional, and projects onto a proper algebraic subvariety Y of $\overline{X}_1$. Therefore $f_{[1]}(\mathbb{C})$ is contained in Y, and the Theorem is proved. □

1.4.   Complement of curves in $\mathbb{P}^2$.   In this section we will consider the case $(\mathbb{P}^2, C)$ where C is a plane curve of degree d. We will estimate the associated 2-jet log-threshold. We start with a vanishing theorem of log-symmetric differentials (similar of that of Sakai [30]).

Lemma 1.4.1.   *Let C be a smooth curve of degree d in $\mathbb{P}^2$, m a nonnegative integer and $k \in \mathbb{Z}$. Then*

$$H^0(\mathbb{P}^2, S^m \overline{T}_{\mathbb{P}^2}^\star \otimes \mathcal{O}(k)) = 0 \quad \text{for all} \quad k \le \min(m-1, d-3).$$

In particular, for $d \ge 4$, $(\mathbb{P}^2, C)$ is of log-general type and we have the estimate

$$(d-3)\overline{\theta}_{1,m} \ge \min\left(1, \frac{d-2}{m}\right).$$

Proof.   We consider the natural ramified covering $X \subset \mathbb{P}^3$ over $\mathbb{P}^2$ associated to C (if C is given by $P(z_0, z_1, z_2) = 0$, then X is defined by $z_3^d = P(z_0, z_1, z_2)$), let L be the hyperplane section in X over C. Then we have an injective morphism (by taking pullbacks)

$$H^0(\mathbb{P}^2, S^m \Omega_{\mathbb{P}^2}(\log C)(k)) \hookrightarrow H^0(X, S^m \Omega_X(\log L)(k)),$$

the last group is contained in $H^0(X, S^m \Omega_X(m + k))$ which vanishes for $k \le \min(m-1, d-3)$ by Lemma 5.1 in [10].

Now we have $\overline{K}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(d-3)$, consequently, there are no nonzero sections in $H^0(\mathbb{P}^2, S^m\overline{T}^\star_{\mathbb{P}^2} \otimes \mathcal{O}(t\overline{K}_{\mathbb{P}^2}))$ unless $t(d-3) \geq \min(m, d-2)$, whence the lower bound for $\overline{\theta}_{1,m}$. $\square$

Using the above vanishing lemma we obtain a lower bound on the 2-jet log-threshold

Lemma 1.4.2. *Let C is a curve of degree $d \geq 4$ in $\mathbb{P}^2$. Suppose that the 2-jet base locus $\overline{B}_2$ associated with $(X, D) = (\mathbb{P}^2, C)$ is of the form $Z_\sigma = Z_0 \cup \Gamma_2$, where Z_0 an irreducible component and Z_σ is the set of zeros of a section $\sigma \in H^0(\overline{X}_2, \mathcal{O}_{\overline{X}_2}(m_0) \otimes \mathcal{O}(t_0 \overline{K}_X))$. Then for $m_0 \geq 6$ we have the estimate*

$$\overline{\theta}_2 = \overline{\theta}_{2,m_0} \geq \max\left(\frac{-1}{m_0}; \min\left(\frac{1}{2(d-3)} - \frac{1}{6}, \frac{d-2}{2m_0(p_0-1)(d-3)} - \frac{1}{6} \right) \right),$$

where $p_0 = [m_0/3]$.

Proof. Observe that σ can be considered as a global holomorphic section of the bundle $E_{2,m_0}\overline{T}^\star_X \otimes \mathcal{O}(t_0\overline{K}_X)$. By the filtration of $E_{2,m_0}\overline{T}^\star_X$, we have a short exact sequence

$$0 \rightarrow S_0^m\overline{T}^\star_X \rightarrow E_{2,m_0}\overline{T}^\star_X \rightarrow E_{2,m_0-3}\overline{T}^\star_X \otimes \mathcal{O}(\overline{K}_X) \rightarrow 0.$$

Multiply all terms by $\mathcal{O}(t_0\overline{K}_X)$ and consider the associated sequence in cohomology. As $t_0 < 0$ and by Lemma 1.4.1, the first H^0 group vanishes and we get an injection

$$H^0\big(X, E_{2,m_0}\overline{T}^\star_X \otimes \mathcal{O}(t_0\overline{K}_X)\big) \lhook\joinrel\longrightarrow H^0\big(X, E_{2,m_0-3}\overline{T}^\star_X \otimes \mathcal{O}((t_0+1)\overline{K}_X)\big).$$

