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VARIETIES FOR MODULES OVER A BLOCK OF A FINITE GROUP

Dedicated to Professor Yukio Tsushima on his Sixtieth Birthday

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1. Introduction

Let p be a prime number and let k be an algebraically closed field of characteristic p. Let G be a finite group and b a block of kG. In [10], M. Linckelmann introduces two notions: the *transfer maps* in Hochschild cohomology of symmetric algebras, and the *cohomology ring of the block* b. Using the transfer map, he shows that the cohomology ring $H^*(G, b, D_\gamma)$ of the block b is embedded into the Hochschild cohomology ring $HH^*(kGb)$ of the block algebra kGb, which gives a block version of the well-known embedding from the usual cohomology ring $H^*(G, k)$ into the Hochschild cohomology ring $HH^*(kG)$ (see [10, Proposition 4.5]). Around the same time, M. Linckelmann [11] introduces a cohomological variety which is a block version of Carlson's module variety [4]. As is well known, for a finitely generated kG-module U, Carlson's variety $V_G(U)$ is defined to be the maximal ideal spectrum of $H^*(G, k)/I^*_G(U)$, where $I^*_G(U)$ is the annihilator of the action of $H^*(G, k)$ on $\mathrm{Ext}^*_{kG}(U, U)$ induced by the cup product. On the other hand, for a bounded complex U of kGb-modules, Linckelmann's variety $V_{G,b}(U)$ is defined to be the maximal ideal spectrum of $H^*(G, b, D_\gamma)/I^*_{G,b,D_\gamma}(U)$, where $I^*_{G,b,D_\gamma}(U)$ is the kernel of the algebra homomorphism $H^*(G, b, D_\gamma) \to \mathrm{Ext}^*_{kGb}(U, U)$ induced from the above embedding $H^*(G, b, D_\gamma) \to HH^*(kGb)$ (see Section 2 for details).

In this paper we study the variety $V_{G,b}(U)$. The following is the main result, which gives a characterization of the ideal $I^*_{G,b,D_\gamma}(U)$ in terms of the cohomology ring of a defect group D of b.

Theorem 1.1. *Let b be a block of kG, D_γ a defect pointed group of b and let $i \in \gamma$ be a source idempotent of b. Let U be a bounded complex of kGb-modules. Then we have that*

$$I^*_{G,b,D_\gamma}(U) = H^*(G, b, D_\gamma) \cap I^*_D(iU).$$

In particular if b is a nilpotent block of kG, then we have that $V_{G,b}(U) = V_D(iU)$.

In [11], M. Linckelmann shows that for any bounded complex U of kGb-modules, there exists a finite surjective map $V_{G,b}(U) \to V_G(U)$ (see Theorem 2.4 below). From Theorem 1.1, we can show the following reverse version of this result.

Corollary 1.2. *For any bounded complex U of kGb-modules, there exists a finite surjective map $\iota_{D,b}^{*}\colon V_D(iU) \to V_{G,b}(U)$. Further, the dimensions of $V_D(iU)$ and $V_{G,b}(U)$ coincide.*

In [11], Linckelmann also shows that the varieties $V_{G,b}(U)$ are invariant under splendid stable and derived equivalences in the sense of Linckelmann (for the definition of these notions, see [12]). Applying Theorem 1.1 we consider the question whether the varieties $V_{G,b}(U)$ are invariant under the Brauer correspondence betweeen blocks b in kG with defect group D and blocks b_0 in $kN_G(D)$ with defect group D. As a partial answer, we have the following.

Theorem 1.3. *Suppose that D is abelian. Let U be an indecomposable kGb-module with vertex D and let W be the Green correspondent of U with respect to $(G, D, N_G(D))$. Then we have that $V_{G,b}(U) = V_{N_G(D),b_0}(W)$.*

Let Z be a central p-subgroup of G and set $G/Z = \bar{G}$. Then the natural k-algebra homomorphism $kG \to k\bar{G}$ gives a one to one correspondence from all blocks b of kG onto all blocks $\bar{b}$ of $k\bar{G}$. Applying Theorem 1.1 again, we have the following.

Theorem 1.4. *Let U be a bounded complex of $k\bar{G}\bar{b}$-modules. Regarding U as a complex of kGb-modules through $kGb \to k\bar{G}\bar{b}$, we can define an affine map $\mathrm{inf}^{*}_{\bar{b},b}\colon V_{G,b}(U) \to V_{\bar{G},\bar{b}}(U)$ induced from the inflation $\mathrm{inf}_{\bar{b},b}$.*

Finally, we consider the inverse images of the affine maps above also. See Propositions 5.2 and 5.3 below.

All modules considered in this paper are assumed to be finitely generated left modules, unless otherwise stated. We end the introduction with fixing notations. Let H be a subgroup of a finite group G. For a group algebra RG over a commutative ring R, $(RG)_H$ denotes the regular $RG - RH$-bimodule and, similarly, ${}_H(RG)$ denotes the regular $RH - RG$-bimodule. For the conjugation given by an element x of G, we use the left notation: ${}^xH = xHx^{-1}$ and ${}^xa = xax^{-1}$ for $a \in RG$. Further, $c_x\colon H^*(H,k) \to H^*({}^xH,k)$ denotes the conjugation map given by x. For complexes X and Y, $X \mid Y$ means that X is isomorphic to a direct summand of Y.

2. Preliminaries

Let A and B be algebras over a commutative ring R. By an $A - B$-bimodule, we mean a bimodule on which the left and right actions of R coincide, in other words, an $A \otimes_R B^o$-module (B^o is the R-algebra opposite to B). For an R-module M and a complex C of R-modules, we set $M^* = \mathrm{Hom}_R(M, R)$ and $C^* = \mathrm{Hom}_R(C, R)$, their R-uals. Let U be a right bounded complex of A-modules. We denote by $\mathscr{P}(U)$ a projective resolusion of U, that is, $\mathscr{P}(U)$ is a right bounded complex of projective A-modules with a map of chain complexes $\mathscr{P}(U) \to U$ which is an isomorphism on homology. Consider A as an $A \otimes_R A^o$-module (i.e., $A - A$-bimodule). Further we also consider A as a complex with the degree 0 component A and all other components being zero. Then we denote by $\mathscr{P}_A$ a projective resolution of the complex A. Let U and V be right bounded complexes of A-modules. Notice that $\mathrm{Ext}_A^n(U, V) = H^n(\mathrm{Hom}_A(\mathscr{P}(U), V))$ by definition. Further, we have the following isomorphisms (see, e.g., [3, I, Section 2.7] or [8, Chapter 6]):

$$\begin{aligned}\mathrm{Ext}_A^n(U, V) &= H^n(\mathrm{Hom}_A(\mathscr{P}(U), V)) \cong H^n(\mathrm{Hom}_A(\mathscr{P}(U), \mathscr{P}(V)) \\ &\cong \mathrm{Hom}_{K(A)}(\mathscr{P}(U), \mathscr{P}(V)[n]),\end{aligned}$$

where $K(A)$ is the homotopy category of complexes of A-modules and $\mathscr{P}(V)[n]$ is the complex obtained by moving $\mathscr{P}(V)$ to the left by n places. Suppose that the R-algebra A is projective as an R-module. Then we can define Hochschild cohomology $HH^n(A)$ to be the cohomology $H^n(\mathrm{Hom}_{A\otimes A^o}(\mathscr{P}_A, A)) = \mathrm{Ext}^n_{A\otimes A^o}(A, A)$. So we have an isomorphism

$$HH^n(A) \cong \mathrm{Hom}_{K(A\otimes A^o)}(\mathscr{P}_A, \mathscr{P}_A[n]).$$

Note that if homogeneous components of U are projective as R-modules and if A also is projective as an R-module, then we see from the Künneth theorem that $\mathscr{P}_A \otimes_A U$ becomes a projective resolution of U. So we have an algebra homomorphism

$$\alpha_U\colon HH^*(A) \to \mathrm{Ext}_A^*(U, U)$$

induced by the functor $-\otimes_A U$. Here, the homomorphism α_U maps the homotopy class of a chain map $\zeta\colon \mathscr{P}_A \to \mathscr{P}_A[n]$ to that of $\zeta \otimes \mathrm{Id}_U$ (n is a nonnegative integer).

