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ON MANIFOLDS WITH TRIVIAL LOGARITHMIC TANGENT BUNDLE

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1. Introduction

By a classical result of Wang [15] a connected compact complex manifold X has holomorphically trivial tangent bundle if and only if there is a connected complex Lie group G and a discrete subgroup Γ such that X is biholomorphic to the quotient manifold G/Γ. In particular X is homogeneous. If X is Kähler, G must be commutative and the quotient manifold G/Γ is a compact complex torus.

The purpose of this note is to generalize this result to the non-compact Kähler case. Evidently, for arbitrary non-compact complex manifold such a result can not hold. For instance, every domain over $\mathbb{C}^n$ has trivial tangent bundle, but many domains have no automorphisms.

So we consider the "*open case*" in the sense of Iitaka ([7]), i.e. we consider manifolds which can be compactified by adding a divisor.

Following a suggestion of the referee, instead of only considering Kähler manifolds we consider manifolds in class $\mathcal{C}$ as introduced in [5]. A compact complex manifold X is said to be class in $\mathcal{C}$ if there is a surjective holomorphic map from a compact Kähler manifold onto X. Equivalently, X is bimeromorphic to a Kähler manifold ([14]). For example, every Moishezon manifold is in class $\mathcal{C}$.

We obtain the following characterization:

Main Theorem. *Let $\bar{X}$ be a connected compact complex manifold, D a closed analytic subset and $X = \bar{X} \setminus D$. Assume that $\bar{X}$ is in class $\mathcal{C}$ as introduced in* [5] (*also called* "*weakly Kähler*").

Then the following conditions are equivalent:

(1) *D is a divisor which is locally s.n.c.* (*see definitions in* §2 *below*) *and the logarithmic tangent bundle $T(-\log D)$ is a holomorphically trivial vector bundle on $\bar{X}$.*

(2) *There is a complex semi-torus G acting effectively on $\bar{X}$ with X as open orbit such that the all the isotropy groups are themselves semi-tori.*

Moreover, if one (*hence both*) *of these conditions are fulfilled, then D is*

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a s.n.c. divisor and there is a short exact sequence of complex Lie groups

$$1 \to (\mathbb{C}^*)^d \to G \to \mathrm{Alb}(\overline{X}) \to 1$$

where $\mathrm{Alb}(\overline{X})$ *denotes the Albanese torus of* $\overline{X}$ *and* $d = \dim(X) - (1/2)b_1(X)$.

In the algebraic category we have the following result.

Corollary 1. *Let* $\overline{X}$ *be a non-singular complete algebraic variety defined over* $\mathbb{C}$, D *a divisor with only simple normal crossings as singularities and let* $X = \overline{X} \setminus D$.

Then the following conditions are equivalent:

(1) *The logarithmic tangent bundle* $T(-\log D)$ *is a trivial vector bundle on* $\overline{X}$.

(2) *There is a semi-abelian variety* T *acting on* $\overline{X}$ *with* X *as open orbit.*

Corollary 2. *Let* X *be a nonsingular algebraic variety defined over* $\mathbb{C}$.

Then the following are equivalent:

(1) *There exists a completion* $X \hookrightarrow \overline{X}$ *such that* $D = \overline{X} \setminus X$ *is a s.n.c. divisor and* $T(-\log D)$ *is trivial.*

(2) X *is isomorphic to a semi-abelian variety.*

Proof. This follows from the preceding result because every semi-abelian variety admits an equivariant completion by a s.n.c. divisor. □

2. Terminology

A *complex semi-torus* G is a connected complex Lie group which can be realized as a quotient of a vector group $(\mathbb{C}^n, +)$ by a discrete subgroup Γ such that Γ generates $\mathbb{C}^n$ as complex vector space. Equivalently, a connected complex Lie group G is a *semi-torus* if and only if there exists a short exact sequence of complex Lie groups

$$0 \to L \to G \to T \to 0$$

where T is a compact complex torus and $L \simeq (\mathbb{C}^*)^d$ for some $d \in \mathbb{N}$.

We also need the notion of a divisor with only simple normal crossings as singularities ("s.n.c. divisor").

A divisor D on a complex manifold $\overline{X}$ is called "locally s.n.c." if for every point $x \in \overline{X}$ there exists local coordinates $z_1, \dots, z_n$ and a number $d \in \{0, \dots, n\}$ such that in a neighbourhood of x the divisor D equals the zero divisor of the holomorphic function $\prod_{i=1}^d z_i$.