By assumption on $\overline{B}_2$ we must have $t_0 + 1 \geq 0$, this gives the firt part of the estimate. Let now $m_0 = 3p_0 + q_0$, $0 \leq q_0 \leq 2$ a positive integer. Then there is a (nonlinear) discriminant mapping

$$\Delta: E_{2,m_0}\overline{T}^\star_X \otimes \mathcal{O}(t_0\overline{K}_X) \rightarrow S^{(p_0-1)(3p_0+2q_0)}\overline{T}^\star_X \otimes \mathcal{O}((p_0+2t_0)(p_0-1)\overline{K}_X).$$

In fact, write an element of $E_{2,m}\overline{T}^\star_X$ in the form

$$P(f) = \sum_{0 \leq j \leq p} a_j \cdot f'^{\,3(p-j)+q}\, W^j$$

where the a_j is viewed as an element of $S^{3(p-j)+q}\overline{T}^\star_X \otimes \overline{K}^j_X$, and $W \in \overline{K}^{-1}_X$. The discriminant $\Delta(P)$ is then calculated by interpreting P as a polynomial in the indeterminate W.

Applying this to σ, we obtain $2(p_0 - 1)t + p_0(p_0 - 1) \geq (p_0 - 1)(3p_0 + 2q_0) \times \theta_{1,(p_0-1)(3p_0+2q_0)}$ and this implies

$$\frac{t}{m} \geq \frac{3p_0 + 2q_0}{2m_0}\theta_{1,(p_0-1)(3p_0+2q_0)} - \frac{p_0}{2m_0}.$$

Using Lemma 1.4.1 again we obtain the remainer part of the estimate. $\qquad \square$

We now turn to the question of the existence of 2-jet differentials of small degree. Recall from the filtration of the bundle of 2-jet differentials that we have an exact sequence

$$0 \longrightarrow S^m\overline{T}_X^\star \longrightarrow E_{2,m}\overline{T}_X^\star \xrightarrow{\Phi} E_{2,m-3}\overline{T}_X^\star \otimes \overline{K}_X \to 0.$$

We have the following "proportionality" lemma.

Lemma 1.4.3. *Let (X, D) be a nonsingular surface of log-general type. Then, for all sections*

$$P_i \in H^0(X, E_{2,m_i}\overline{T}_X^\star \otimes \mathcal{O}_X(t_i \overline{K}_X))$$

with $m_i = 3, 4, 5$ and $t_i \in \mathbb{Q}$, the section $\beta_1 P_2 - \beta_2 P_1$ associated with $\beta_i = \Phi(P_i)$ can be considered as a section in

$$H^0(X, S^{m_1+m_2-3}\overline{T}_X^\star \otimes \mathcal{O}_X((1 + t_1 + t_2)\overline{K}_X)),$$

and it vanishes when $1 + t_1 + t_2 < (m_1 + m_2 - 3)\overline{\theta}_{1,m_1+m_2-3}$.

Proof. The section $\beta_1 P_2 - \beta_2 P_1$ is contained in $H^0(X, E_{2,m_1+m_2-3}\overline{T}_X^\star \otimes \mathcal{O}_X((1 + t_1 + t_2)\overline{K}_X))$, its image by Φ is zero. Then it can be considered as a section in $H^0(X, S^{m_1+m_2-3}\overline{T}_X^\star \otimes \mathcal{O}_X((1 + t_1 + t_2)\overline{K}_X))$. $\qquad \square$

Now we have the following application of the proportionality lemma.

Lemma 1.4.4. *Let C be a generic curve of degree $d \geq 7$. Then*

$$\overline{\theta}_{2,m} \geq -\frac{1}{2m} + \frac{1 - (3 + \varepsilon)/2m}{d - 3} \qquad \text{for } m = 3, 4, 5,$$

where $\varepsilon := d \mod 2$.