Suppose that A and B are symmetric R-algebras from now and let X be a bounded complex of $A - B$-bimodules which are projective as left A-modules and as right B-modules. M. Linckelmann gives a graded R-linear map $t_X\colon HH^*(B) \to HH^*(A)$ which is called the *transfer map* associated with X (see [10, Definition 2.9]). He then shows the following connection between transfer maps in Hochschild cohomology and ordinary cohomology of finite groups: it is known that there exists an embedding of $H^*(G, R)$ into $HH^*(RG)$, that is, an injective graded R-algebra homomorphism

$$\delta_G\colon H^*(G, R) \to HH^*(RG)$$

induced by the diagonal induction functor $\mathrm{Ind}_{\triangle G}^{G\times G}$ (see [10, Proposition 4.5]). Through these embeddings, the following diagrams are commutative

$$\begin{tikzcd} H^*(G,R) \arrow[r, "\mathrm{res}_{G,H}"] \arrow[d, "\delta_G"'] & H^*(H,R) \arrow[d, "\delta_H"] \\ HH^*(RG) \arrow[r, "t_{H(RG)}"'] & HH^*(RH), \end{tikzcd} \qquad \begin{tikzcd} H^*(H,R) \arrow[r, "\mathrm{tr}_{H,G}"] \arrow[d, "\delta_H"'] & H^*(G,R) \arrow[d, "\delta_G"] \\ HH^*(RH) \arrow[r, "t_{(RG)_H}"'] & HH^*(RG), \end{tikzcd}$$

where $\mathrm{res}_{G,H}$ and $\mathrm{tr}_{H,G}$ denote the restriction map and the transfer map on ordinary cohomology of finite groups, respectively. He also gives a notion of X-stable elements as follows.

DEFINITION 2.1 (Linckelmann [10, Definition 3.1]). An element $[\zeta] \in HH^*(A)$ is said to be X-*stable* if there is $[\tau] \in HH^*(B)$ such that the following diagram is homotopy commutative for any nonnegative integer n

$$\begin{tikzcd} \mathscr{P}_A \otimes_A X \arrow[r, "\simeq"] \arrow[d, "\zeta_n \otimes \mathrm{Id}_X"'] & X \otimes_B \mathscr{P}_B \arrow[d, "\mathrm{Id}_X \otimes \tau_n"] \\ \mathscr{P}_A[n] \otimes_A X \arrow[r, "\simeq"'] & X \otimes_B \mathscr{P}_B[n], \end{tikzcd}$$

where ζ_n and τ_n are the components in degree n of ζ and τ, respectively, and where the horizontal arrows are given by the natural homotopy equivalences $\mathscr{P}_A \otimes_A X \simeq X \otimes_B \mathscr{P}_B$ lifting the natural isomorphism $A \otimes_A X \cong X \otimes_B B$. We denote by $HH^*_X(A)$ the set of X-stable elements in $HH^*(A)$.

Then, using the notion of X-stable elements, the transfer map t_X is characterized as follows [10, Lemma 3.4 and Theorem 3.6]. Let $t^0_X \colon Z(B) \to Z(A)$ be the linear map obtained by the degree zero component of t_X, composing with the natural isomorphisms $Z(B) \cong HH^0(B)$ and $Z(A) \cong HH^0(A)$. Set $t^0_X(1_B) = \pi_X$. Similarly, set $t^0_{X^*}(1_A) = \pi_{X^*}$ for the transfer map $t_{X^*} \colon HH^*(A) \to HH^*(B)$. If $[\zeta] \in HH^n(A)$ is X-stable and $[\tau] \in HH^n(B)$ is the element corresponding to $[\zeta]$ in Definition 2.1, then it holds that (i) $t_X([\tau]) = \pi_X[\zeta]$, (ii) $[\tau]$ is the X^*-stable element and $t_{X^*}([\zeta]) = \pi_{X^*}[\tau]$.

From the above, if π_X is invertible, then we see that

$$T_X = (\pi_X)^{-1} t_X \colon HH^*_{X^*}(B) \to HH^*_X(A)$$

is a surjective graded R-algebra homomorphism. Further, if both π_X and π_{X^*} are invertible, then it holds that $T_{X^*} \circ T_X([\tau]) = [\tau]$ for $[\tau] \in HH^*(B)$ and that $T_X \circ T_{X^*}([\zeta]) = [\zeta]$ for $[\zeta] \in HH^*(A)$.

For example, let $X = {}_H(RG)$. Then we have from [10, Proposition 4.8] that $\mathrm{Im}(\delta_G) \subseteq HH^*_{(RG)_H}(RG)$. Further, we see from [10, Example 2.6 and Definition 3.1] that $\pi_{H(RG)} = 1_{RH}$. Thus we see that $t_{H(RG)}\colon HH^*_{(RG)_H}(RG) \to HH^*_{H(RG)}(RG)$ is a surjective graded R-algebra homomorphism in the above commutative diagram for $\mathrm{res}_{G,H}$ and $t_{H(RG)}$.

Using Puig's pointed group theory, M. Linckelmann defined cohomology rings of blocks. We recall here some definitions and results on Puig's pointed group theory [15]. Let k be an algebraically closed field of prime characteristic p. For a p-subgroup P of G, let $\mathrm{Br}^G_P\colon (kG)^P \to kC_G(P)$ be the *Brauer homomorphism* of P, which is a surjective algera homomorphism. A *point* of P on kG is a $((kG)^P)^\times$-conjugacy class γ of primitive idempotents in $(kG)^P$ and if $\mathrm{Br}^G_P(\gamma) \neq 0$, γ is called a *local point* of P on kG. A *block* of kG is a primitive idempotent b in the center $Z(kG)$. A *defect group D* of the block b is a maximal p-subgroup of G such that $\mathrm{Br}^G_D(b) \neq 0$. For a defect group D of b, there is a primitive idempotent $i \in (kGb)^D$ such that $\mathrm{Br}^G_D(i) \neq 0$. So the point γ that contains i is a local point of D on kG contained in the block b. The idempotent i is called a *source idempotent* of the block b and the pair (D, γ), which we denote by D_γ, is called a *defect pointed group* of the block b. Since $\mathrm{Br}^G_D(i)$ is a primitive idempotent in $kC_G(D)$, there is a unique block e_D of $kC_G(D)$ such that $\mathrm{Br}^G_D(i)e_D = \mathrm{Br}^G_D(i)$. The pair (D, e_D) is the *maximal b-Brauer pair* which corresponds to D_γ. In general, a *Brauer pair* of G is a pair (Q, f) where Q is a p-subgroup of G and f is a block of $kC_G(Q)$. For a Brauer pair (Q, f) and a maximal b-Brauer pair (D, e_D) corresponding to D_γ, if Q is a subgroup of D and f satisfies $\mathrm{Br}^G_Q(\gamma)f = \mathrm{Br}^G_Q(\gamma)$, then $(Q, f) \le (D, e_D)$. It is known that for any subgroup Q of D, there is a unique block e_Q such that $(Q, e_Q) \le (D, e_D)$. For more details, see [16, Section 40].

DEFINITION 2.2 (Linckelmann [10, Definition 5.1]). Let G be a finite group, b a block of kG and D_γ a defect pointed group of b. The *cohomology ring of the block b of kG associated with D_γ* is the subring $H^*(G, b, D_\gamma)$ of $H^*(D, k)$ which consists of all $[\zeta] \in H^*(D, k)$ satisfying $\mathrm{res}_{x^{-1}D,Q} \circ c_{x^{-1}}([\zeta]) = \mathrm{res}_{D,Q}([\zeta])$ for any subgroup Q in D and for any $x \in G$ with ${}^x(Q, e_Q) = ({}^xQ, {}^xe_Q) \le (D, e_D)$.

If $D_\gamma{}'$ is another defect pointed group of b, then there exists $g \in G$ such that $D_\gamma{}' = {}^gD_\gamma$ (see [16, Proposition 40.13]), and we see that the conjugation map c_g induces an isomorphism $H^*(G, b, D_\gamma) \cong H^*(G, b, D_\gamma{}')$.