It is called a "divisor with only simple normal crossings as singularities" or "s.n.c. divisor" if in addition every irreducible component of D is smooth.

The definition of "locally s.n.c" implies that, given a locally s.n.c. divisor D on a manifold $\overline{X}$ and a point $p \in \overline{X}$ there is an open neighborhood W of p in $\overline{X}$ such that all irreducible components of $W \cap D$ are smooth.

Hence a locally s.n.c. divisor D is necessarily s.n.c. unless it contains an irreducible component which is not everywhere locally irreducible.

Let $\bar{X}$ be a compact complex manifold with a locally s.n.c. divisor D. There is a stratification as follows: $X_0 = X = \bar{X} \setminus D$, $X_1 = D \setminus \mathrm{Sing}(D)$ and for $k > 1$ the stratum X_k is the non-singular part of $\mathrm{Sing}(X_{k-1})$. If in local coordinates D can be written as $\left\{z : \prod_{i=1}^{d} z_i = 0\right\}$, then $z = (z_1, \ldots, z_n) \in X_k$ iff $\#\{i : 1 \le i \le d,\ z_i = 0\} = k$.

Let D be an effective divisor on a complex manifold $\bar{X}$. Then the sheaf $\Omega^1(\log D)$ of *logarithmic* 1-*forms* with respect to D is defined as the $\mathcal{O}_{\bar{X}}$-module subsheaf of the sheaf of meromorphic one-forms on $\bar{X}$ which is locally generated by all df/f where f is a section $\mathcal{O}_{\bar{X}} \cap \mathcal{O}_X^*$.

This sheaf is always coherent. It is locally free if D is a locally s.n.c. divisor. In fact, if $D = \{z_1 \cdots z_d = 0\}$, then $\Omega^1(\log D)$ is locally the free $\mathcal{O}_{\bar{X}}$-module over $dz_1/z_1, \ldots, dz_d/z_d, dz_{d+1}, \ldots, dz_n$.

For a *locally s.n.c.* divisor D on a complex manifold we define the *logarithmic tangent bundle* $T(-\log D)$ as the dual bundle of $\Omega^1(\log D)$.

Then $T(-\log D)$ can be identified with the sheaf of those holomorphic vector fields V on $\bar{X}$ which fulfill the following property:

V_x is tangent to X_k at x for every k and every $x \in X_k$.

In local coordinates: If $D = \left\{z : \prod_{i=1}^{d} z_i = 0\right\}$, then $T(-\log D)$ is the locally free sheaf generated by the vector fields $z_i(\partial/\partial z_i)$ $(1 \le i \le d)$ and $\partial/\partial z_i$ $(d < i \le n)$.

3. The proof of the main theorem

Proof. (1) $\Rightarrow$ (2):

Triviality of $T(-\log D)$ implies that the sheaf of logarithmic one-forms $\Omega^1(\log D)$ is trivial as well.

Let $V = \Omega^1(\bar{X}, \log D)$ and $V^* = \Gamma(\bar{X}, T(-\log D))$. By [4], [10] every logarithmic one-form $\omega \in \Omega^1(\bar{X}, \log D)$ is closed if X is Kähler. For an arbitrary manifold X in class $\mathcal{C}$ there is always a holomorphic surjective bimeromorphic map $p\colon X' \to \bar{X}$ from some compact Kähler manifold X'. Moreover, by blowing-up X' if necessary, we may assume that $p^{-1}(D)$ is a s.n.c divisor on X'. Now the aforementioned result for Kähler manifolds implies that $d(p^*\omega) = 0$ for every $\omega \in \Omega^1(\bar{X}, \log D)$. Since p is biholomorphic on some open subset, we obtain $d\omega = 0$. Therefore closedness of logarithmic one-forms holds not only for Kähler compact complex manifolds, it holds for all manifolds in class $\mathcal{C}$.