Proof. We consider the curve

$$C_a = \{z_0^{k_0}(z_0^{d-k_0} + a\, z_1^{k_1} z_2^{k_2}) + z_1^d + z_2^d = 0\},$$

where k_0, k_1, k_2, ≥ 0 are integers with $\sum k_i = d$ and a a complex number such that C_a is non singular. We put

$$s_0 = z_0^{k_0}(z_0^{d-k_0} + a\, z_1^{k_1} z_2^{k_2}), \qquad s_i = z_i^d, \quad i = 1, \ 2.$$

Using Nadel's method [27], we solve the linear system

$$\sum_{0 \leq k \leq 2} \widetilde{\Gamma}_{ij}^k \frac{\partial s_l}{\partial z_k} = \frac{\partial^2 s_l}{\partial z_i \partial z_j}, \qquad 0 \leq i, \ j, \ l \leq 2,$$

and get in this way a homogeneous meromorphic connection of degree -1 on $\mathbb{C}^3$. One can check that this connection descends to a partial projective meromorphic connection $\nabla = (\Gamma_{ij}^k)$ on $\mathbb{P}^2$ such that C_a is totally geodesic (see [14] and [9]), the pole divisor of the connection ∇ is given by

$$B = \{z_0 z_1 z_2 (d\, z_0^{k_1+k_2} + a k_0 z_1^{k_1} z_2^{k_2}) = 0\}.$$

Then, this connection can be seen as a meromorphic connection on $\overline{T}_{\mathbb{P}^2}$. In fact, if we take two tangent vector fields u and v to C_a, the vector field $\nabla_u v$ is also tangent to C_a by construction.

Consequently, by taking the Wronskian operator

$$W_\nabla(f) = f' \wedge f''_\nabla, \qquad f'' = \nabla_{f'} f',$$

we get a section

$$P_1 \in H^0 E_{2,3} \overline{T}_{\mathbb{P}^2}^\star \otimes \mathcal{O}(-\overline{K}_{\mathbb{P}^2}) \otimes \mathcal{O}(B) = H^0 E_{2,3} \overline{T}_{\mathbb{P}^2}^\star \otimes \mathcal{O}(t_1 \overline{K}_{\mathbb{P}^2}),$$

where $t_1 = (3 + k_1 + k_2)/(d - 3) - 1$. Remark that $p = 3 + k_1 + k_2$ can be taken equal to any integer in $[3, d + 3]$.

We take $p = [(d + 1)/2]$ (the biggest integer less or equal to $(d + 1)/2$), so that

$$\frac{1}{2} + t_1 = \frac{3 + \varepsilon}{2(d - 3)}, \quad \text{where} \quad \varepsilon = d \mod 2, \quad \varepsilon \in \{0, 1\}.$$

The integer p must be at least equal to 3, thus our choice is permitted if $d \geq 7$. We claim that $\mathbb{P}^2$ has no non trivial section in

$$H^0(\mathbb{P}^2, E_{2,m} \overline{T}_{\mathbb{P}^2}^\star \otimes \mathcal{O}(t\overline{K}_{\mathbb{P}^2})), \qquad m = m_2 = 3, \ 4, \ 5$$

if $1/2 + t < (m - 3/2 - \varepsilon/2)/(d - 3)$. We assume the contrary, so that $P_2 \in H^0(\mathbb{P}^2, E_{2,m} \overline{T}_{\mathbb{P}^2}^\star \otimes \mathcal{O}(t\overline{K}_{\mathbb{P}^2}))$, $P_2 \neq 0$. Then, for $m_1 = 3$, $m_2 = m$ and $t_2 = t$, our choices imply

$$1 + t_1 + t_2 < \frac{m}{d - 3} \leq (m_1 + m_2 - 3)\overline{\theta}_{1, m_1 + m_2 - 3},$$

as $\overline{\theta}_{1,m} \geq 1/(d-3)$, for all $m = 3, 4, 5$ and $d \geq 7$ (by 1.4.1). By Lemma 1.4.3, we get a meromorphic connection with logarithmic pole on C_a associated with a Wronskian operator $P_2/\beta_2 = P_1/\beta_1$. As P_1/β_1 is an irreducible fraction with $\operatorname{div}\beta_1 = B$, we conclude that $\beta_2/\beta_1 \in H^0(\mathbb{P}^2, S^{m-3}\overline{T}^{\star}_{\mathbb{P}^2} \otimes \mathcal{O}((t_2 - t_1)\overline{K}_{\mathbb{P}^2}))$ must be holomorphic, hence

$$t_2 \geq t_1 + (m-3)\overline{\theta}_{1,m-3} \geq t_1 + \frac{(m-3)}{d-3}.$$

On the other hand

$$t_2 = t < -\frac{1}{2} + \frac{m - 3/2 - \varepsilon/2}{d-3} = t_1 + \frac{m-3-\varepsilon}{d-3},$$

which yieldes a contradiction. By the Zariski semicontinuity of cohomology, the group

$$H^0(\mathbb{P}^2, E_{2,m}\overline{T}^{\star}_{\mathbb{P}^2} \otimes \mathcal{O}(t\overline{K}_{\mathbb{P}^2}))$$

vanishes for a generic curve C of degree $d \geq 7$, unless

$$\frac{t}{m} \geq -\frac{1}{2m} + \frac{1 - (3+\varepsilon)/2m}{d-3}.$$

this yields the estimate.   □

As a corollary we obtain the main result of this first part.