With the notation of Definition 2.2, let $i \in \gamma$. Consider kGi as a $kGb - kD$-bimodule and ikG as a $kD - kGb$-bimodule. By [10, Theorem 5.6], π_{kGi} and π_{ikG} are invertible in $Z(kGb)$ and $Z(kD)$, respectively. Thus $T_{kGi} = (\pi_{kGi})^{-1}t_{kGi}\colon HH^*_{ikG}(kD) \to HH^*_{kGi}(kGb)$ is a graded k-algebra isomorphism (notice that $(kGi)^* \cong ikG$ as $kD - kGb$-bimodules). Furthermore from [10, Corollary 3.8 and Proposition 5.4], we have an injective graded k-algebra homomorphism $\delta_D\colon H^*(G, b, D_\gamma) \to HH^*_{ikG}(kD)$, where δ_D is the restriction of the diagonal embedding $H^*(D, k) \to HH^*(kD)$ stated before. Therefore, we obtain an injective graded k-algebra homomorphism

$$T_{kGi} \circ \delta_D \colon H^*(G, b, D_\gamma) \to HH^*_{kGi}(kGb).$$

Let U be a finitely generated kG-module. In [4], J.F. Carlson introduces a notion of the variety of U as follows. There is a k-algebra homomorphism $\gamma_U \colon H^*(G,k) \to \mathrm{Ext}^*_{kG}(U,U)$ mapping $[\zeta] \in H^*(G,k)$ to $[\zeta] \cup I \in \mathrm{Ext}^*_{kG}(U,U)$, where $\cup$ is a cup product and I is the identity element of $\mathrm{Ext}^*_{kG}(U,U)$. Let $I^*_G(U)$ be the kernel of γ_U. Then $V_G(U)$, the *variety* of U, is the maximal ideal spectrum of the quotient $H^*(G,k)/I^*_G(U)$. On the other hand, $\mathscr{P}(k) \otimes_k U$ is a projective resolution of U, where $\mathscr{P}(k)$ is a projective resolution of the trivial kG-module k (see e.g. [11, 2.9]). So we see that γ_U is induced by the functor $- \otimes_k U$ through the isomorphisms $H^n(G,k) \cong \mathrm{Hom}_{K(kG)}(\mathscr{P}(k), \mathscr{P}(k)[n])$ and $\mathrm{Ext}^n_{kG}(U,U) \cong \mathrm{Hom}_{K(kG)}(\mathscr{P}(k) \otimes_k U, \mathscr{P}(k) \otimes_k U[n])$. Similarly, for any bounded complex U of kG-modules, the functor $- \otimes_k U$ induces a k-algebra homomorphism

$$\gamma_U \colon H^*(G,k) \to \mathrm{Ext}^*_{kG}(U,U).$$

We also write $I^*_G(U)$ for the kernel of γ_U, and $V_G(U)$ for the variety of U. As a remarkable fact in this direction, we have from [11, 2.9] that the above k-algebra homomorphism γ_U is equal to the composite of k-algebra homomorphisms

$$H^*(G,k) \xrightarrow{\delta_G} HH^*(kG) \xrightarrow{\alpha_U} \mathrm{Ext}^*_{kG}(U,U).$$

For a bounded complex U of kGb-modules, M. Linckelmann [11] gives a notion of the variety of U associated with the block b.

DEFINITION 2.3 (Linckelmann [11, Definition 4.1]). Let G be a finite group, b a block of kG, D_γ a defect pointed group of b, and i a source idempotent in γ. For any bounded complex U of kGb-modules, denote by $I^*_{G,b,D_\gamma}(U)$ the kernel in $H^*(G, b, D_\gamma)$ of the composite of k-algebra homomorphisms

$$H^*(G, b, D_\gamma) \xrightarrow{T_{kGi} \circ \delta_D} HH^*(kGb) \xrightarrow{\alpha_U} \mathrm{Ext}^*_{kGb}(U,U)$$

and let $V_{G,b}(U)$ be the maximal ideal spectrum of $H^*(G,b,D_\gamma)/I^*_{G,b,D_\gamma}(U)$. We also let $V_{G,b}$ be the maximal ideal spectrum of $H^*(G, b, D_\gamma)$.

For another defect pointed group $D_\gamma{}'$ of b, there exists $g \in G$ such that $c_g \colon H^*(G,b,D_\gamma) \cong H^*(G,b,D_\gamma{}')$, as stated above. So the isomorphism class of the variety $V_{G,b}(U)$ does not depend on the choice of D_γ.

M. Linckelmann shows the following connection between varieties associated with blocks and Carlson's module varieties. By the definition, it is clear that the restriction map $\mathrm{res}_{G,D}$ induces an algebra homomorphism $\rho_b \colon H^*(G,k) \to H^*(G,b,D_\gamma)$.

Let $\rho_b^*\colon V_{G,b} \to V_G(k)$ be the affine map defined by ρ_b, where $V_G(k)$ is the maximal ideal spectrum of $H^*(G,k)$.

Theorem 2.4 (Linckelmann [11, Corollary 4.4]). *For any bounded complex U of kGb-modules, it holds that $I_G^*(U) = \rho_b^{-1}(I_{G,b,D_\gamma}^*(U))$. Thus, $\rho_b^*\colon V_{G,b}(U) \to V_G(U)$ is a finite surjective map and the dimensions of $V_{G,b}(U)$ and $V_G(U)$ coincide. In particular, if b is the principal block of kG, then the above map is an isomorphism.*

3. Proof of Theorem 1.1 and its applications

In this section we give a proof of Theorem 1.1. Further, applying this theorem we consider the question whether the varieties $V_{G,b}(U)$ are invariant under the block correspondence in Brauer's first main theorem (with the assumption that defect groups are abelian). Theorem 1.1 is based on the following theorem which is the reverse version of Linckelmann's theorem [11, Theorem 5.1].

Theorem 3.1. *Let A, B be symmetric algebras over a commutative ring R, and let X be a bounded complex of A-B bimodules whose components are projective as left and right modules. If π_{X^*} is invertible in $Z(B)$, then for any bounded complex U of A-modules there is a commutative diagram of graded R-algebra homomorphisms*

$$\begin{tikzcd} HH_X^*(A) \arrow[r, "\alpha_U"] \arrow[d, "T_{X^*}"'] & \operatorname{Ext}_A^*(U,U) \arrow[d, "\beta_{X^*,U}"] \\ HH_{X^*}^*(B) \arrow[r, "\alpha_{X^*\otimes_A U}"'] & \operatorname{Ext}_B^*(X^*\otimes_A U, X^*\otimes_A U), \end{tikzcd}$$

where the horizontal maps are induced by the functors $-\otimes_A U$ and $-\otimes_B (X^*\otimes_A U)$, respectively, and where the right vertical map is induced by the functor $X^*\otimes_A -$.

Theorem 3.1 implies the next proposition. With the notation in Section 2, let $X = kGi$ regarded as a $kGb - kD$-bimodule, and so $X^* = (kGi)^* \cong ikG$ as $kD - kGb$-bimodules. Further, for a bounded complex U of kGb-modules, let $\gamma_{iU}\colon H^*(D,k) \to \operatorname{Ext}_{kD}^*(iU, iU)$ be the k-algebra homomorphism induced by the functor $-\otimes_k iU$, where $iU = ikG \otimes_{kGb} U$ is considered as a complex of kD-modules.

Proposition 3.2. *Let U be a bounded complex of kGb-modules. The following diagram of graded k-algebra homomorphisms is commutative.*

$$\begin{tikzcd}
H^*(G,b,D_\gamma) \arrow[r, "\alpha_U \circ T_{kGi} \circ \delta_D"] \arrow[d, "\iota"'] & \mathrm{Ext}^*_{kGb}(U,U) \arrow[d, "i \circ \mathrm{res}_{G,D}"] \\
H^*(D,k) \arrow[r, "\gamma_{iU}"'] & \mathrm{Ext}^*_{kD}(iU,iU),
\end{tikzcd}$$

where the right vertical map is the composite of the restriction homomorphism and the projection from $\mathrm{Ext}^*_{kD}(U,U)$ *onto* $\mathrm{Ext}^*_{kD}(iU,iU)$*, and the left vertical map is the inclusion map.*

Proof. From [10, Theorem 5.6], π_{ikG} is invertible in $Z(kD)$. So by Theorem 3.1, we have that $\alpha_{iU} \circ T_{ikG} = \beta_{ikG,U} \circ \alpha_U$. Now, since π_{kGi} is also invertible, T_{kGi} and T_{ikG} are mutually inverse k-algebra isomorphisms from [10, Theorem 3.6]. Thus it follows that $\alpha_{iU} = \beta_{ikG,U} \circ \alpha_U \circ T_{kGi}$. That is, we have the following commutative diagram of graded k-algebra homomorphisms:

$$\begin{tikzcd}
HH^*_{kGi}(kGb) \arrow[r, "\alpha_U"] & \mathrm{Ext}^*_{kGb}(U,U) \arrow[d, "\beta_{ikG,U}"] \\
HH^*_{ikG}(kD) \arrow[u, "T_{kGi}"] \arrow[r, "\alpha_{iU}"'] & \mathrm{Ext}^*_{kD}(iU,iU).
\end{tikzcd}$$

From this diagram, we can form the commutative diagram

$$\begin{tikzcd}
H^*(G,b,D_\gamma) \arrow[r, "\delta_D"] \arrow[d, "\iota"'] & HH^*_{ikG}(kD) \arrow[r, "\alpha_U \circ T_{kGi}"] \arrow[d, equal] & \mathrm{Ext}^*_{kGb}(U,U) \arrow[d, "\beta_{ikG,U}"] \\
H^*(D,k) \arrow[r, "\delta_D"'] & HH^*_{ikG}(kD) \arrow[r, "\alpha_{iU}"'] & \mathrm{Ext}^*_{kD}(iU,iU).
\end{tikzcd}$$

Now, it is clear that the functor $ikG \otimes_{kGb} -$ gives the algebra homomorphism $i \circ \mathrm{res}_{G,D}$ and so $\beta_{ikG,U} = i \circ \mathrm{res}_{G,D}$. Moreover, from [11, 2.9], the composite $\alpha_{iU} \circ \delta_D$ is equal to the algebra homomorphism γ_{iU} given by the functor $- \otimes_k iU$. So the proposition follows. $\square$

The following lemma is a block variety version of [3, II, Proposition 5.7.5] on Carlson's varieties $V_G(U)$.

