A GROUP-THEORETIC CHARACTERIZATION OF THE SPACE OBTAINED BY OMITTING THE COORDINATE HYPERPLANES FROM THE COMPLEX EUCLIDEAN SPACE

Dedicated to Professor Makoto Namba on his sixtieth birthday

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Introduction

In the study of the holomorphic automorphism group $\text{Aut}(M)$ of a complex manifold $M$, it seems to be natural to direct our attention not only to the abstract group structure of $\text{Aut}(M)$ but also to its topological group structure equipped with the compact-open topology. In fact, a well-known theorem of H. Cartan says that the topological group of the holomorphic automorphisms of a bounded domain in $\mathbb{C}^n$ has the structure of a Lie group, and this result enables us to make various kinds of detailed studies of bounded domains in $\mathbb{C}^n$. On the other hand, in contrast to the case of bounded domains, the holomorphic automorphism group $\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^l)$ of the unbounded domain $\mathbb{C}^k \times (\mathbb{C}^*)^l$ is terribly big when $k+l \geq 2$, and cannot have the structure of a Lie group. But, by looking at topological subgroups of $\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^l)$ with Lie group structures, we can find a lead to apply the Lie group theory to the investigation of the problems related to the structure of $\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^l)$. In the present paper, we try to approach from this standpoint to the fundamental problem of what complex manifold has the holomorphic automorphism group isomorphic to $\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^l)$ as topological groups. Namely, we prove the following result with the aid of the theory of Reinhardt domains developed in Shimizu [8], [9] (cf. Kruzhilin [6]).

Main Theorem. Let $M$ be a connected Stein manifold of dimension $n$. Assume that $\text{Aut}(M)$ is isomorphic to $\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^{n-k})$ as topological groups. Then $M$ is biholomorphically equivalent to $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$.

As a consequence of the above theorem, we can obtain the fundamental result on the topological group structure of $\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^l)$.

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Corollary. If two pairs \((k, l)\) and \((k', l')\) of nonnegative integers do not coincide, then the topological groups \(\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^l)\) and \(\text{Aut}(\mathbb{C}^{k'} \times (\mathbb{C}^*)^{l'})\) are not isomorphic.

It should be remarked that, as shown in Ahern and Rudin [1], the groups \(\text{Aut}(\mathbb{C}^n)\) and \(\text{Aut}(\mathbb{C}^m)\) are isomorphic as abstract groups precisely when \(n = m\). Also, as a consequence of the study of \(U(n)\)-actions on complex manifolds of dimension \(n\), a related result to our Main Theorem has been obtained by Isaev and Kruzhilin [4].

This paper is organized as follows. In Section 1, we collect some preliminary facts. In particular, two main tools for our study are given. One is a tool to obtain the normal form of some compact group action on a Reinhardt domain, and the other is a tool for the standardization of torus actions on complex manifolds. Section 2 is devoted to the proof of the Main Theorem and its corollary. Our method used in Section 2 has interesting applications. As one of such applications, we discuss in Section 3 a new approach to the study of \(U(n)\)-actions on complex manifolds of dimension \(n\).

1. Lie group actions, Reinhardt domains and torus actions

We begin with a basic fact on Lie group actions on complex manifolds. Let \(M\) be a complex manifold. An automorphism of \(M\) means a biholomorphic mapping of \(M\) onto itself. We denote by \(\text{Aut}(M)\) the topological group of all automorphisms of \(M\) equipped with the compact-open topology. Let \(G\) be a Lie group and consider a continuous group homomorphism \(\rho: G \to \text{Aut}(M)\). Then the mapping

\[
G \times M \ni (g, p) \longrightarrow (\rho(g))(p) \in M
\]

is continuous. It follows from Akhiezer [2] that this mapping is actually of class \(C^\omega\), and therefore \(G\) acts on \(M\) as a Lie transformation group. In view of this, when a continuous group homomorphism \(\rho: G \to \text{Aut}(M)\) is given, we say that \(G\) acts on \(M\) as a Lie transformation group through \(\rho\). Also, the action of \(G\) on \(M\) is called effective if \(\rho\) is injective.