Theorem 1.4.5. *Every non algebraically degenerate holomorphic entire map into the complement of a generic curve of degree $d \geq 15$ in $\mathbb{P}^2$ is a leaf of a multi-foliation on $\mathbb{P}^2$.*

Proof. Let C be a genecic curve of degree $d \geq 15$ in $\mathbb{P}^2$. If $\overline{B}_2$ is not as in Lemma 1.4.2, then we are done (we have two independent sections), so suppose that $\overline{B}_2 = Z \cup \Gamma_2$ (with the notations of Lemma 1.4.2). If $m_0 = 3, 4, 5, 6, 7$ (we have clearly $m_0 \geq 3$) we apply 1.4.4 and 1.4.2 to get the estimate on the 2-jet threshold

$$\overline{\theta}_2 \geq \frac{(3-\varepsilon)}{6(d-3)} - \frac{1}{6},$$

where $\varepsilon = d \mod 2$. According to this estimate,

$$(13 + 12\overline{\theta}_2)\overline{c}_1^2 - 9\overline{c}_2 \geq (d-3)(2d - 27 - 2\varepsilon) - 27$$

this is positive when $d \geq 15$ and we can apply Lemma 1.3.2 to obtain the statement.

When $m_0 \geq 8$ we apply the estimate in Lemma 1.4.2, we obtain $\overline{\theta}_2 \geq -1/8$. It is easy to verify that $(13 - 3/2)\overline{c}_1^2 - 9\overline{c}_2 = (d-3)(2, 5d - 34, 5) - 27$ is positive for $d \geq 15$ and we can again apply Lemma 1.3.2.   □

2. Entire leaves of foliations on log-general type surfaces

In this part we will generalize the main result in [24], we will follow basically the strategy in [6] with a little improvement due to the "convenience" of the logarithmic formalism.

2.1. Singularities of foliations on surfaces.

Let X be a compact complex surface. Recall from [18] that we have a bijective correspondence between a holomorphic foliation $\mathcal{F}$ on the surface X with isolated singularities and a locally free subsheaf of the tangent sheaf of X denoted $T_{\mathcal{F}}$. In this case we have an exact sequence of sheaves

$$0 \longrightarrow T_{\mathcal{F}} \longrightarrow T_X \longrightarrow N_{\mathcal{F}}.I_Z \longrightarrow 0,$$

where $N_{\mathcal{F}}$ is called the normal bundle of the foliation and I_Z an ideal supported on the singularity set Z of $\mathcal{F}$.

The elements of Z are the points where local vector fields defining $\mathcal{F}$ vanish. Suppose that $\mathcal{F}$ is given around a singular point p by a vector field v, then we note by λ_1 and λ_2 the eigenvalues of the linear part of v and we make the following

DEFINITION 2.1.1. The singularity p is called reduced if the linear part $(Dv)(p)$ is nonzero (say $\lambda_2 \neq 0$) and the quotient $\lambda = \lambda_1/\lambda_2$ is not a positive rational number.

A reduced singularity at p is called nondegenerate if $\lambda_1\lambda_2 \neq 0$ and a saddle-node otherwise. The importance of those singularities comes from the following well-known theorem

Theorem 2.1.2 ([31]). *There exist a sequence of blow-ups* $\sigma: \tilde{X} \to X$ *such that the foliation* $\sigma^\star \mathcal{F}$ *has only reduced singularities.*

Now let D be a normal crossing divisor on X, we say that the foliation $\mathcal{F}$ defines a logarithmic foliation on (X, D) if $\mathcal{F}$ is tangent to each component of D. The sheaf injection from $T_{\mathcal{F}} \hookrightarrow T_X$ factors to a sheaf injection

$$0 \longrightarrow T_{\mathcal{F}} \hookrightarrow T_X(-\log D) \longrightarrow N_{\mathcal{F}}(-D).I_{Z'} \longrightarrow 0,$$

where Z' is the set of logarithmic singularities of $\mathcal{F}$ which is obviously contained in Z. The bundle $N_{\mathcal{F}}(-D)$ will be called the logarithmic normal bundle of $\mathcal{F}$, and denoted by $\overline{N}_{\mathcal{F}}$.