Thus

$$0 = d\omega(x, y) = x(\omega(y)) - y(\omega(x)) - \omega([x, y])$$

for $\omega \in V$, $x, y \in V^*$. Now $\omega(y)$ and $\omega(x)$ are global holomorphic functions on a compact manifold and therefore constant. Hence $x(\omega(y)) = 0 = y(\omega(x))$. It follows that

$\omega([x,y]) = 0$ for all $\omega \in V$, $x, y \in V^*$. Thus V^* is a *commutative* Lie algebra of holomorphic vector fields on $\overline{X}$. Let $G \subset \mathrm{Aut}(\overline{X})$ denote the subgroup generated by the one-parameter groups corresponding to vector fields $v \in V^*$. Recall that the sections in $T(-\log D)$ can be regarded as the vector fields which are tangent to X_k at every $x \in X_k$ for all k. It follows that the G-orbits in $\overline{X}$ are precisely the connected components of the strata X_k. In particular G has an open orbit, namely $X = X_0 = \overline{X} \setminus D$. Furthermore the closed orbits of G are the connected components of the unique closed stratum X_d where d is the largest natural number with $X_d \neq \emptyset$.

The existence of an open orbit implies that G acts transitively on the Albanese torus $\mathrm{Alb}(\overline{X})$. Therefore, all the fibers of $\overline{X} \to \mathrm{Alb}(\overline{X})$ are isomorphic. Let $\overline{X} \to Y \to \mathrm{Alb}(\overline{X})$ be the Stein factorization. Since the Stein factorization is canonical, it is compatible with the $\mathrm{Aut}(\overline{X})$-action. For this reason $Y \to \mathrm{Alb}(X)$ is a finite holomorphic map of G-spaces. Hence G acts transitively on Y. It follows that Y is a compact complex space which is a quotient of a connected commutative complex Lie group, in other words, Y must be a compact complex torus. By the universality property of the Albanese torus this implies $Y = \mathrm{Alb}(\overline{X})$.

Thus the fibers of $\overline{\rho}\colon \overline{X} \to \mathrm{Alb}(\overline{X})$ are connected. Let H be the kernel of $G \mapsto \mathrm{Alb}(\overline{X})$. Recall that G is commutative. It follows that the H-orbits in $\overline{X}$ are precisely the intersections of G-orbits in $\overline{X}$ with fibers of the map $\overline{\rho}\colon \overline{X} \to \mathrm{Alb}(\overline{X})$. Moreover, H acts freely on an open orbit in each fiber of $\overline{\rho}$. This implies that H is connected.

Now let Z be a closed G-orbit (i.e. a connected component of the smallest stratum X_d). Then the fibers of $\overline{\rho}|_Z$ are closed H-orbits.

If $\overline{X}$ is Kähler, a result of Sommese ([13], prop. 1) implies that closed orbits of H are fixed points. Due to Fujiki ([5]) the same assertion holds for an arbitrary manifold $\overline{X}$ in class $\mathcal{C}$.

It follows that $\overline{\rho}|_Z\colon Z \to \mathrm{Alb}(\overline{X})$ is biholomorphic.

For each irreducible component D_i of D and each $\omega \in V$ we may regard the *residue* $\mathrm{res}_i(\omega)$. This residue is given as integral of ω over a small loop around D_i. A priori, it is a holomorphic function on D_i. But, since D_i is compact, this holomorphic function is constant (Alternatively, one may also use a calculation in local coordinates which shows that $d\omega = 0$ forces $\mathrm{res}_i(\omega)$ to be locally constant).

Let $n = \dim \overline{X}$ and $g = \dim \mathrm{Alb}(\overline{X})$.

Fix $p \in Z$. Near p, Z is the intersection of $d = n - g$ irreducible components $D_1, \ldots, D_d$ of D. We choose a basis $(\omega_1, \ldots, \omega_n)$ of V such that

$$\mathop{\mathrm{res}}_i(\omega_j)_p = \begin{cases} 2\pi i & \text{if } i = j \leq d, \\ 0 & \text{if } i \neq j \text{ or } j > d. \end{cases}$$

Now we choose a point $q \in X$ near p and define local coordinates z_i near p via

$$z_i(x) = \begin{cases} \exp\left(\int_q^x \omega_i\right) & \text{if } i \leq d, \\ \int_p^x \omega_i & \text{if } i > d. \end{cases}$$

Then in these local coordinates we can describe the ω_i as follows:

$$\omega_i = \begin{cases} \dfrac{dz_i}{z_i} & \text{if } i \leq d, \\ dz_i & \text{if } i > d. \end{cases}$$