We now recall basic concepts and results on Reinhardt domains (cf. [8], [9]). We denote by \(U(k)\) the unitary group of degree \(k\). Write \(T^n = (U(1))^n\). The \(n\)-dimensional torus \(T^n\) acts as a group of automorphisms on \(\mathbb{C}^n\) by the standard rule

\[
\alpha \cdot z = (\alpha_1 z_1, \ldots, \alpha_n z_n)
\]

for \(\alpha = (\alpha_1, \ldots, \alpha_n) \in T^n\) and \(z = (z_1, \ldots, z_n) \in \mathbb{C}^n\).

By definition, a Reinhardt domain \(D\) in \(\mathbb{C}^n\) is a domain in \(\mathbb{C}^n\) which is stable under this action of \(T^n\). Each element \(\alpha\) of \(T^n\) then induces an automorphism \(\pi_\alpha\) of \(D\) given by \(\pi_\alpha(z) = \alpha \cdot z\), and the mapping \(\rho_D\) sending \(\alpha\) to \(\pi_\alpha\) is an injective continuous group homomorphism of \(T^n\) into \(\text{Aut}(D)\). The subgroup \(\rho_D(T^n)\) of \(\text{Aut}(D)\) is denoted by \(T(D)\).

Let \(f\) be a holomorphic function on a Reinhardt domain \(D\) in \(\mathbb{C}^n\). Then \(f\) can be
expanded uniquely into a “Laurent series”

\[ f(z) = \sum_{\nu \in \mathbb{Z}^n} a_\nu z^\nu, \]

which converges absolutely and uniformly on any compact set in \( D \), where \( z = (z_1, \ldots, z_n), \nu = (\nu_1, \ldots, \nu_n) \), and \( z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n} \). The following lemma is a consequence of the uniqueness of the Laurent series expansion.

**Lemma 1.1.** If \( f \) satisfies the condition that, for some \( \nu_0 \in \mathbb{Z}^n \),

\[ f(\alpha \cdot z) = \alpha^{\nu_0} f(z) \quad \text{for all } \alpha \in T^n \text{ and all } z \in D, \]

then \( f \) has the form \( f(z) = a_{\nu_0} z^{\nu_0} \).

Proof. Since we have

\[ f(\alpha \cdot z) = \sum_{\nu \in \mathbb{Z}^n} \alpha^{\nu} a_\nu z^\nu \quad \text{and} \quad \alpha^{\nu_0} f(z) = \sum_{\nu \in \mathbb{Z}^n} \alpha^{\nu_0} a_\nu z^\nu, \]

it follows from the assumption that, for every \( \nu \in \mathbb{Z}^n \), we have

\[ \alpha^{\nu_0} a_\nu = \alpha^{\nu_0} a_\nu \quad \text{for all } \alpha \in T^n. \]

This implies that if \( a_\nu \neq 0 \), then \( \nu = \nu_0 \), and our lemma is proved. \( \square \)

We denote by \( \Pi(C^n) \) the group of all automorphisms of \( C^n \) of the form

\[ C^n \ni (z_1, \ldots, z_n) \mapsto (\alpha_1 z_1, \ldots, \alpha_n z_n) \in C^n, \]

where \((\alpha_1, \ldots, \alpha_n) \in (C^*)^n\). For a Reinhardt domain \( D \) in \( C^n \), we denote by \( \Pi(D) \) the subgroup of \( \Pi(C^n) \) consisting of all elements of \( \Pi(C^n) \) leaving \( D \) invariant. Identifying \( \Pi(C^n) \) with the multiplicative group \((C^*)^n\), we see that, when \( \Pi(D) \) is regarded as a topological subgroup of \( \text{Aut}(D) \), it is isomorphic to a closed Lie subgroup of \((C^*)^n\). Using Lemma 1.1, we obtain a characterization of \( \Pi(D) \) as a subgroup of \( \text{Aut}(D) \).

**Lemma 1.2.** Let \( D \) be a Reinhardt domain in \( C^n \). Then \( \Pi(D) \) is the centralizer \( C_{\text{Aut}(D)}(T(D)) \) of \( T(D) \) in \( \text{Aut}(D) \).