To a logarithmic foliation $(\mathcal{F}, D)$ we associate a (singular) surface $\tilde{X}$ in the 1-jet logarithmic space $\overline{X}_1$, called the logarithmic graph of $(\mathcal{F}, D)$, which consists of the adherence of the liftings of all leaves or equivalently the blow-up of X along the ideal $I_{Z'}$.

When $\mathcal{F}$ has only reduced singularities, and if p is a nondegenerate singularity in

Z', then the surface $\tilde{X}$ is smooth around p and isomorphic around $\pi^{-1}(p)$ to the blow up of X at p. In the case p is a saddle node of multiplicity d (i.e., the first terms of v in local coordinates are $z(\partial/\partial z) + w^d(\partial/\partial w)$), then $\tilde{X}$ has a singular point of type A_{d-1} (i.e., it is given in local coordinates by $z^d = xy$).

We finish this section by the following useful natural formula

$$\mathcal{O}_{\overline{X}_1}|_{\tilde{X}} = \pi^{\star}(T_{\mathcal{F}}) \otimes \mathcal{O}\left(\sum_{p \in Z'} d_p E_p\right),$$

where d_p is the multiplicity of p and E_p the fibre $\pi^{-1}(p)$.

2.2. The log-tautological inequality and consequences.

Recall from [24] that to a holomorphic curve $f : \mathbb{C} \to X$, where X is supposed to be endowed with a Kähler form ω, we can associate a closed positive current in the following way: for every 2-form η and $r > 0$ we define

$$T_r(\eta) = \int_0^r \frac{dt}{t} \int_{D(t)} f^{\star}\eta,$$

where $D(t)$ is the disc of radius t. Then we consider the positive currents Φ_r defined by

$$\Phi_r(\eta) := \frac{T_r(\eta)}{T_r(\omega)} \ \forall \eta \in A^2(X).$$

The family $\{\Phi_r\}_{r>0}$ is bounded and we can see easily that there is a closed positive current Φ in its adherence. When $f(\mathbb{C})$ is not contained in a hypersurface Y, we prove, using the Lelong-Poincaré formula, that Φ has positive intersection with Y. As a consequence, for such an f, the current Φ is actually numerically effective.

Remark that this construction is independent from the dimension of X. Hence, we can associate a positive current Φ_1 on $\overline{X}_1$ to f_1, the lifting of f. Now if we suppose that $f^{-1}(D)$ is finite, we have the following logarithmic tautological inequality (see [34] and [26])

$$\mathcal{O}_{\overline{X}_1}(-1).\Phi_1 \geq 0,$$

As a consequence of this inequality we have $\pi_{\star}\Phi_1 = \Phi$. From now on we will suppose that there is a logarithmic foliation $\mathcal{F}$ on (X, D) with reduced singularities such that D is the union of its algebraic leaves and f is a Zariski dense entire leaf (in this case $f^{-1}(D)$ is finite). Let $\nu(\Phi, p) = [\Phi_1].[d_p E_p]$, where d_p and E_p are defined in the previous section. We apply the logarithmic tautological inequality to f_1 which gives

$$\pi^{\star}(T_{\mathcal{F}}) \otimes \mathcal{O}\left(\sum_{p \in Z'} d_p E_p\right).\Phi_1 \geq 0.$$

As $\pi_\star \Phi_1 = \Phi$, then we obtain

$$T_\mathcal{F}.\Phi \geq -\sum_{p \in Z'} \nu(\Phi, p).$$

This last inequality is exactly the refined tautological inequality in [24] because the intersection of algebraic leaves are not counted in Z', in fact those points are smooth from the logarithmic point of view. So we have the following

Observation 2.2.1. *The logarithmic tautological inequality applied to the triple* $(X, \mathcal{F}, D)$ *implies the refined tautological inequality.*