It follows that there is a biholomorphic map from a neighbourhood of p to a neighbourhood of 0 in $\mathbb{C}^n$ taking the fundamental vector fields of Lie(H) into vector fields of the form $\sum_{i \leq d} a_i w_i(\partial/\partial w_i)$. This implies in particular that H contains a totally real compact subgroup $K = (S^1)^d$ acting as

$$K \ni (\theta_1, \ldots, \theta_d) \colon z \mapsto (\theta_1 z_1, \ldots, \theta_d z_d, z_{d+1}, \ldots, z_n).$$

Thus H is a connected commutative complex Lie group of dimension d containing a totally real compact subgroup of real dimension d. It follows that H is a complex semi-torus. On the other hand, the above description of the H-vector fields in local coordinates also implies that H admits an almost faithful representation on the tangent space of each fixed point of H. Therefore H is a semi-torus of dimension d which admits an almost faithful representation. It follows that H must be isomorphic to $(\mathbb{C}^*)^d$.

As a consequence we obtain that G admits a short exact sequence of complex Lie groups in the form

$$1 \to (\mathbb{C}^*)^d \to G \to \mathrm{Alb}(\overline{X}) \to 1$$

Thus G is a semi-torus as well.

(1) $\Leftarrow$ (2):

Let $v_1, \ldots, v_n$ be a basis for the vector space of G-fundamental vector fields on $\overline{X}$. The complement $D = \overline{X} \setminus X$ of the open orbit X can be characterized as the set of those points where the vector fields v_i fail to span the tangent bundle $T_{\overline{X}}$. Thus D is defined by the vanishing of

$$w = \bigwedge_{i=1}^n v_i \in \Gamma\left(\overline{X}, \Lambda^n T_{\overline{X}}\right).$$

Since $\Lambda^n T_{\overline{X}}$ is a line bundle, it is clear that D is of pure codimension one.

Now let $p \in \operatorname{supp} D$ and $G_p = \{a \in G : a(p) = p\}$. By assumption, G_p is a semi-torus and therefore reductive. This implies that the G_p-action can be linearized near p, i.e. there is an G_p-equivariant biholomorphism between an open neighbourhood Ω of p in $\bar{X}$ and an open neighbourhood of 0 in the vector space $W = T_p\bar{X}$. It follows that a neighbourhood of p in D is isomorphic to a union of vector subspaces of codimension one in W. From the assumption that the G_p-action is effective, one can deduce that this is a transversal union, i.e. $D \cap \Omega$ is a simple normal crossings divisor in the neighbourhood Ω of p. Thus D is locally s.n.c.

Recall that the isotropy groups are required to be semi-tori. In particular, they are reductive. Therefore the action of every isotropy subgroup is linearizable in some open neighbourhood. This implies that for every point on $\bar{X}$ we can find a system of local coordinates in which the G-fundamental vector fields are simply

$$z_1\frac{\partial}{\partial z_1}, \dots, z_d\frac{\partial}{\partial z_d}, \frac{\partial}{\partial z_{d+1}}, \dots, \frac{\partial}{\partial z_n},$$

where d equals the dimension of the isotropy group. Hence we have

$$\mathbf{T}(\bar{X}, D) \cong \bar{X} \times \mathrm{Lie}(G).$$

We have already seen that D is locally s.n.c. In order to show that it is s.n.c., it suffices to verify that each irreducible component of D is everywhere locally irreducible. In other words, we have to show that, given an open neighbourhood Ω and two irreducible components D_i, D_j of $D \cap \Omega$ these two components are not contained in the same irreducible component of D unless $D_i = D_j$. Using the local coordinates $(z_i)_i$ introduced above, the irreducible components of $D \cap \Omega$ are given as

$$D_i = \{p \in \Omega : z_i(p) = 0\}$$

with i running from 1 to d. For each such i there is a G-fundamental vector field of the form

$$z_i\frac{\partial}{\partial z_i}.$$

This vector field vanishes identically on D_i, but not on any D_j with $j \neq i$. By the identity principle this vector field also vanishes on the (global) irreducible component of D containing D_i. As a consequence, for $i \neq j$ it is not possible that D_i and D_j are contained in the same global irreducible component of D. $\square$

REMARK 1. In the proof for the direction "(1) $\Leftarrow$ (2)" we did not employ the Kähler assumption. Therefore this part of the theorem is valid even without requiring X to be Kähler.

4. Examples

4.1. Toric varieties. The easiest examples of equivariant compactifications of $(\mathbb{C}^*)^d$ with trivial logarithmic tangent bundles are $\mathbb{P}_d(\mathbb{C})$ and $\mathbb{P}_1(\mathbb{C})^d$. More examples are obtained from the theory of toric varieties, see e.g. [6], [12].