Proof. It is immediate that \( \Pi(D) \subset C_{\text{Aut}(D)}(T(D)) \). To prove the reverse inclusion, let \( \varphi \) be any element of \( C_{\text{Aut}(D)}(T(D)) \) and write \( \varphi = (\varphi_1, \ldots, \varphi_n) \), where \( \varphi_1, \ldots, \varphi_n \) are holomorphic functions on \( D \). Then, for every \( i = 1, \ldots, n \), we have

\[ \varphi_i(\alpha \cdot z) = \alpha_i \varphi_i(z) = \alpha^{\varphi_i} \varphi_i(z) \quad \text{for all } \alpha \in T^n \text{ and all } z \in D. \]
where each \( e_i \) denotes the element of \( \mathbb{Z}^n \) whose \( i \)-th component is equal to 1 and whose components except the \( i \)-th are all equal to 0. By Lemma 1.1, it follows from this property that every function \( \varphi_f(z) \) has the form

\[
\varphi_f(z) = a_{e_i}z^e = a_{e_i}z^i.
\]

This implies that \( \varphi \in \Pi(D) \), and the reverse inclusion \( C_{\text{Aut}(g)}(T(D)) \subset \Pi(D) \) is shown, as desired.

The argument used in Shimizu [9] for determining the automorphisms of bounded Reinhardt domains has the following consequence, which plays a crucial role in our study.

**Proposition 1.1.** Let \( D \) be a bounded Reinhardt domain in \( \mathbb{C}^n \) and suppose that

\[
D \cap \{ z_i = 0 \} \neq \emptyset, \quad 1 \leq i \leq m,
\]

\[
D \cap \{ z_i = 0 \} = \emptyset, \quad m + 1 \leq i \leq n.
\]

If \( G \) is a connected compact subgroup of \( \text{Aut}(D) \) containing \( T(D) \), then there exists a transformation

\[
\varphi : \mathbb{C}^m \times (\mathbb{C}^*)^{n-m} \ni (z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) \in \mathbb{C}^m \times (\mathbb{C}^*)^{n-m},
\]

\[
\begin{cases}
w_i = r_i z_{\sigma'(i)}(z'')^{\nu''_i}, & 1 \leq i \leq m, \\
w_i = r_i z_{\sigma''(i)}, & m + 1 \leq i \leq n,
\end{cases}
\]

such that, for \( \tilde{D} = \varphi(D) \) and \( \tilde{G} = \varphi G \varphi^{-1} \subset \text{Aut}(\tilde{D}) \), one has

\[
\tilde{G} = U(k_1) \times \cdots \times U(k_s) \times U(k_{s+1}) \times \cdots \times U(k_t),
\]

\[
k_1 + \cdots + k_s + k_{s+1} + \cdots + k_t = n,
\]

\[
k_1 + \cdots + k_s = m,
\]

\[
k_{s+1} = \cdots = k_t = 1,
\]

where \( r_1, \ldots, r_n \) are positive constants, \( \sigma' \) and \( \sigma'' \) are permutations of \( \{1, \ldots, m\} \) and \( \{m+1, \ldots, n\} \), respectively, \( z'' \) denotes the coordinates \( (z_{m+1}, \ldots, z_n) \), and \( \nu''_1, \ldots, \nu''_m \) are elements of \( \mathbb{Z}^{n-m} \).

We give a useful form of this proposition as a corollary.

**Corollary.** In the above proposition, if \( G \) is isomorphic to \( U(k) \times (U(1))^{n-k} \) as topological groups and if \( k \geq 2 \), then \( m \geq k \).
Proof. Since \( G \) is necessarily isomorphic to \( U(1) \times (U(1))^{n-k} \) as Lie groups, we have \( \dim G = k^2 + (n - k) \). On the other hand, Proposition 1.1 implies that \( \dim G = \dim \tilde{G} = k_1^2 + \cdots + k_s^2 + (n - m) \). Therefore, if \( m < k \), then it follows that
\[
k^2 = k_1^2 + \cdots + k_s^2 + (k - m) \quad \text{and} \quad k = k_1 + \cdots + k_s + (k - m).
\]
By noting that \( k \geq 2 \) and \( k - m > 0 \), this is a contradiction. Thus we obtain \( m \geq k \).

We recall the fundamental result on torus actions on complex manifolds, which is a part of Barrett, Bedford and Dadok [3, Theorem 1].