Now we iterate this construction: (X, D) will be replaced by $(\tilde{X}, \tilde{D})$, where $\tilde{D}$ is the union of all algebraic leaves of the induced foliation on $\tilde{X}$ (we replace $\tilde{X}$ by its desingularization if necessary). The last step is almost the same as in [24] and [6]. Let $(X^{(n)}, \mathcal{F}^{(n)})$ be the foliated surface obtained after n iterations of this process and let $\pi^{(n)}$ denotes the canonical morphism from $X^{(n)}$ to $X^{(0)} = X$. As the singularities are reduced, we have

$$\mathcal{F}^{(n)} = (\pi^{(n)})^\star(\mathcal{F}).$$

Suppose (for simplicity) that we start from $Z' = \{p\}$, only one point, of multiplicity d_p. Then, by construction, there are exactly two points q_1^n and q_2^n on $\mathcal{F}^{(n)}$ that can possibly be logarithmic singularities, one necessarily of multiplicity d_p while the other of multiplicity 1. So the log-tautological inequality gives

$$T_\mathcal{F}.\Phi = T_{\mathcal{F}^{(n)}}.\Phi^{(n)} \geq -\nu(\Phi^{(n)}, q_1^{(n)}) - \nu(\Phi^{(n)}, q_2^{(n)}).$$

Now we prove that $\nu(\Phi^{(n)}, q_i^{(n)})$ tends to zero as n tends to infinity, in fact, by comparing $\Phi^{(n)}$ and Φ, we obtain the inequality

$$0 \leq [\Phi^{(n)}]^2 \leq [\Phi]^2 - \sum_{j=0, i=1,2}^{n-1} \nu(\Phi^{(j)}, q_i^{(j)}).$$

Consequently, we obtain

Theorem 2.2.2 ([24]). *Let $\mathcal{F}$ a holomorphic foliation (with reduced singularities) on a compact surface X and Φ the current associated to a Zariski dense entire leaf. Then we have the intersection inequality $T_\mathcal{F}.\Phi \geq 0$.*

2.3. Positivity of the log-normal bundle on leaves. Recall first the following result from [6].

Theorem 2.3.1 ([6]). *Let $\mathcal{F}$ be a holomorphic foliation (with reduced singularities) on a compact surface X and Φ be an diffuse current (i.e., with zero Lelong numbers exept at a finite set of points). Suppose that Φ is $\mathcal{F}$-invariant. Then, we have the intersection inequality*

$$N_{\mathcal{F}}(-D).\Phi \geq 0, \text{ for all invariant divisor } D.$$

The proof of the previous theorem consists of an explicit construction of a closed 2-form which represents the Chern class of $N_{\mathcal{F}}(-D)$. The intersection is computed by integrating this form along the support of Φ which is a union of leaves. This integration is concentrated around singularities for which structure and holonomy are well understood.

Now using this, the generalization of the positivity of the normal bundle on the current associated with a Zariski dense leaf to the log-case is immediate from the following

Corollary 2.3.2. *Let $(\mathcal{F}, D)$ be a holomorphic log-foliation with reduced singularities and Φ an $\mathcal{F}$-invariant current such that the support of Φ_{alg} is contained in D. Then we have the following intersection inequality for the logarithmic normal bundle*

$$\overline{N}_{\mathcal{F}}.\Phi \geq 0.$$

Proof. By Theorem 2.3.1, it remains to prove that $\overline{N}_{\mathcal{F}}.\Phi_{\mathrm{alg}} \geq 0$. We use the same observation as in [6]: Let C be a component of Φ_{alg}, then

$$N_{\mathcal{F}}(-D).C = C.C + Z(C, \mathcal{F}) - D.C,$$

where $Z(C, \mathcal{F})$ is the total multiplicity of the singularities of $\mathcal{F}$ along C (cf. [5] lemme 3). This number is at least equal to the intersection $(D - C).C$, so we obtain $N_{\mathcal{F}}(-D).C \geq 0$. This is true for every component in the support of Φ_{alg}, which concludes the proof. $\square$

As a consequence we have the following

Theorem 2.3.3. *Let (X, D) be a surface of log-general type with a logarithmic foliation $(\mathcal{F}, D)$, then an entire leaf of $\mathcal{F}$ must be algebraically degenerate.*

Proof. By the Seidenberg theorem we can suppose that $\mathcal{F}$ has only reduced singularities. Suppose that $\mathcal{F}$ has a Zariski dense entire leaf. Then, by Theorem 2.2.2 and Corollary 2.3.2, as $K_X^{-1} = T_{\mathcal{F}} \otimes N_{\mathcal{F}}$, we obtain

$$K_X \otimes \mathcal{O}(D').\Phi \leq 0,$$

where D' is the union of D and the support of Φ_{alg}. Remark that this implies that the same inequality remains true on the original surface before blowing up the singularities (this is will be used in Theorem 2.4.2 below). As $K_X \otimes \mathcal{O}(D) \hookrightarrow K_X \otimes \mathcal{O}(D')$, the latter bundle is big and, hence, has the decomposition $\mathcal{O}(A+E)$, where A is ample and E is effective. But Φ is numerically effective, so we obtain an obvious contradiction. $\square$

2.4. Entire leaves on a surface of log general type. In this section we will generalize the main theorem in [24]. We now need to consider foliations that are not necessarily logarithmic.