Now let A be a complex semi-torus admitting a short exact sequence

$$1 \to (\mathbb{C}^*)^d \to A \to T \to 1 \tag{1}$$

where T is a compact complex torus. Let $L \hookrightarrow \overline{L}$ be a smooth equivariant compactification of $L = (\mathbb{C}^*)^d$. Then a smooth equivariant compactification of A can be constructed as a fiber product: $\overline{A} = (A \times \overline{L})/\sim$ where $(a, x) \sim (a', x')$ iff there exists an element $g \in L$ such that $a \cdot g^{-1} = a'$ and $g \cdot x = x'$.

This construction preserves the Kähler condition:

Lemma. *If $\overline{L}$ is Kähler, then $\overline{A}$ is Kähler, too.*

Proof. The fiber bundle (1) is given by locally constant transition functions with values in the maximal compact subgroup K of L, acting by multiplication. By averaging we may assume that the Kähler metric on $\overline{L}$ is K-equivariant. Thus the associated Kähler form ω_1 induces a closed semi-positive $(1,1)$-form on $\overline{A}$ such that the restriction to the tangent bundle of any fiber is positive. Taking sum of this $(1,1)$-form and the pull-back of a Kähler form on T yields a Kähler form on $\overline{A}$. □

(This is a special case of a general result of Blanchard [2] which implies that for any holomorphic fiber bundle of compact complex manifolds $E \to B$ with typical fiber F and $b_1(F) = 0$ the Kähler property for both B and F implies the Kähler property for E.)

In this way we see that every semi-torus admits a smooth equivariant Kähler compactification. On the other hand, our Main Theorem implies that every smooth equivariant Kähler compactification of a semi-torus arises in this way.

In contrast, non-Kähler compactifications may arise in many ways, see e.g. [9] and the examples given further below in this article.

4.2. Class $\mathcal{C}$ versus Kähler. A compact complex manifold is projective iff it is both Kähler and Moishezon. On the other hand, every algebraic variety is birational to a projective variety and therefore in class $\mathcal{C}$. As a consequence, a compact algebraic manifold is always in class $\mathcal{C}$ and it is Kähler iff it is projective.

Hence a compact smooth non-projective algebraic toric variety (see [12], p. 84 for the existence of such toric varieties) yields an example of a compact complex manifold $\overline{X}$ with s.n.c. divisor D such that the logarithmic tangent bundle $T(-\log D)$ is trivial and $\overline{X}$ is in class $\mathcal{C}$, but not Kähler.

4.3. Non-Kähler examples. If X is a compact complex manifold with trivial tangent bundle, and G the connected component of its automorphism group, then X is Kähler iff X is in class $\mathcal{C}$ iff G is commutative.

In the logarithmic case, there is such a conclusion only in one direction: If X is a compact complex manifold with s.n.c. divisor D such that $T(-\log D)$ is trivial, then the Kähler assumption implies that the connected component of $\mathrm{Aut}(X, D)$ is commutative. On the other hand the commutativity does not imply the Kähler property as we will see by the example given below.

Let $\alpha, \beta \in \mathbb{C}$ with $|\alpha|, |\beta| > 1$. We define a $\mathbb{Z}$-action on $\mathbb{C}^2 \setminus \{(0, 0)\}$ by

$$(z_1, z_2) \mapsto (\alpha^n z_1, \beta^n z_2).$$

Then the quotient of $\mathbb{C}^2 \setminus \{(0, 0)\}$ by this $\mathbb{Z}$-action is a so-called *Hopf surface*. Such a Hopf surface $\bar{X}$ is diffeomorphic to $S^1 \times S^3$. In particular $\dim H^1(\bar{X}, \mathbb{C})$ is odd and therefore $\bar{X}$ can not be Kähler. Moreover, it can not be in class $\mathcal{C}$.

Now let T be the quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the subgroup

$$\{(\alpha^n, \beta^n) : n \in \mathbb{Z}\}.$$

Then T is a complex semi-torus and $\bar{X}$ is an equivariant compactification of T. The isotropy groups at the two non-open orbits are isomorphic to $\mathbb{C}^*$.

Thus all the isotropy groups are semi-tori and consequently the logarithmic tangent bundle is trivial (see Remark 1).