**Standardization Theorem.** Let \( M \) be a connected Stein manifold of dimension \( n \). Assume that \( T^n \) acts effectively on \( M \) as a Lie transformation group through \( \rho \). Then there exist a biholomorphic mapping \( F \) of \( M \) into \( \mathbb{C}^n \) and a continuous group automorphism \( \theta \) of \( T^n \) such that
\[
F((\rho(\alpha))(p)) = \theta(\alpha) \cdot F(p) \quad \text{for all} \quad \alpha \in T^n \text{ and all } p \in M.
\]
Consequently, \( D := F(M) \) is a Reinhardt domain in \( \mathbb{C}^n \), and one has \( F\rho(T^n)F^{-1} = T(D) \).

To apply the Standardization Theorem to our study, we need a lemma.

**Lemma 1.3.** In the Standardization Theorem, if \( M = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \), then we have \( D = F(M) = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \) after a suitable permutation of coordinates, if necessary.

Proof. We first show that \( D \cap (\mathbb{C}^*)^n = D - \{z_1 \cdots z_n = 0\} = (\mathbb{C}^*)^n \). Suppose contrarily that \( D \cap (\mathbb{C}^*)^n \neq (\mathbb{C}^*)^n \). Since \( D \cap (\mathbb{C}^*)^n \) is a Stein manifold, the logarithmic image of the Reinhardt domain \( D \cap (\mathbb{C}^*)^n \) is a convex domain contained in a half space of \( \mathbb{R}^n \). Hence, there exists a nonconstant bounded plurisubharmonic function \( u \) on \( D \cap (\mathbb{C}^*)^n \). Since \( u \) extends to the whole of \( D \), we have a nonconstant bounded plurisubharmonic function on \( D \). This contradicts the fact that \( D \) is biholomorphically equivalent to \( M = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \). Thus we obtain \( D \cap (\mathbb{C}^*)^n = (\mathbb{C}^*)^n \).

Since \( D \) is a Stein manifold, it follows from what we have shown above that, after a suitable permutation of coordinates, \( D \) has the form \( D = \mathbb{C}^h \times (\mathbb{C}^*)^{n-h} \) (cf. [7, p. 46, Theorem 1.5]). Note that \( \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \) and \( \mathbb{C}^h \times (\mathbb{C}^*)^{n-h} \) are homeomorphic precisely when \( k = h \). Therefore we have \( D = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \), because \( D \) and \( M \) are biholomorphically equivalent.
2. The characterization of $C^k \times (C^*)^l$: Proof of the Main Theorem and its corollary

For brevity, we write $X_{k,l} = C^k \times (C^*)^l$ and $\Omega_k = X_{k,n-k}$.

Now, as in the Main Theorem stated in the introduction, let $M$ be a connected Stein manifold of dimension $n$ and assume that there exists an isomorphism $\Phi: \text{Aut}(\Omega_k) \rightarrow \text{Aut}(M)$. Since $\Omega_k$ is a Reinhardt domain in $C^n$, we have the injective continuous group homomorphism $\rho_{\Omega_k}: T^n \rightarrow \text{Aut}(\Omega_k)$. Thus we obtain an injective continuous group homomorphism $\Phi \circ \rho_{\Omega_k}: T^n \rightarrow \text{Aut}(M)$. Hence, by the Standardization Theorem, there exists a biholomorphic mapping $F$ of $M$ into $C^n$ such that $D := F(M)$ is a Reinhardt domain in $C^n$ and we have $F(\Phi \circ \rho_{\Omega_k}(T^n))F^{-1} = T(D)$. Therefore we may assume that $M$ is a Reinhardt domain $D$ in $C^n$ and we have an isomorphism $\Phi: \text{Aut}(\Omega_k) \rightarrow \text{Aut}(D)$ such that $\Phi(T(\Omega_k)) = T(D)$.

We show that $(C^*)^l \subset D$. Since $\Phi: \text{Aut}(\Omega_k) \rightarrow \text{Aut}(D)$ is a group isomorphism and since $\Phi(T(\Omega_k)) = T(D)$, we see that $\Phi$ gives rise to a topological group isomorphism $\Phi: \text{Aut}(\Omega_k)(T(\Omega_k)) \rightarrow \text{Aut}(D)(T(D))$ between the centralizers. Moreover, by Lemma 1.2 we have $\text{Aut}(\Omega_k)(T(\Omega_k)) = \Pi(\Omega_k)$, and it is immediate that $\Pi(\Omega_k) = \Pi(C^n)$. On the other hand, again by Lemma 1.2 we have $\text{Aut}(D)(T(D)) = \Pi(D)$. Therefore we obtain

$$2n = \dim \Pi(C^n) = \dim \text{Aut}(\Omega_k)(T(\Omega_k)) = \dim \text{Aut}(D)(T(D)) = \dim \Pi(D).$$