Lemma 3.3. *Let* U_1 *and* U_2 *be bounded complexes of* kGb*-modules. Then we have that* $I^*_{G,b,D_\gamma}(U_1 \oplus U_2) = I^*_{G,b,D_\gamma}(U_1) \cap I^*_{G,b,D_\gamma}(U_2)$*, and hence we have that* $V_{G,b}(U_1 \oplus U_2) = V_{G,b}(U_1) \cup V_{G,b}(U_2)$*.*

Proof. Notice that $\mathscr{P}_{kGb} \otimes_{kGb} (U_1 \oplus U_2) \cong (\mathscr{P}_{kGb} \otimes_{kGb} U_1) \oplus (\mathscr{P}_{kGb} \otimes_{kGb} U_2)$. Under this isomorphism, we have the decomposition $\zeta \otimes \mathrm{Id}_{U_1 \oplus U_2} = (\zeta \otimes \mathrm{Id}_{U_1}) \oplus (\zeta \otimes \mathrm{Id}_{U_2})$ for a chain map $\zeta \colon \mathscr{P}_{kGb} \to \mathscr{P}_{kGb}[n]$. Thus for the projection

$$\mu_i \colon \mathrm{Ext}^*_{kGb}(U_1 \oplus U_2, U_1 \oplus U_2) \longrightarrow \mathrm{Ext}^*_{kGb}(U_i, U_i) \ (i = 1,\ 2),$$

it follows that $\mu_i \circ \alpha_{U_1 \oplus U_2} = \alpha_{U_i}$ $(i = 1,\ 2)$, respectively. Then these commutations show that $I^*_{G,b,D_\gamma}(U_1 \oplus U_2) \subseteq I^*_{G,b,D_\gamma}(U_1) \cap I^*_{G,b,D_\gamma}(U_2)$. Moreover, if $\zeta \otimes \mathrm{Id}_{U_i}$ $(i = 1,\ 2)$ is homotopic to 0, respectively, then $\zeta \otimes \mathrm{Id}_{U_1 \oplus U_2}$ is also homotopic to 0. So the lemma follows. $\square$

The following lemma is well-known. We include here a proof for completeness.

Lemma 3.4. *Let U be a bounded complex of kGb-modules, D_γ a defect pointed group of b and let $i \in \gamma$. Then U is isomorphic to a direct summand of $kGi \otimes_{kD} iU$.*

Proof. Since $\mathrm{Tr}^G_D(i) = \Sigma_{x \in [G/D]} x i x^{-1}$ is invertible in $Z(kGb)$ (see [10, Theorem 5.6]), we have $\beta = (\mathrm{Tr}^G_D(i))^{-1} \in Z(kGb)$. Now, consider the chain map $t \colon U \to kGi \otimes_{kD} iU$ consisting of kG-homomorphisms $t_n \colon U_n \to kGi \otimes_{kD} iU_n$ defined by

$$u \longmapsto \sum_{x \in [G/D]} xi \otimes i\beta x^{-1} u \quad \text{for } u \in U_n,$$

where U_n is the degree n component of U. On the other hand, there exists a chain map $s \colon kGi \otimes_{kD} iU \to U$ consisting of natural homomorphisms $s_n \colon kGi \otimes_{kD} iU_n \to U_n$ induced by the action of kG on U_n. Then, since $\mathrm{Tr}^G_D(i\beta) = \mathrm{Tr}^G_D(i)\beta = b$, we have $s_n \circ t_n(u) = u$ for $u \in U_n$. So the lemma follows. $\square$

Proof of Theorem 1.1. From [11, Theorem 5.1], there exists a commutative diagram

$$\begin{tikzcd}
HH^*_{ikG}(kD) \arrow[r, "\alpha_{iU}"] \arrow[d, "T_{kGi}"'] & \mathrm{Ext}^*_{kD}(iU, iU) \arrow[d, "\beta_{kGi,iU}"] \\
HH^*_{kGi}(kGb) \arrow[r, "\alpha_{kGi \otimes_{kD} iU}"'] & \mathrm{Ext}^*_{kGb}(kGi \otimes_{kD} iU, kGi \otimes_{kD} iU).
\end{tikzcd}$$

Now, from Lemma 3.4, we have the canonical projection μ: $\mathrm{Ext}^*_{kGb}(kGi \otimes_{kD} iU, kGi \otimes_{kD} iU) \longrightarrow \mathrm{Ext}^*_{kGb}(U, U)$. Then from Lemma 3.3 it follows that $\mu \circ \alpha_{kGi \otimes_{kD} iU} =$

α_U. Thus the above diagram induces the following commutative diagram:

$$\begin{tikzcd}
H^*(D,k) \arrow[r, "\delta_D"] & HH^*_{ikG}(kD) \arrow[r, "\alpha_{iU}"] \arrow[d, equal] & \mathrm{Ext}^*_{kD}(iU, iU) \arrow[d, "\mu\circ\beta_{kGi,iU}"] \\
H^*(G,b,D_\gamma) \arrow[u, "\iota"] \arrow[r, "\delta_D"'] & HH^*_{ikG}(kD) \arrow[r, "\alpha_U\circ T_{kGi}"'] & \mathrm{Ext}^*_{kGb}(U,U).
\end{tikzcd}$$

We note that $\alpha_{iU}\circ\delta_D=\gamma_{iU}$. Hence, we have that $H^*(G,b,D_\gamma)\cap I^*_D(iU)\subseteq I^*_{G,b,D_\gamma}(U)$. Conversely, by Proposition 3.2 we have that $I^*_{G,b,D_\gamma}(U)\subseteq H^*(G,b,D_\gamma)\cap I^*_D(iU)$. Hence we have $I^*_{G,b,D_\gamma}(U)=H^*(G,b,D_\gamma)\cap I^*_D(iU)$. In particular, if b is a nilpotent block of kG, then $H^*(G,b,D_\gamma)=H^*(D,k)$ (see [11, 3.6]) and so we have that $I^*_{G,b,D_\gamma}(U)=I^*_D(iU)$ and $V_{G,b}(U)=V_D(iU)$. This completes the proof. □

We came to know during the circulation of this paper without the last section that essentially the same fact as Theorem 1.1 was shown in M. Linckelmann [13] and Corollary 3.5 below also was obtained in it.

As is well known, the nilradical $\sqrt{0}$ of $H^*(G,k)$ is the intersection of all maximal ideals of $H^*(G,k)$ (see [14]). Recall that $H^*(G,k)/\sqrt{0}$ is a finitely generated commutative k-algebra and $H^*(D,k)$ is finitely generated as a module over $H^*(G,k)$ (via the restriction map) (see [3, II] and [5]). Thus $H^*(D,k)$ is also finitely generated (that is, Noetherian) as a module over $H^*(G,b,D_\gamma)$ and so, by [3, II, Section 5.4] and [14, Section 9, Lemma 2], we have a finite surjective affine map $\iota_{D,b}\colon V_D\to V_{G,b}$ induced by the inclusion map $\iota\colon H^*(G,b,D_\gamma)\to H^*(D,k)$, where V_D is the maximal ideal spectrum of $H^*(D,k)$ and $V_{G,b}$ is the maximal ideal spectrum of $H^*(G,b,D_\gamma)$. Then, Proposition 3.2 shows that for any bounded complex U of kGb-modules, we can define the finite affine map

$$\iota_{D,b}\colon V_D(iU)\longrightarrow V_{G,b}(U).$$

Proof of Corollary 1.2. By Theorem 1.1, it is clear that $\iota_{D,b}$ is a surjective map. Further, Theorem 1.1 shows also that the Krull dimensions of $H^*(D,k)/I^*_D(iU)$ and $H^*(G,b,D_\gamma)/I^*_{G,b,D_\gamma}(U)$ coincide. So the dimensions of $V_G(iU)$ and $V_{G,b}(U)$ coincide. □

With the notation of Theorem 1.1, let U_0 be a bounded complex of kD-modules such that $U_0\mid iU$ and $U\mid kGi\otimes_{kD}U_0$. Then, from the proof of Theorem 1.1 we have also

$$I^*_{G,b,D_\gamma}(U)=H^*(G,b,D_\gamma)\cap I^*_D(U_0).$$

Hence $\iota_{D,b}\colon V_D(U_0)\to V_{G,b}(U)$ is also a finite surjective map.