4.4. The noncommutative case. Let $\bar{X}$ be a non-Kähler compact complex manifold with a locally s.n.c. divisor D such that $T(-\log D)$ is trivial.

In this case it is still true that there is a connected complex Lie group G with $\dim(G) = \dim(\bar{X})$ acting on $\bar{X}$ with $X = \bar{X} \setminus D$ as open orbit. However, G might be non-commutative and the G-action on the open orbit is only almost free, i.e., the isotropy group at a point of the open orbit is not necessarily trivial, but at least discrete.

The easiest such examples, with $D = \emptyset$, are obtained by considering discrete subgroups Γ in connected complex Lie groups with compact quotient G/Γ. By a result of Borel (see [3]) every semisimple Lie group contains such a "cocompact" discrete subgroup Γ. Such complex quotients have been studied in [16].

Next let us give an example with $D \neq \emptyset$. Recall that $SL_2(\mathbb{C})$ contains discrete cocompact subgroups Γ such that $b_1(Y) > 0$ for $Y = SL_2(\mathbb{C})/\Gamma$ (see [8]). Then $H^1(Y, \mathcal{O})$ is a vector space of positive dimension equal to $b_1(Y) > 0$ and the induced action of $SL_2(\mathbb{C})$ on $H^1(Y, \mathcal{O})$ is trivial (see [1]). For any $\alpha \in H^1(Y, \mathcal{O})$ let α' denote the image via

$$\exp\colon H^1(Y, \mathcal{O}) \to H^1(Y, \mathcal{O}^*).$$

Then α' defines a topologically trivial $\mathbb{C}^*$-principal bundle. Since the $SL_2(\mathbb{C})$-action on $H^1(Y, \mathcal{O})$ is trivial, this is a homogeneous bundle. If G denotes the connected component of the group of $\mathbb{C}^*$-principal bundle automorphisms, we thus obtain a short exact sequence

$$1 \to \mathbb{C}^* \to G \to SL_2(\mathbb{C}) \to 1.$$

Such a sequence is necessary split. Hence $G \simeq \mathbb{C}^* \times SL_2(\mathbb{C})$. Now consider the compactification $\overline{X}$ of the total space X of this bundle given by adding a 0- and a ∞-*section*. Then $\overline{X}$ has a trivial logarithmic tangent bundle $T(-\log D)$ for $D = \overline{X} \setminus X$ and G acts on $\overline{X}$ with X as open orbit.

4.5. The condition on the isotropy groups. Next we present an example to show that the condition on the isotropy groups in property (2) in the Main Theorem can not be dropped.

To see this, we first note that a complex-analytic semi-torus may contain closed complex Lie subgroups which are not semi-tori. The easiest such example is obtained as follows: Embedd the additive group (which is not a semi-torus) into the semi-torus $\mathbb{C}^* \times \mathbb{C}^*$ via

$$\mathbb{C} \ni t \overset{f}{\mapsto} (e^t, e^{it}) \in \mathbb{C}^* \times \mathbb{C}^*.$$

To verify that the image is indeed a *closed* subgroup, consider its pre-image in the universal covering of $\mathbb{C}^* \times \mathbb{C}^*$. If we realize the universal covering by $\pi\colon (z_1, z_2) \mapsto (e^{z_1}, e^{z_2})$, then

$$\pi^{-1}(f(\mathbb{C})) = \{(t, it) : t \in \mathbb{C}\} + (2\pi i\mathbb{Z})^2 = \left\{(z_1, z_2) : \frac{z_2 - iz_1}{2\pi i} \in \mathbb{Z}[i]\right\}.$$

Hence $f(\mathbb{C})$ is closed and $(\mathbb{C}^* \times \mathbb{C}^*)/f(\mathbb{C}) \simeq \mathbb{C}/\mathbb{Z}[i]$. In this way the complex manifold $X = \mathbb{C}^* \times \mathbb{C}^*$ can be realized as a $\mathbb{C}$-principal bundle over the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i]$. The embedding $\mathbb{C} \hookrightarrow \mathbb{P}_1(\mathbb{C})$ induces a compactification $\overline{X}$ of X by adding a ∞-section to the $\mathbb{C}$-principal bundle $X \to E$. Thus we obtain an equivariant compactification where the isotropy group for a point at $\overline{X} \setminus X$ is isomorphic to $(\mathbb{C}, +)$ and therefore not a semi-torus. If we now look at the vector field corresponding to this action of the additive group $(\mathbb{C}, +)$, we see that it vanishes of order two at the ∞-section. This implies that, regarded as a section in the logarithmic tangent bundle, it does vanish at the ∞-section with multiplicity one. In particular, the logarithmic tangent bundle admits a holomorphic section which vanishes somewhere, but not everywhere. Hence the logarithmic tangent bundle can not be trivial.