Since $\Pi(D)$ is a closed Lie subgroup of $\Pi(C^n)$, it follows that $\Pi(D) = \Pi(C^n)$. By taking a point $z_0$ in $D \cap (C^*)^l$, this shows that

$$(C^*)^l = \Pi(C^n) \cdot z_0 = \Pi(D) \cdot z_0 \subset D,$$

as required.

Since $D$ is a Stein manifold by assumption, we see from the result of the preceding paragraph that $D$ has the form $D = \Omega_0$ after a suitable permutation of coordinates.

When $n = 1$, we have $D = \Omega_0 = C^*$ or $D = \Omega_1 = C$. Moreover, since Aut($C^*$) and Aut($C$) are not isomorphic, the condition that Aut($\Omega_k$) and Aut($D$) are isomorphic implies that, according to the cases of $k = 0$ and $k = 1$, we must have $D = \Omega_0$ and $D = \Omega_1$. This proves the Main Theorem when $n = 1$. Therefore, in what follows, we assume that $n \geq 2$.

We show that $h \geq k$. When $k = 0$, there is nothing to prove. To prove our assertion when $k \neq 0$, we divide into the two cases of $k = 1$ and $k \geq 2$.

First consider the case of $k \geq 2$. Noting that Aut($\Omega_k$) contains the subgroup $U(k) \times (U(1))^{n-k}$, we set $G = \Phi(U(k) \times (U(1))^{n-k})$, which is a connected compact subgroup of Aut($D$) containing $T(D)$, because $U(k) \times (U(1))^{n-k} \supset T(\Omega_k)$ and $\Phi(T(\Omega_k)) = T(D)$. Take a relatively compact subdomain $U$ of $D$ and put

$$D_0 = \{ g(z) \in D \mid g \in G, z \in U \} = \bigcup_{g \in G} g(U) = \bigcup_{z \in U} G \cdot z.$$
Then $D_0$ is a bounded Reinhardt domain contained in $D$ and $G$ can be regarded as a connected compact subgroup of the Lie group $\text{Aut}(D_0)$ containing $T(D_0)$. Recalling that $G$ is isomorphic to $U(k) \times (U(1))^{n-k}$ and $k \geq 2$, we can apply the corollary to Proposition 1.1 to $D_0$ and $G \subset \text{Aut}(D_0)$. Therefore, after a suitable permutation of coordinates, we have for some $m \geq k$,

$$\emptyset \neq D_0 \cap \{z_i = 0\} \subset D \cap \{z_i = 0\}, \quad 1 \leq i \leq m.$$ 

This implies that $\Omega_{\Omega_0} \subset D$, and, when we write $D = \Omega_{\Omega_0}$, we must have $\lambda \geq m \geq k$, as required.

Now consider the case of $k = 1$. It suffices to show that $\text{Aut}(\Omega_0)$ and $\text{Aut}(\Omega_0)$ are not isomorphic. Suppose contrarily that we have an isomorphism $\Phi: \text{Aut}(\Omega_0) \rightarrow \text{Aut}(\Omega_0)$. Then, by the Standardization Theorem and Lemma 1.3, we may assume that we have an isomorphism $\Phi: \text{Aut}(\Omega_0) \rightarrow \text{Aut}(\Omega_0)$ such that $\Phi(T(\Omega_0)) = T(\Omega_0)$. For $s = 0, 1$, let us set

$$T'(\Omega_s) = \{(1, \alpha_2, \ldots, \alpha_n) \in T(\Omega_s) \mid \alpha_2, \ldots, \alpha_n \in U(1)\}.$$ 