Corollary 3.5. *Let U be an indecomposable kGb-module with vertex D. Suppose that the dimension of a source of U is not a multiple of p. Then we have that $V_{G,b}(U) = V_{G,b}$.*

Proof. We see from Lemma 3.4 that $U \mid kGi \otimes_{kD} iU$. Thus there exists an indecomposable kD-module U_0 such that $U_0 \mid iU$ and $U \mid kGi \otimes_{kD} U_0$, so that U_0 is a source of U. By the assumption, we see that $\sqrt{I_D^*(U_0)} = \sqrt{0}$ in $H^*(D,k)$ from [3, II, Corollary 5.8.5]. Further, by the above it follows that $\sqrt{I^*_{G,b,D_\gamma}(U)} = H^*(G,b,D_\gamma) \cap \sqrt{I_D^*(U_0)}$. Thus we have that $\sqrt{I^*_{G,b,D_\gamma}(U)} = \sqrt{0}$ in $H^*(G,b,D_\gamma)$, that is, $V_{G,b}(U) = V_{G,b}$. $\square$

As is well known, $b_0 = \mathrm{Br}_D^G(b)$ is a block of $kN_G(D)$ with the defect group D and is called the Brauer correspondent of b. Let D_{γ_0} be a defect pointed group of b_0, and let $i_0 \in \gamma_0$. Since $\mathrm{Br}_D^{N_G(D)}(i_0)$ is primitive in $kC_G(D)$, there is a primitive idempotent $i \in (kG)^D$ such that $i = i_0 i = i i_0$ and $\mathrm{Br}_D^G(i) = \mathrm{Br}_D^{N_G(D)}(i_0)$. Then i belongs to the block b. Indeed, $\mathrm{Br}_D^G(ib) = \mathrm{Br}_D^G(i)\,\mathrm{Br}_D^G(b) = \mathrm{Br}_D^{N_G(D)}(i_0)\,\mathrm{Br}_D^{N_G(D)}(b_0) = \mathrm{Br}_D^{N_G(D)}(i_0) \neq 0$. Let γ be the $((kG)^D)^\times$-conjugacy class of i. Then we obtain a defect pointed group D_γ of b. Let (D, e_D) be the maximal b-Brauer pair corresponding to D_γ. Note that (D, e_D) is also the maximal b_0-Brauer pair corresponding to D_{γ_0}.

Corollary 3.6. *With the above notation, let b_0 be the Brauer correspondent of b and let U be a bounded complex of kGb-modules. Suppose that D is abelian. Then we have $I^*_{N_G(D),b_0,D_{\gamma_0}}(b_0U) \subseteq I^*_{G,b,D_\gamma}(U)$, and hence we have that $V_{G,b}(U) \subseteq V_{N_G(D),b_0}(b_0U)$.*

Proof. Theorem 1.1 shows that $I^*_{G,b,D_\gamma}(U) = H^*(G,b,D_\gamma) \cap I_D^*(iU)$ and $I^*_{N_G(D),b_0,D_{\gamma_0}}(b_0U) = H^*(N_G(D),b_0,D_{\gamma_0}) \cap I_D^*(i_0U)$. As in [10, 5.2.3], if D is abelian, then we have that $H^*(G,b,D_\gamma) = H^*(D,k)^{N_G(D,e_D)/C_G(D)}$ and similarly $H^*(N_G(D),b_0,D_{\gamma_0}) = H^*(D,k)^{N_G(D,e_D)/C_G(D)}$. So $H^*(G,b,D_\gamma) = H^*(N_G(D),b_0,D_{\gamma_0})$. Moreover, from [3, II, Proposition 5.7.5] it follows that $I_D^*(i_0U) \subseteq I_D^*(iU)$ since $i_0U = iU \oplus (i_0 - i)U$. Thus we have that $I^*_{N_G(D),b_0,D_{\gamma_0}}(b_0U) \subseteq I^*_{G,b,D_\gamma}(U)$ and so $V_{G,b}(U) \subseteq V_{N_G(D),b_0}(b_0U)$. $\square$

For Carlson's module varieties, the following is known. Let P be an abelian Sylow p-subgroup of G, so that $N_G(P)$ controls the p-fusion in G. Let U be an indecomposable kG-module with vertex P and let W be the Green correspondent of U with respect to $(G, P, N_G(P))$. Then it follows that $V_G(U) \cong V_{N_G(P)}(W)$ (see [2, Theorem 2.26.9] and [3, I, Proposition 3.8.4]). Therefore, in this case, we see from Theorem 2.4 that if b is the principal block of kG, then $V_{G,b}(U) \cong V_{N_G(P),b_0}(W)$ for any indecomposable kGb-module U with vertex P and the Green correspondent W of U.

Extending this fact to any block b of kG, we obtain Theorem 1.3 as follows from Lemma 3.7 below. Note that D is not necessarily abelian in Lemma 3.7.

With the above notation, let $T = T(e_D)$ be the inertial group of e_D in $N_G(D)$, that is, $T = \{x \in N_G(D); {}^x e_D = e_D\}$. Notice that $T = N_G(D, e_D) = N_G(D_\gamma)$ (see [16, Proposition 40.13]), and e_D is also a block of kT. Using the Clifford theory of blocks, we have the following.

Lemma 3.7. *Let U be an indecomposable kGb-module with vertex D. Let W be the Green correspondent of U with respect to $(G, D, N_G(D))$. Let $i \in \gamma$ and $i_0 \in \gamma_0$.*
(i) *We can choose a source U_0 of U such that $U_0 \mid iU$ and $U \mid kGi \otimes_{kD} U_0$, and a source W_0 of W such that $W_0 \mid i_0 W$ and $W \mid kN_G(D)i_0 \otimes_{kD} W_0$.*
(ii) *Let U_0 and W_0 be the sources satisfying the conditions of* (i). *Then there exists an element $t \in T$ such that $tU_0 \cong W_0$ as kD-modules.*

Proof. First of all, we note that if i and i' lie in γ, then $iU \cong i'U$ and $kGi \cong kGi'$ as kD-modules, and likewise for $i_0 \in \gamma_0$. Indeed, let $i' = {}^z i$ for an element z in $((kG)^D)^\times$. Then, mapping $a \in iU$ to $za \in i'U$, we have the isomorphism $iU \cong i'U$ as kD-modules. By applying this argument, we obtain all other isomorphisms. Thus, the choices of U_0 and W_0 in (i) depend only on the points γ and γ_0. As we have shown in Corollary 3.5, (i) now follows from Lemma 3.4.

Next we prove (ii). Suppose that $i \in \gamma$ satisfies $i = i_0 i = i i_0$. Now, since W is the Green correspondent of U, we have that $b_0 U \cong W \oplus W'$ and so $i_0 U \cong i_0 W \oplus i_0 W'$ where any indecomposable direct summand of W' (and so $i_0 W'$) does not have vertex D. Here, from $iU \mid i_0 U$, we see $U_0 \mid i_0 U$. Then since U_0 has vertex D, we see $U_0 \mid i_0 W$. Further, we can choose $i_0 \in \gamma_0$ such that $e_D i_0 = i_0 e_D = i_0$. Indeed, let i'_0 be a primitive idempotent in $(kN_G(D))^D$ such that $e_D i'_0 = i'_0 e_D = i'_0$ and $\mathrm{Br}_D^{N_G(D)}(i'_0) \neq 0$. Let γ'_0 be a point of D on $kN_G(D)$ containing i'_0. Then from $\mathrm{Br}_D^{N_G(D)}(i'_0)e_D = \mathrm{Br}_D^{N_G(D)}(i'_0)$, $D_{\gamma'_0}$ is the defect pointed group of b_0 corresponding to (D, e_D), because the relation $\mathrm{Br}_D^{N_G(D)}(i'_0)e_D = \mathrm{Br}_D^{N_G(D)}(i'_0)$ defines a bijection between maximal b_0-Brauer pairs and defect pointed groups of b_0 (see [16, Proposition 40.13]). Thus we see that $\gamma'_0 = \gamma_0$. By this choice of $i_0 \in \gamma_0$, we have that $i_0 W \mid e_D W$. So it holds that $U_0 \mid e_D W$ and also $W_0 \mid e_D W$. Let $e_D = e_T$ and $N_G(D) = N$. Using the Clifford theory, we show that the kT-module $e_T W$ is indecomposable and has vertex D. Since $b_0 = \sum_{x \in N/T} {}^x e_T$ and ${}^x e_T {}^y e_T = 0$ for $x \not\equiv y \pmod T$, we have that $W = b_0 W = kN \otimes_{kT} e_T W$ and so $e_T W$ is indecomposable and vertices of $e_T W$ contain D. On the other hand, since $W \mid kNi_0 \otimes_{kD} W_0$ and $e_T i_0 = i_0$, we see that $W \mid kNe_T \otimes_{kD} W_0$ so that $W = b_0 W \mid b_0 kNe_T \otimes_{kD} W_0$. Here since $b_0 kN = \bigoplus_{(x,y) \in [N \times N/T \times T]} x(kTe_T)y^{-1}$, we see that $e_T(b_0 kN)e_T = kTe_T$. Thus $e_T W \mid e_T(b_0 kN)e_T \otimes_{kD} W_0 = kTe_T \otimes_{kD} W_0$ and so $e_T W \mid kT \otimes_{kD} W_0$. Therefore we see that $e_T W$ has vertex D. Further, from $W_0 \mid e_T W$, we see that W_0 is a source of $e_T W$. Now, we also have $U_0 \mid e_T W$ and so we see that there is an element $t \in T$ such that $tU_0 \cong W_0$ as kD-modules. $\square$