REMARK 2. This example was first used by Serre for an entirely different purpose: Via GAGA the $\mathbb{C}$-principal bundle over E is algebraic. In this way one obtains an exotic algebraic structure on the complex manifold $\mathbb{C}^* \times \mathbb{C}^*$. This yields an example of an algebraic variety which is not affine (in fact every regular function is constant), although it is Stein as a complex manifold.

5. An application

Let $U \to \Delta$ be a family of projective manifolds. By our characterization of log-parallelizable manifolds we obtain an easy proof that the set of all $t \in \Delta$ for which U_t is an equivariant compactification of a semi-abelian variety forms a *constructible* subset of Δ.

Proposition. *Let $\pi\colon U \to \Delta$ be a smooth projective connected surjective map between Kähler complex manifolds and let D be a hypersurface on U which does not contain any fiber of π.*

Let S denote the set of all $t \in \Delta$ for which the fiber $U_t = \pi^{-1}(\{t\})$ is an equivariant algebraic compactification of a semi-abelian variety with $U_t \setminus D$ as open orbit.

Then S is constructible, i.e., there is a finite family of pairs of closed analytic subsets $Y_i \subset Z_i \subset \Delta$ such that $S = \bigcup_i Z_i \setminus Y_i$.

Proof. By our theorem, S coincides with the set of all $t \in \Delta$ with the property that $\pi^{-1}(\{t\}) \cap D$ is a s.n.c. divisor and furthermore the logarithmic tangent bundle is trivial.

The fiber dimension $\dim_x(\pi^{-1}(\pi(x)))$ is Zariski semicontinuous, because π is proper. Hence there is no loss in generality in assuming that the fiber dimension is constant. Let r denote this fiber dimension. Then for every $p \in \Delta$ the number of irreducible components of $\pi^{-1}(p) \cap D$ equals $\dim H^{2r-2}(\pi^{-1}(p) \cap D, \mathbb{C})$. Using the resolution of the sheaf of local constant functions $\underline{\mathbb{C}}$ by coherent sheaves via the holomorphic de Rham complex

$$0 \to \underline{\mathbb{C}} \to \mathcal{O} \to \Omega^1 \to \cdots$$

combined with the semicontinuity results for coherent sheaves it follows that Δ decomposes as a finite union of constructible sets along which the number of irreducible components of $\pi^{-1}(p) \cap D$ is constant.

Now a divisor with m irreducible components in a compact complex manifold F is a *s.n.c.* divisor iff all the irreducible components are smooth and meet transversally. These are Zariski open conditions (as long as the number of irreducible components does not jump).

Put together, these arguments show that the set Σ of all $t \in \Delta$ for which the fiber $\pi^{-1}(p) \cap D$ is a s.n.c. divisor constitutes a constructible subset of Δ.

Thus there is no loss of generality in assuming that $\pi^{-1}(p) \cap D$ is always a s.n.c. divisor.

Let E denote the bundle of logarithmic vertical vector fields. Then E is a vector bundle of rank r where $r = \dim(U) - \dim(\Delta)$. By our Main Theorem $t \in S$ iff $E|_{U_t}$ is holomorphically trivial. By the semi-continuity theorem

$$Z = \{t \in \Delta : \dim \Gamma(U_t, E|_{U_t}) \geq r\}$$

is a closed analytic subset of Δ and

$$\Omega = \{t \in \Delta : \dim \Gamma(U_t, E|_{U_t}) = r\}$$

is Zariski open in Z. Moreover, for $t \in \Omega$ every section of $E|_{U_t}$ extends to some neighbourhood of U_t in $\pi^{-1}(\Omega)$. Thus the set W of all points in $\pi^{-1}(\Omega)$ where the sections of $E|_{U_t}$ fail to span $E|_{U_t}$ is a closed analytic subset. Since π is proper, $S = \Omega \setminus \pi(W)$ is Zariski open in Z. $\square$

REMARK 3. For a similar result by different methods, compare [11].

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