Then $\Phi(T'(\Omega_0))$ is an $(n-1)$-dimensional subtorus of $T(\Omega_0)$, and, after a suitable change of coordinates by a transformation of the form

$$\Omega_0 = (\mathbb{C}^*)^{\nu_1} \ni (z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) \in (\mathbb{C}^*)^{\nu_n} = \Omega_0, \quad w_i = z_i^{\nu_i}, \quad 1 \leq i \leq n,$$

where $\nu_1, \ldots, \nu_n$ are elements of $\mathbb{Z}^n$, we have $\Phi(T'(\Omega_0)) = T'(\Omega_0)$. Since $\Phi: \text{Aut}(\Omega_0) \rightarrow \text{Aut}(\Omega_0)$ is a group isomorphism, we see that $\Phi$ maps the centralizer $Z_1$ of $T'(\Omega_0)$ in $\text{Aut}(\Omega_0)$ onto the centralizer $Z_0$ of $T'(\Omega_0)$ in $\text{Aut}(\Omega_0)$. Therefore, for the groups $Z_0$ and $Z_1$, their commutator groups $[Z_0, Z_0]$ and $[Z_1, Z_1]$ must be isomorphic. To derive a contradiction, it is sufficient to see that $[Z_0, Z_0]$ is an abelian group, while $[Z_1, Z_1]$ is not an abelian group. We verify this only in the case of $\eta = 2$, because the verification in the case of $\eta > 2$ is almost identical. Using a method similar to that in the proof of Lemma 1.2, we can show that $Z_1$ and $Z_0$ are the groups of all elements

$$g_1 \in \text{Aut}(\Omega_0) = \text{Aut}(\mathbb{C} \times \mathbb{C}^*) \quad \text{and} \quad g_0 \in \text{Aut}(\Omega_0) = \text{Aut}((\mathbb{C}^*)^2)$$

having the forms

$$(*) \quad g_1(z) = (\alpha z_1 + \beta, f(z_1)z_2)$$

and

$$g_0(z) = (\alpha z_1, f(z_1)z_2),$$
respectively, where \( \alpha \in \mathbb{C}^* \), \( \beta \in \mathbb{C} \), and \( f(z_1) \) is a nowhere vanishing holomorphic function that is defined on \( \mathbb{C} \) for \( g_1 \) and on \( \mathbb{C}^* \) for \( g_0 \). Take any two transformations \( K_{\alpha, \beta, f} \) and \( K_{\alpha', \beta', f'} \) of the form (1) given by

\[
K_{\alpha, \beta, f}(z) = (\alpha z_1 + \beta, f(z_1)z_2) \quad \text{and} \quad K_{\alpha', \beta', f'}(z) = (\alpha' z_1 + \beta', f'(z_1)z_2)
\]

and write \( [K_{\alpha, \beta, f}, K_{\alpha', \beta', f'}](z) = (K_1(z), K_2(z)) \) in terms of the coordinates in \( \mathbb{C}^2 \), where \( [\varphi, \psi] := \varphi^{-1} \circ \psi^{-1} \circ \varphi \circ \psi \) denotes the commutator of transformations \( \varphi \) and \( \psi \). Then we have

\[
K_1(z) = \frac{\alpha \alpha' z_1 + \alpha \beta' - \beta \alpha' + \beta - \beta'}{\alpha \alpha'},
\]

\[
K_2(z) = \frac{f((\alpha \alpha' z_1 + \alpha \beta' - \beta \alpha' + \beta - \beta')/\alpha \alpha')}{f'((\alpha \alpha' z_1 + \alpha \beta' + \beta - \beta')/\alpha \alpha')}
\]

As a consequence, considering the case of \( (\beta, \beta') = (0, 0) \), we have

\[
(\star\star) \quad [K_{\alpha, 0, f}, K_{\alpha', 0, f'}](z) = \left( z_1, \frac{f((\alpha \alpha' z_1) f'(z_1)z_2)}{f(z_1) f'(\alpha \alpha' z_1)} \right).
\]

Now it follows immediately from (\star\star) that \( [Z_0, Z_0] \) is abelian. On the other hand, consider three elements

\[
P(z) = (\alpha z_1 + \beta, z_2), \quad Q(z) = (z_1, z_2 \exp z_1), \quad \text{and} \quad R(z) = (\gamma z_1, z_2 \exp z_1)
\]

in \( Z_1 \). Then, using the computation result above, we obtain

\[
[P, Q](z) = (z_1, z_2 \exp \{(1 - \alpha)z_1 - \beta\}),
\]

\[
[P, R](z) = \left( \frac{\alpha \gamma z_1 + \beta (1 - \gamma)}{\alpha \gamma}, z_2 \exp \left\{(1 - \alpha)z_1 - \frac{\beta}{\gamma}\right\} \right),
\]

and therefore \( [[P, Q], [P, R]] \) is not the identity mapping whenever \( \beta(\alpha - 1)(\gamma - 1) \neq 0 \). This implies that \( [Z_1, Z_1] \) is not abelian, and our assertion that \( \text{Aut}(\Omega_1) \) and \( \text{Aut}(\Omega_0) \) are not isomorphic is shown.