Proof of Theorem 1.3. Let U_0 and W_0 be the sources chosen in Lemma 3.7. Then, as noted preceding to Corollary 3.5, we see that $I^*_{G,b,D_\gamma}(U) = H^*(G,b,D_\gamma) \cap I^*_D(U_0)$ and $I_{N_G(D),b_0,D_{\gamma_0}}(W)^* = H^*(N_G(D),b_0,D_{\gamma_0}) \cap I^*_D(W_0)$. Now, let $t \in T$ be the element obtained in Lemma 3.7 and let $c_t\colon H^*(D,k) \to H^*(D,k)$ be the conjugation map given by t. Then, since $T = N_G(D,e_D)$, we see from Definition 2.2 that c_t is the identity map on $H^*(G,b,D_\gamma)$. Further it is easy to see that $c_t \circ \gamma_{U_0} = \gamma_{tU_0} \circ c_t$ (for γ_{U_0} and γ_{tU_0} see Section 2) and so we have that $c_t(I^*_D(U_0)) = I^*_D(tU_0) = I^*_D(W_0)$. Here, since D is abelian, we have that $H^*(G,b,D_\gamma) = H^*(N_G(D),b_0,D_{\gamma_0})$ (see the proof of Corollary 3.6). Therefore it follows that $I^*_{G,b,D_\gamma}(U) = c_t(I^*_{G,b,D_\gamma}(U)) = c_t(H^*(G,b,D_\gamma) \cap I^*_D(U_0)) = H^*(G,b,D_\gamma) \cap I^*_D(W_0) = H^*(N_G(D),b_0,D_{\gamma_0}) \cap I^*_D(W_0) = I_{N_G(D),b_0,D_{\gamma_0}}(W)$, that is, we have that $V_{G,b}(U) = V_{N_G(D),b_0}(W)$. $\square$

4. Module varieties and quotient groups

Let G be a finite group and let Z be a central p-subgroup of G. We denote G/Z by $\bar{G}$ and let $f\colon kG \to k\bar{G}$ be the natural k-algebra homomorphism. As is well known, f gives a one to one correspondence from all blocks of kG onto all blocks of $k\bar{G}$. In this section we consider relations of the varieties under this correspondence. For a subgroup H of G, its image in $\bar{G}$ is denoted by $\bar{H}$. In general, the mark $^-$ will be attached to the quantities associated with $\bar{G}$ and $k\bar{G}$. Let P be a p-subgroup of G. Consider k-algebra homomorphisms $f^P\colon (kG)^P \to (k\bar{G})^{\bar{P}}$ and composite $f_{C_G(P)}\colon kC_G(P) \to k\overline{C_G(P)} \to kC_{\bar{G}}(\bar{P})$ induced by f, where $(kG)^P$ is the subalgebra of kG consisting of all P-fixed elements and $(k\bar{G})^{\bar{P}}$ likewise. Then $\mathrm{Br}_{\bar{P}}^{\bar{G}} \circ f^P = f_{C_G(P)} \circ \mathrm{Br}_P^G$ as is well known. Now, since Z is a central p-subgroup we see that $\overline{C_G(P)}$ is a normal subgroup of $C_{\bar{G}}(\bar{P})$ and $C_{\bar{G}}(\bar{P})/\overline{C_G(P)}$ is a p-group. Indeed, for $x \in G$, let $x = x_p x_{p'}$ (where x_p is the p-component and $x_{p'}$ is the p-regular component of x) and if $\bar{x} \in C_{\bar{G}}(\bar{P})$, then we see $x_{p'} \in C_G(P)$ so that $C_{\bar{G}}(\bar{P})/\overline{C_G(P)}$ is a p-group.

Lemma 4.1. *With the notation above, if i is a primitive idempotent in $(kG)^P$, then $f^P(i)$ remains a primitive idempotent in $(k\bar{G})^{\bar{P}}$.*

Proof. We follow the terminology and arguments in Külshammer-Puig [9]. Let $\tilde{f}^P$ and $\widetilde{f_{C_G(P)}}$ be the exomorphisms determined by f^P and $f_{C_G(P)}$, respectively. From [9, Theorem 3.16], it follows that $\tilde{f}^P$ is a strict semicovering for any p-subgroup P of G if and only if $\widetilde{f_{C_G(P)}}$ is a strict semicovering for any p-subgroup P of G. Now, from [9, Example 3.9] we see that $\widetilde{f_{C_G(P)}}$ is a strict semicovering for any P, since $C_{\bar{G}}(\bar{P})/\overline{C_G(P)}$ is a p-group. From the fact that $\tilde{f}^P$ is a strict semicovering, we see that $f^P(i) \neq 0$ and that if n and m are the number of primitive idempotents decomposing unity elements of $(kG)^P$ and $(k\bar{G})^{\bar{P}}$, respectively, then $n = m$. So $f^P(i)$ is primitive. $\square$

Let b be a block of kG, D_γ a defect pointed group of b, and let $i \in \gamma$. We set $f(b) = \bar{b}$, $D/Z = \bar{D}$ and $f^D(i) = \bar{i}$. Then since Z is a central p-subgroup of G, we see that $\bar{b}$ is a block of $k\bar{G}$ and $\bar{D}$ is a defect group of $\bar{b}$. Further, $\bar{i}$ is primitive by Lemma 4.1. Also, since $k\overline{C_G(D)} = k(C_G(D)/Z) \cong kC_G(D)/J(kZ)kC_G(D)$ (where $J(kZ)$ is the radical of kZ), we see that $\mathrm{Br}_{\bar{D}}^{\bar{G}} \circ f^D(i) = f_{C_G(D)} \circ \mathrm{Br}_D^G(i) \neq 0$ and so $\bar{i}$ is local. Therefore, let $\bar{\gamma}$ be the $((k\bar{G})^{\bar{D}})^\times$-conjugacy class of $\bar{i}$, then $\bar{D}_{\bar{\gamma}}$ is a defect pointed group of $\bar{b}$. Recall that $H^*(G, b, D_\gamma)$ consists of all $[\zeta] \in H^*(D, k)$ satisfying $\mathrm{res}_{{}^{x^{-1}}D,Q} \circ c_{x^{-1}}([\zeta]) = \mathrm{res}_{D,Q}([\zeta])$ for any subgroup Q of D and for any $x \in G$ with ${}^x(Q, e_Q) \leq (D, e_D)$.