Lemma 2.4.1. *Let $\mathcal{F}$ be a foliation on a surface X with a non-invariant curve C. Suppose that there is a Zariski-dense entire leaf $f : \mathbb{C} \to X \setminus (C \setminus P)$, where P is a finite set of points. Then there exists a sequence of blow ups $\sigma : \tilde{X} \to X$ of points of $f(\mathbb{C})$ such that if we denote by $\tilde{C}$ the strict transform of C and $\tilde{f}$ the lifting to $\tilde{X}$ of f, the support of an associated current $\tilde{\Phi}$ to $\tilde{f}$ is disjoint from C. In particular, $\tilde{\Phi}.\tilde{C} = 0$.*

Proof. By the Seidenberg theorem, using a sequence of blow ups $\tilde{X} \to X$, we can reduce the singularities of $\mathcal{F}$ in an open neighbourhood of $f(\mathbb{C})$ and suppose that the leaves of the induced foliation $\tilde{\mathcal{F}}$ are smooth in a neighbourhood of $\tilde{f}$ the lifting of f. This is done by blowing up points of $f(\mathbb{C})$ which are a singular points of the foliation. We can also suppose that these leaves are transverse to $\tilde{C}$ by blowing up the tangency points and that $\tilde{f}$ does not inersect $\tilde{C}$ by blowing the intersection points. Let $\mathcal{L}$ be the leaf containing the image of $\tilde{f}$, then as f is Zariski-dense, $\mathcal{L}$ intersects $\tilde{C}$ on at most one point ($\mathcal{L}$ is parametrized by $\mathbb{C}$ or $\mathbb{C}^\star$). Blowing up this point if it exists, we can suppose that $\mathcal{L}$ does not intersect $\tilde{C}$. We will prove that the topological closure of $\mathcal{L}$, which we denote by K, does not intersect $\tilde{C}$. Remark that K is a union of leaves. Let p a point on $\tilde{C} \cup K$, so there is a leaf $\mathcal{L}_p$ in K passing by p. Now $\mathcal{L}$ accumulates on $\mathcal{L}_p$ and this leaf is trasverse to $\tilde{C}$. As consequence in a neighbourhood of p, the number of intersection points of $\mathcal{L}$ with C is infinite, a contradiction. Finally, the support of $\tilde{\Phi}$ is contained in this closure so does not intersect $\tilde{C}$. $\square$

As a consequence of Theorem 2.3.3 and the previous lemma we obtain the following generalization of the main theorem of [24].

Theorem 2.4.2. *Let X be a surface with a foliation $\mathcal{F}$ and a divisor D such that (X, D) is of log-general type. Then every entire curve $f : \mathbb{C} \to X \setminus (D \setminus P)$ contained in a leaf of $\mathcal{F}$, where P is a finite set of points not contained in the singular locus of D, is algebraically degenerate.*

Proof.   Suppose that we have a Zariski-dense entire curve contained in a leaf of $\mathcal{F}$. Let $D' \subset D$ be the union of noninvariant components of D. By Lemma 2.4.1, there exists a sequence of blow-ups such that, if we denote $\tilde{X}$ the surface obtained, by $\tilde{D}$ (resp. $\tilde{D}'$) the strict transform of D (resp. of D') and by $\tilde{\Phi}$ an associated current to the lifting of f to $\tilde{X}$, we have $\tilde{\Phi}.\tilde{D}' = 0$.