Summarizing our results obtained so far, we have shown that if \( M \) is a connected Stein manifold of dimension \( n \) and if the topological groups \( \text{Aut}(M) \) and \( \text{Aut}(\Omega_k) \) are isomorphic, then \( M \) is biholomorphically equivalent to \( \Omega_h \) with \( h \geq k \).

To complete the proof of our Main Theorem, it is sufficient to see \( h = k \). Suppose contrarily that \( h \neq k \). Then, for the connected Stein manifold \( \Omega_k \) of dimension \( n \), we have that \( \text{Aut}(\Omega_k) \) and \( \text{Aut}(\Omega_h) \) are isomorphic. By letting \( M = \Omega_k \), an application of what we have shown just above yields that \( \Omega_k \) is biholomorphically equivalent to \( \Omega_p \) with \( p \geq h \). Since \( k < h \leq p \), this contradicts the fact that \( \Omega_h \) and \( \Omega_t \) are not homeomorphic when \( s \neq t \). We thus obtain \( h = k \), and our Main Theorem is proved. \( \square \)
It remains to prove the corollary to the Main Theorem. If $k + l = k' + l'$, then it is immediate from the Main Theorem that $\text{Aut}(X_{k,l})$ and $\text{Aut}(X_{k',l'})$ are isomorphic precisely when $(k, l) = (k', l')$. To prove the corollary in the case of $k + l \neq k' + l'$, we need the following lemma.

**Lemma 2.1.** Let $M$ be a connected Stein manifold of dimension $n$. If $N > n$, then there is no injective continuous group homomorphism of the torus $T^N$ into the topological group $\text{Aut}(M)$.

**Proof.** Suppose contrarily that we have an injective continuous group homomorphism $\rho$ of $T^N$ into $\text{Aut}(M)$. Choose an $n$-dimensional subtorus $T^n$ of $T^N$. By the Standardization Theorem, there exists a biholomorphic mapping $F: M \to D$ of $M$ onto a Reinhardt domain $D$ in $\mathbb{C}^n$ such that $F\rho(T^n)F^{-1} = T(D)$. Set $G = F\rho(T^N)F^{-1}$ and take a relatively compact subdomain $U$ of $D$. Then $D_0 := \{g(z) \in D \mid g \in G, z \in U\}$ is a bounded Reinhardt domain in $\mathbb{C}^n$ and $G$ can be regarded as a connected compact subgroup of the Lie group $\text{Aut}(D_0)$ containing $T(D_0)$. Since $G$ is isomorphic to $T^N$ and $N > n = \dim T(D_0)$, $G$ is a torus in $\text{Aut}(D_0)$ containing $T(D_0)$ properly. But, by [8, Section 4, Proposition 1], $T(D_0)$ is a maximal torus in $\text{Aut}(D_0)$, that is, any torus in $\text{Aut}(D_0)$ containing $T(D_0)$ must coincide with $T(D_0)$. This is a contradiction, and our assertion is proved.

Suppose $k+l \neq k'+l'$, say, $k+l < k'+l'$, and write $n = k+l$, $n' = k'+l'$. If there exists an isomorphism $\Phi: \text{Aut}(X_{k',l'}) \to \text{Aut}(X_{k,l})$, then we have an injective continuous group homomorphism $\Phi \circ \rho_{X_{k',l'}}$ of $T^{n'}$ into $\text{Aut}(X_{k,l})$. Since $X_{k,l}$ is a connected Stein manifold of dimension $n < n'$, this contradicts the above lemma. Therefore, $\text{Aut}(X_{k,l})$ and $\text{Aut}(X_{k',l'})$ are not isomorphic, and the proof of the corollary is completed. □

3. $U(n)$-actions on a Stein manifold of dimension $n$

The method used in the preceding section can be applied to the study of $U(n)$-actions on a complex manifold $M$ of dimension $n$. The following theorem gives a different approach from Kaup [5], Isaev and Kruzhilin [4]. In the case where $\text{Aut}(M)$ is not a Lie group, we cannot obtain various results on the conjugacy of subgroups of $\text{Aut}(M)$ by applying the conjugacy theorems in the Lie group theory, in general. However, even when $\text{Aut}(M)$ is not a Lie group, we have a conjugacy result on $\text{Aut}(M)$ in a case, as is shown in our theorem below.