Lemma 4.2. *Let Z be a central p-subgroup of G. In the definition of $H^*(G, b, D_\gamma)$, we can assume that Q contains Z.*

Proof. Let (Q, e_Q) be a b-Brauer pair such that $(Q, e_Q) \leq (D, e_D)$. Since Z is a central p-subgroup, we have that $\mathrm{Br}_{QZ}^G = \mathrm{Br}_Q^G$. Thus, $\mathrm{Br}_{QZ}^G(i)e_Q = \mathrm{Br}_Q^G(i)e_Q = \mathrm{Br}_Q^G(i) = \mathrm{Br}_{QZ}^G(i)$, and so $e_{QZ} = e_Q$ and $(QZ, e_Q) \leq (D, e_D)$. Also, for an element $x \in G$, it is clear that ${}^x(Q, e_Q) \leq (D, e_D)$ if and only if ${}^x(QZ, e_Q) \leq (D, e_D)$. Further, if $\mathrm{res}_{{}^{x^{-1}}D,QZ} \circ c_{x^{-1}}([\zeta]) = \mathrm{res}_{D,QZ}([\zeta])$, then we have that $\mathrm{res}_{{}^{x^{-1}}D,Q} \circ c_{x^{-1}}([\zeta]) = \mathrm{res}_{D,Q}([\zeta])$. So we can replace Q by QZ in the definition. $\square$

Proposition 4.3. *With the notation above, let $[\bar{\zeta}]$ be an element of $H^*(\bar{G}, \bar{b}, \bar{D}_{\bar{\gamma}})$. Then $\mathrm{inf}_{\bar{D},D}([\bar{\zeta}])$ belongs to $H^*(G, b, D_\gamma)$.*

Proof. Let (Q, e_Q) be a b-Brauer pair such that $(Q, e_Q) \leq (D, e_D)$ and that Q contains Z. For the block e_Q, since Z is a central p-subgroup of $C_G(Q)$, $\overline{e_Q}$ is the block of $k\overline{C_G(Q)}$. Further, since $C_{\bar{G}}(\bar{Q})/\overline{C_G(Q)}$ is a p-group, we have a unique block $e_{\bar{Q}}$ of $kC_{\bar{G}}(\bar{Q})$ which covers $\overline{e_Q}$. Then, since $(Q, e_Q) \leq (D, e_D)$, we have $(\bar{Q}, e_{\bar{Q}}) \leq (\bar{D}, e_{\bar{D}})$. Note that $(\bar{D}, e_{\bar{D}})$ is the maximal $\bar{b}$-Brauer pair corresponding to $\bar{D}_{\bar{\gamma}}$. Indeed, $e_{\bar{Q}}\ \overline{e_Q} = \overline{e_Q}$ and so it holds that $\mathrm{Br}_{\bar{Q}}^{\bar{G}}(\bar{i})e_{\bar{Q}} = \overline{\mathrm{Br}_Q^G(i)}\overline{e_Q} = \overline{\mathrm{Br}_Q^G(i)} = \mathrm{Br}_{\bar{Q}}^{\bar{G}}(\bar{i})$. Also, we see that if ${}^x(Q, e_Q) \leq (D, e_D)$ for $x \in G$, then ${}^{\bar{x}}(\bar{Q}, e_{\bar{Q}}) \leq (\bar{D}, e_{\bar{D}})$. Thus, for an element $[\bar{\zeta}] \in H^*(\bar{G}, \bar{b}, \bar{D}_{\bar{\gamma}})$, it follows that

$$\begin{aligned}
\mathrm{res}_{{}^{x^{-1}}D,Q} \circ c_{x^{-1}}\left(\mathrm{inf}_{\bar{D},D}([\bar{\zeta}])\right) &= \mathrm{res}_{{}^{x^{-1}}D,Q} \circ \mathrm{inf}_{{}^{\bar{x}^{-1}}\bar{D},\,{}^{x^{-1}}D} \circ c_{\bar{x}^{-1}}\left([\bar{\zeta}]\right) \\
&= \mathrm{inf}_{\bar{Q},Q}\left(\mathrm{res}_{{}^{\bar{x}^{-1}}\bar{D},\bar{Q}} \circ c_{\bar{x}^{-1}}([\bar{\zeta}])\right) \\
&= \mathrm{inf}_{\bar{Q},Q}\left(\mathrm{res}_{\bar{D},\bar{Q}}([\bar{\zeta}])\right) \\
&= \mathrm{res}_{D,Q}\left(\mathrm{inf}_{\bar{D},D}([\bar{\zeta}])\right).
\end{aligned}$$

So we conclude that $\mathrm{inf}_{\bar{D},D}([\bar{\zeta}]) \in H^*(G, b, D_\gamma)$. $\square$

From the above proposition, we can define an inflation map $\mathrm{inf}_{\bar{b},b}$: $H^*(\bar{G}, \bar{b}, \bar{D}_{\bar{\gamma}}) \to H^*(G, b, D_\gamma)$. Then, since $H^*(G, b, D_\gamma)/\sqrt{0}$ is a finitely generated commutative k-algebra (see [11, Theorem 4.2]), the inflation $\inf_{\bar{b},b}$ induces an affine map $\inf^*_{\bar{b},b}\colon V_{G,b} \to V_{\bar{G},\bar{b}}$. Further, for any bounded complex U of $k\bar{G}\bar{b}$-modules, which may also be considered as a complex of kGb-modules through $kGb \to k\bar{G}\bar{b}$, we have Theorem 1.4.

Proof of Theorem 1.4. For the convenience of the proof, when we consider U as a complex of $k\bar{G}\bar{b}$-modules, we denote it by $\bar{U}$. Then Theorem 1.1 shows that $I^*_{\bar{G},\bar{b},\bar{D}_{\bar{\gamma}}}(\bar{U}) = H^*(\bar{G},\bar{b},\bar{D}_{\bar{\gamma}}) \cap I^*_{\bar{D}}(\bar{i}\bar{U})$ and that $I^*_{G,b,D_\gamma}(U) = H^*(G,b,D_\gamma) \cap I^*_D(iU)$. Note that for the chain maps ζ and τ representing $[\zeta] \in \mathrm{Ext}^n_{k\bar{D}}(\bar{i}\bar{U}, \bar{i}\bar{U})$ and $[\tau] \in \mathrm{Ext}^n_{kD}(iU, iU)$, $\inf_{\bar{D},D}([\zeta]) = [\tau]$ if and only if the following diagram is homotopy commutative:

$$\begin{tikzcd}
\mathscr{P}(iU) \arrow[r, "\tau"] \arrow[d] & \mathscr{P}(iU)[n] \arrow[d] \\
\mathscr{P}(\bar{i}\bar{U}) \arrow[r, "\zeta"'] & \mathscr{P}(\bar{i}\bar{U})[n],
\end{tikzcd}$$

where the left vertical map is a chain map lifting the identity map $iU \to \bar{i}\bar{U}$ and the right vertical map is its shift. It is easy to see that $\inf_{\bar{D},D} \circ \gamma_{\bar{i}\bar{U}} = \gamma_{iU} \circ \inf_{\bar{D},D}$, so that we have $\inf_{\bar{D},D}(I^*_{\bar{D}}(\bar{i}\bar{U})) \subseteq I^*_D(iU)$ and so $\inf_{\bar{b},b}(I^*_{\bar{G},\bar{b},\bar{D}_{\bar{\gamma}}}(\bar{U})) \subseteq I^*_{G,b,D_\gamma}(U)$. Therefore we can define an affine map $\inf^*_{\bar{b},b}\colon V_{G,b}(U) \to V_{\bar{G},\bar{b}}(\bar{U})$. □

5. Inverse images

Under the assumption that the defect groups are abelian, we consider the inverse images of the affine maps given in Corollary 1.2 and Theorem 1.4. Our consideration is based on Linckelmann's following stratification theorem for block varieties of modules [13]. Let b be a block of kG and D_γ a defect pointed group of b. For any subgroup Q of D, the composite graded algebra homomorphism r_Q : $H^*(G,b,D_\gamma) \xrightarrow{\iota} H^*(D,k) \xrightarrow{\mathrm{res}_{D,Q}} H^*(Q,k)$ induces a finite affine map of varieties

$$r_Q^*\colon V_Q \to V_{G,b},$$

where ι is the inclusion and $\mathrm{res}_{D,Q}$ is the restriction map. In particular, $r_D^*\colon V_D \to V_{G,b}$ is finite surjective (this map is denoted by $\iota_{D,b}$ in the previous sections). Let U be a finitely generated kGb-module and let $i \in \gamma$. Following [13], we now define the following subvarieties of V_Q and $V_{G,b}$:

$$\begin{aligned}
V_Q^+ &= V_Q - \cup_{R<Q}(\mathrm{res}_{Q,R})^*\, V_R, & V_Q^+(iU) &= V_Q(iU) \cap V_Q^+ \\
V_{G,Q} &= r_Q^*\, V_Q, & V_{G,Q}^+ &= r_Q^*\, V_Q^+ \\
V_{G,Q}(U) &= r_Q^*\, V_Q(iU), & V_{G,Q}^+(U) &= r_Q^*\, V_Q^+(iU)
\end{aligned}$$

Theorem 5.1 (Linckelmann [13, Theorem 4.2 and Proposition 4.3]). *Let U be a finitely generated kGb-module.*
(i) *The variety $V_{G,b}(U)$ is the disjoint union of locally closed subvarieties $V^+_{G,E}(U)$, where E runs over the set of subgroups of D such that (E, e_E) runs over the set of representatives of the G-conjugacy classes of those b-Brauer pairs contained in (D, e_D) for which E is elementary abelian and $C_D(E)$ is a defect group of the block e_E.*
(ii) *Let E be an elementary abelian subgroup of D such that $C_D(E)$ is a defect group of e_E. The group $W(E) = N_G(E, e_E)/C_G(E)$ acts on the variety $V^+_E(iU)$, and r^*_E induces an inseparable isogeny $V^+_E(iU)/W(E) \to V^+_{G,E}(U)$.*
(iii) *Suppose that U is indecomposable with D as a vertex and a source of dimension prime to p. Then, for any subgroup Q of D we have $V_Q(iU) = V_Q$ and $V^+_Q(iU) = V^+_Q$. Further, for any subgroup Q of D we have $V_{G,Q}(U) = V_{G,Q}$ and $V^+_{G,Q}(U) = V^+_{G,Q}$.*

It is known that a simple kG-module U in b of height 0 satisfies the condition in (iii) (see [7, Corollary 4.6]). So (i) and (ii) give a stratification of $V_{G,b}$ also. Using the above theorem, we show the following which is a block variety version of Avrunin and Scott [1, Theorem 3.1].