Let $\tilde{D}_a$ be the union of all algebraic leaves of $\tilde{\mathcal{F}}$ (the foliation obtained on $\tilde{X}$), then we have

$$(K_{\tilde{X}} + \tilde{D}' + \tilde{D}_a).\tilde{\Phi} = (K_{\tilde{X}} + \tilde{D}_a).\tilde{\Phi}.$$

As $\tilde{\mathcal{F}}$ induces a logarithmic foliation on $(\tilde{X}, \tilde{D}_a)$, using the proof of Theorem 2.3.3 we obtain $(K_{\tilde{X}} + \tilde{D}_a).\tilde{\Phi} \leq 0$, which is equivalent to

$$(K_{\tilde{X}} + \tilde{D}' + \tilde{D}_a).\tilde{\Phi} \leq 0.$$

As $\tilde{\Phi}$ is nef we get a contradiction if we prove that the divisor $K_{\tilde{X}} + \tilde{D}' + \tilde{D}_a$ is big. To see this, remark that it contains the divisor $K_{\tilde{X}} + \tilde{D}$ which is big because we blown up just smooth points of D. In fact if we denote by π the composition of those blowing-ups we have $K_{\tilde{X}} + \tilde{D} = \pi^\star(K_X + D)$.   □

Actually, we think that the condition on P in the previous theorem could be omitted by using a more sophisticated argument. As a consequence of this theorem we obtain the following

Corollary 2.4.3.   *Let (X, D) be a log-surface of log-general type such that its logarithmic Chern classes verify $\bar{c}_1^2 > \bar{c}_2$. Then every entire curve $f : \mathbb{C} \to X \setminus (D \setminus P)$, where P is a finite set of points not contained in the singular locus of D and such that $f^{-1}(P)$ is finite, is algebraically degenerate.*

Proof.   We apply Riemann-Roch to symmetric powers of $\overline{T}_X$, the Euler characteristic is positive with our condition on Chern classes. Using Serre duality and effectiveness of $K_X + D$, the h^2 term is bounded by the h^0 term (as in [23]). As a consequence we get that $\mathcal{O}_{\overline{X_1}}(1)$ is big, and we apply the tautological inequality to get a contradiction if the lifting of f to $\overline{X_1}$ is not contained in B_1 (in the case P is empty we can alternatively apply Theorem 1.1.3). Then we get a foliation $\mathcal{F}$ on a (singular) surface $\tilde{X}$ in X_1 which is ramified over X (the foliation is defined by the intersection of V_1 and the tangent space of the irreducible component $\tilde{X}$ of $\overline{B}_1$ which contains the lifting of f to 1-jets). Let $\tilde{D}$ be the divisor on $\tilde{X}$ over D, then $(\tilde{X}, \tilde{D})$ is of log-general type and f can be lifted in $\tilde{X} \setminus \tilde{D}$ as a leaf of the foliation $\mathcal{F}$.   □

3. Proof of the Main Theorem

By the results of Theorem 1.4.5, if C is a generic plane curve of degree $d \geq 15$, then there is a ramified cover $\tilde{X} \subset \overline{X}_1$ over $X = \mathbb{P}_{\mathbb{C}}^2$ with a foliation $\mathcal{F}$ such that every entire curve f in $\mathbb{P}_{\mathbb{C}}^2 \backslash C$ is such that $f_{[2]}(\mathbb{C})$ is contained in $\tilde{X}$ as a leaf of $\mathcal{F}$. Morover, $f_{[2]}(\mathbb{C})$ is contained in $\tilde{X} \setminus \tilde{C}$ where $\tilde{C}$ is the (reduced) divisor in $\tilde{X}$ over C.

A log model of $(\tilde{X}, \tilde{C})$ is clealy of log-general type. In fact, recall that given two log-manifolds (X, D_X) and (Y, D_Y), a holomorphic map $\psi \colon X \to Y$ such that $\psi^{-1}D_Y \subset D_X$ (in the geometric sense) is called a log-morphism. Such a morphism induces (see [20]) a sheaf morphism

$$\psi^{\star} \colon \psi^{\star}\overline{T}_Y^{\star} \to \overline{T}_X^{\star}.$$

If ψ is dominating, then this morphism is clearly injective. Thus we have a natural injection of sheaves $\psi^{\star}\overline{K}_Y \hookrightarrow \overline{K}_X$.

Now, by Theorem 2.4.2 every entire curve f in $X \backslash C$ has algebraically degenerate lifting in $\tilde{X} \setminus \tilde{C}$ and hence itself has its image contained in an algebraic plane curve. Now every algebraic curve in $\mathbb{P}^2$ intersects a very generic curve of degree $d \geq 5$ in at least 3 point (see [32]). So f is constant and $\mathbb{P}_{\mathbb{C}}^2 \setminus C$ is hyperbolic and hyperbolically embedded in $\mathbb{P}_{\mathbb{C}}^2$ (see [16]).

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