**Theorem.** Let $M$ be a connected Stein manifold of dimension $n \geq 2$. Assume that $U(n)$ acts effectively on $M$ as a Lie transformation group through $\rho$. Then $M$ is biholomorphically equivalent to either $B^n$ or $C^n$, where $B^n$ denotes the unit ball in $\mathbb{C}^n$. Moreover, if we identify $M$ with $B^n$ or $C^n$, then there exists an element $\psi$ of $\text{Aut}(M)$ such that $\psi \rho(U(n)) \psi^{-1} = U(n)$. 

Proof. Choose a maximal torus $T^n$ in $U(n)$. By the Standardization Theorem, there exists a biholomorphic mapping $F: M \to D$ of $M$ onto a Reinhardt domain $D$ in $\mathbb{C}^n$ such that $F \rho(T^n) = T(D)$. Set $G = F \rho(U(n))$ and take a relatively compact subdomain $D$ of $D$. Then $D_0 := \{ g(z) \in D \mid g \in G, z \in U \}$ is a bounded Reinhardt domain in $\mathbb{C}^n$ and $G$ can be regarded as a connected compact subgroup of the Lie group $Aut(D_0)$ containing $T(D_0)$. Recalling that $G$ is isomorphic to $U(n)$ and $n \geq 2$, we can apply Proposition 1.1 and its corollary to $D_0$ and $G \subset Aut(D_0)$. Therefore there exists a transformation

$$
\varphi: \mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) \in \mathbb{C}^n,
$$

$$
w_i = r_i^\sigma(i), \quad 1 \leq i \leq n,
$$

such that, for $D_0 = \varphi(D_0)$ and $\tilde{G} = \varphi G \varphi^{-1} \subset Aut(D_0)$, we have $\tilde{G} = U(n)$, where $r_1, \ldots, r_n$ are positive constants and $\sigma$ is a permutation of $\{1, \ldots, n\}$. Putting $\tilde{D} = \varphi(D)$, we see by the uniqueness theorem on holomorphic functions that $U(n) = \tilde{G} \subset Aut(\tilde{D})$, or $g(\tilde{D}) = \tilde{D}$ for all $g \in U(n)$. Since $\tilde{D}$ is a Stein manifold, it follows that $\tilde{D}$ has the form

$$
\tilde{D} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \left| \sum_{i=1}^{n} |z_i|^2 < r \right. \right\},
$$

where $0 < r \leq +\infty$. This shows that $\tilde{D}$, and hence $M$ is biholomorphically equivalent to either $B^n$ or $\mathbb{C}^n$, proving the first assertion.

Now, let us identify $M$ with $B^n$ or $\mathbb{C}^n$. When $M = B^n$, the existence of $\psi \in Aut(M)$ satisfying the relation $\psi \rho(U(n)) \psi^{-1} = U(n)$ is a consequence of the conjugacy of maximal compact subgroups of the simple Lie group $Aut(B^n)$. So, consider the case of $M = \mathbb{C}^n$. Then, by the same reasoning as above, there exist biholomorphic mappings $F: M = \mathbb{C}^n \to D = \mathbb{C}^n$ and $\varphi: C^n \to \mathbb{C}^n$ such that $(\varphi \circ F) \rho(U(n)) (\varphi \circ F)^{-1} = U(n)$. Therefore, the composition $\psi = \varphi \circ F$ is an element of $Aut(\mathbb{C}^n)$ required in the theorem. 

\[\square\]

Added in proof. After the submission of this paper, the authors learned in the letter of August 21, 2002, from Professor A. Isaev that, in the special case of $k = n$, the same result as our Main Theorem had been obtained independently by him (Proc. Steklov Inst. Math. 235 (2001), 103–106).

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References

A GROUP-THOERETIC CHARACTERIZATION OF THE SPACE


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