Proposition 5.2. *Suppose that D is abelian. Let Q be a subgroup of D and let U be a finitely generated kGb-module. Then we have that*

$$V_Q(iU) = (r^*_Q)^{-1}\, V_{G,b}(U).$$

Proof. First we show two facts obtained from the assumption that D is abelian. Let (R, e_R) be a b-Brauer pair contained in (D, e_D). Then e_D and e_R are Brauer correspondents of b, and further e_D is also a Brauer correspondent of e_R. Thus e_R has a defect group D, because e_D has the defect group D. So, for any $(R, e_R) \le (D, e_D)$, e_R satisfies the last condition of Theorem 5.1 (i) (that is, $C_D(R) = D$ is a defect group of e_R). The second assertion is the following. Let $(R', e_{R'})$ also be a b-Brauer pair contained in (D, e_D). Suppose that ${}^x(R', e_{R'}) = (R, e_R)$ for some $x \in G$. Then, by the fusion theorem for Brauer pairs (see [16, Proposition 49.5 and Proposition 49.6]), there is $y \in N_G(D, e_D)$ such that ${}^y r' = {}^x r'$ for all $r' \in R'$. Thus, mapping $a \in {}^x iU$ to $yx^{-1}a \in {}^y iU$, we have the isomorphism ${}^x iU \cong {}^y iU$ as kR-modules. Further, since $N_G(D, e_D) = N_G(D_\gamma)$, there exists $z \in ((kG)^D)^\times$ such that ${}^y i = {}^z i$. Thus, mapping $b \in iU$ to $zb \in {}^y iU$, we have the isomorphism $iU \cong {}^y iU$ as kD-modules. So ${}^x iU \cong iU$ as kR-modules. Therefore the conjugation map $c_x \colon H^*(R', k) \to H^*(R, k)$ induces an isomorphism of subvarieties $V_{R'}(iU) \cong V_R({}^x iU) = V_R(iU)$.

Clearly $r^*_Q\, V_Q(iU) \subseteq V_{G,b}(U)$. Conversely, let $v \in (r^*_Q)^{-1}\, V_{G,b}(U)$. Applying the Quillen stratification theorem (see e.g. [1, Theorem 2.2]) to V_Q, we can choose an elementary abelian p-subgroup E of Q and $s \in V^+_E$ with $(\mathrm{res}_{Q,E})^*(s) = v$. On the other hand, by Theorem 5.1 (i), we can choose an elementary abelian p-subgroup

E' of D and $s' \in V_{E'}^{+}(iU)$ with $r_{E'}^{*}(s') = r_{Q}^{*}(v)$. Then $r_{E'}^{*}(s') = r_{Q}^{*}(v) = r_{E}^{*}(s)$. Thus, by Theorem 5.1 (i) for $V_{G,b}$, we see that b-Brauer pairs (E, e_E) and $(E', e_{E'})$ must be G-conjugate. Let $(E, e_E) = {}^{g}(E', e_{E'})$ for an element g in G. Then since ${}^{g}(E', e_{E'}) = (E, e_E) \leq (D, e_D)$, we see from the definition of $H^*(G, b, D_\gamma)$ that every $[\zeta] \in H^*(G, b, D_\gamma)$ satisfies $c_{g^{-1}} \circ \mathrm{res}_{D,E}([\zeta]) = \mathrm{res}_{D,E'}([\zeta])$ (equivalently, $\mathrm{res}_{{}^{g^{-1}}D,E'} \circ c_{g^{-1}}([\zeta]) = \mathrm{res}_{D,E'}([\zeta])$). Thus we have that $r_{E}^{*}(c_{g^{-1}}^{*}(s')) = r_{E'}^{*}(s') = r_{E}^{*}(s)$. Here, note that $c_{g^{-1}}^{*} \colon V_{E'} \to V_E$ is equal to the natural map induced by the conjugation map $c_g \colon H^*(E', k) \to H^*(E, k)$. We write ${}^{g}s'$ for $c_{g^{-1}}^{*}(s')$. Since ${}^{g}s'$ and s are contained in V_E^{+} and $r_E^{*}({}^{g}s') = r_E^{*}(s)$ and further $C_D(E) = D$ is a defect group of e_E, Theorem 5.1 (ii) for $V_{G,b}$ shows that there exists $h \in N_G(E, e_E)$ with ${}^{hg}s' = s$. Now, since D is abelian, we see from the fact stated above that the conjugation map c_{hg} induces an isomorphism $V_{E'}(iU) \cong V_E(iU)$. Thus s is contained in $V_E(iU)$. Therefore $v = (\mathrm{res}_{Q,E})^{*}(s)$ is contained in $V_Q(iU)$, so the proposition follows. □

Using the above proposition, we show the following, which is a block variety version of [6, Theorem 1].

Proposition 5.3. *With the notation in* Section 4, *suppose that D is abelian. Let U be a finitely generated $k\bar{G}\bar{b}$-module. Then we have that*

$$V_{G,b}(U) = (\mathrm{inf}_{\bar{b},b}^{*})^{-1}\, V_{\bar{G},\bar{b}}(U).$$

Proof. It is clear that there is a commutative diagram as follows:

$$\begin{tikzcd} V_D \arrow[r, "r_D^*"] \arrow[d, "\mathrm{inf}_{\bar{D},D}^*"'] & V_{G,b} \arrow[d, "\mathrm{inf}_{\bar{b},b}^*"] \\ V_{\bar{D}} \arrow[r, "\bar{r}_{\bar{D}}^*"'] & V_{\bar{G},\bar{b}} \end{tikzcd} .$$

Then we have

$$(r_D^{*})^{-1} \circ (\mathrm{inf}_{\bar{b},b}^{*})^{-1}\, V_{\bar{G},\bar{b}}(U) = (\mathrm{inf}_{\bar{D},D}^{*})^{-1} \circ (\bar{r}_{\bar{D}}^{*})^{-1}\, V_{\bar{G},\bar{b}}(U) = (\mathrm{inf}_{\bar{D},D}^{*})^{-1}\, V_{\bar{D}}(\bar{i}U) = V_D(iU),$$

where the second equality holds by Proposition 5.2, and the third holds by [6, Theorem 1]. Thus, since $r_D^{*} \colon V_D \to V_{G,b}$ and its restriction $r_D^{*} \colon V_D(iU) \to V_{G,b}(U)$ are both surjective, it follows that $(\mathrm{inf}_{\bar{b},b}^{*})^{-1}\, V_{\bar{G},\bar{b}}(U) = r_D^{*}\, V_D(iU) = V_{G,b}(U)$. This completes the proof. □

REMARK 5.4. Suppose that b is the principal block of kG. Then we have from Theorem 2.4 that $V_{G,b}(U) = (\mathrm{inf}_{\bar{b},b}^{*})^{-1}\, V_{\bar{G},\bar{b}}(U)$, without the assumption that D is abelian. Indeed, from the relation $\rho_b \circ \mathrm{inf}_{\bar{G},G} = \mathrm{inf}_{\bar{b},b} \circ \rho_{\bar{b}}$, we see that $(\rho_b^{*})^{-1} \circ (\mathrm{inf}_{\bar{G},G}^{*})^{-1}\, V_{\bar{G}}(U) = (\mathrm{inf}_{\bar{b},b}^{*})^{-1} \circ (\rho_{\bar{b}}^{*})^{-1}\, V_{\bar{G}}(U) = (\mathrm{inf}_{\bar{b},b}^{*})^{-1}\, V_{\bar{G},\bar{b}}(U)$. Further, we see from [6, Theorem 1] that $(\rho_b^*)^{-1} \circ (\inf^*_{\bar{G},G})^{-1}\ V_{\bar{G}}(U) = (\rho_b^*)^{-1}\ V_G(U) = V_{G,b}(U)$. So, it follows that $V_{G,b}(U) = (\inf^*_{\bar{b},b})^{-1}\ V_{\bar{G},\bar{b}}(U)$.

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