SOME HOMOTOPY OF THE UNITARY GROUPS DETECTED
BY THE \( K \)-THEORY OF 2-CELL COMPLEXES

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(Received May 2, 2002)

1. Introduction

Let \( k \geq 1 \) and \( m \geq 2k + 1 \). Consider the real Hopf-Whitehead \( J \)-homomorphism
\[ J : \pi_{4k-1}(SO(2m)) \longrightarrow \pi_{2m+4k-1}(S^{2m}) \]. Since the quotient \( SO / SO(2m) \) is \( 2m \)-connected, by real Bott periodicity, we have \( \pi_{4k-1}(SO(2m)) \cong \pi_{4k-1}(SO) \cong \mathbb{Z} \). For \( s \geq m \),
\( BU \) and \( BU(s) \) admit CW-complex structures with the same \((2m+1)\)-skeleton, so, we have isomorphisms \([S^{2m}, BU(s)] \cong [S^{2m}, BU] \cong K^0(S^{2m}) \cong \mathbb{Z} \), using complex Bott periodicity. By the Freudenthal Suspension Theorem, there is an isomorphism
\[ \pi_{2m+4k-1}(S^{2m}) \cong \pi_{4k-1} \]. (We refer to p. 480 in [12], p. 216 in [8], Theorem I in [2], and Theorem VI.2.10 in [4] for the details.) We prove the following result:

Theorem 1.1. For \( k \geq 1 \), \( m \geq 2k + 1 \) and \( m \leq s < m + 2k \), let \( j_{4k-1} \in \pi_{4k-1} \) denote the image of a generator of \( \pi_{4k-1}(SO) \) under the \( J \)-homomorphism, and let \( x_{2m} \) be the Bott generator of \( K^0(S^{2m}) \). Then, the composition \( x_{2m} \circ j_{4k-1} \) represents a non-zero element in \( \pi_{2m+4k-1}(BU(s)) \), whose order is given by

\[
\begin{cases} 
\text{denom} \left( \frac{B_k}{4k} \right), & \text{if } k \text{ is even} \\
\text{denom} \left( \frac{B_k}{4k} \right) \text{ or } \frac{1}{2} \text{denom} \left( \frac{B_k}{4k} \right), & \text{if } k \text{ is odd},
\end{cases}
\]

where \( B_k \) is the \( k \)-th Bernoulli number. When \( k \) is odd and \( s \) is equal to \( m + 2k - 1 \), the order of \( x_{2m} \circ j_{4k-1} \) is \((1/2)\text{denom} \left( \frac{B_k}{4k} \right)\).

Unfortunately, we were unable to determine in full generality the precise order when \( k \) is odd. Notice that for given \( k \) and \( m \), the order might depend on \( s \) (neither could we settle this question.)

We single out that the element \( x_{2m} \circ j_{4k-1} \) of \( \pi_{2m+4k-1}(BU(s)) \) can be written down explicitly by means of the \( J \)-homomorphism and of the real and the complex Bott periodicity isomorphisms. Let us now give some numerical examples, where the indicated homotopy groups of the Grassmannians \( BU(n) \) can for instance be found either

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Research partially supported by Swiss National Fund for Scientific Research.
in Mimura’s survey article [9] or in Lundell’s tables [6].

**Examples 1.2.** i) For \( s = 1 \) and \( m = 3 \), denom \((B_1/4) = 24\) holds and we can take \( s = 3 \) or 4; the corresponding groups are \( \pi_s(BU(3)) \cong \mathbb{Z}/12 \) and \( \pi_s(BU(4)) \cong \mathbb{Z}/24 \). We see that \( x_0 \circ j_3 \) is a generator of the former, but only generates a subgroup of index 2 in the latter. Changing \( m \) yields in each case an element of order 12 or 24 in the first indicated group and of order 12 in the second one:

\[
\text{for } m = 4: \quad \pi_{11}(BU(4)) \cong \frac{\mathbb{Z}}{2} \oplus \frac{\mathbb{Z}}{24} \oplus \frac{\mathbb{Z}}{5}, \\
\pi_{11}(BU(5)) \cong \frac{\mathbb{Z}}{24} \oplus \frac{\mathbb{Z}}{5};
\]

\[
\text{for } m = 5: \quad \pi_{13}(BU(5)) \cong \frac{\mathbb{Z}}{72} \oplus \frac{\mathbb{Z}}{5}, \\
\pi_{13}(BU(6)) \cong \frac{\mathbb{Z}}{144} \oplus \frac{\mathbb{Z}}{5}.
\]

ii) Since denom \((B_2/8) = 24\), for \( k = 2 \) and \( m = 5 \), we get an element of order 24 in the groups

\[
\pi_{17}(BU(5)) \cong \frac{\mathbb{Z}}{2} \oplus \frac{\mathbb{Z}}{48} \oplus \frac{\mathbb{Z}}{5} \oplus \frac{\mathbb{Z}}{7}, \\
\pi_{17}(BU(6)) \cong \frac{\mathbb{Z}}{144} \oplus \frac{\mathbb{Z}}{5} \oplus \frac{\mathbb{Z}}{7}, \\
\pi_{17}(BU(7)) \cong \frac{\mathbb{Z}}{576} \oplus \frac{\mathbb{Z}}{5} \oplus \frac{\mathbb{Z}}{7}, \\
\pi_{17}(BU(8)) \cong \frac{\mathbb{Z}}{1152} \oplus \frac{\mathbb{Z}}{5} \oplus \frac{\mathbb{Z}}{7}.
\]

We observe that even for \( k \) even, the element \( x_{4k+2} \circ j_{4k-1} \) does generally not generate a direct summand in \( \pi_{8k+1}(BU(2k+1)) \).

iii) In Theorem 1.1, the case of most interest for \( k \geq 1 \) fixed is when \( m \) and \( s \) are as small as possible, namely \( m = s = 2k + 1 \): it predicts that \( x_{4k+2} \circ j_{4k-1} \) is of order \( \text{denom } (B_k/4k) \) (or possibly half of it for \( k \) odd) in the group \( \pi_{8k+1}(BU(2k+1)) \). As an illustration, for \( k = 6 \), we get the element \( x_{26} \circ j_{23} \) of order 65520 in \( \pi_{49}(BU(13)) \).

Here is a brief outline of the content of the paper. In Section 2, we study the \( K \)-theory of 2-cell complexes with even dimensional cells, say \( X = S^{2m} \cup_f e^{2m+2} \). In particular, we determine the Chern classes of the elements of \( K^0(X) \) in terms of the Adams \( e \)-invariant of the attaching map \( f \). The connection with the homotopy of \( BU(m) \) is obtained by studying the set of bundles over \( X \) that restrict to a given multiple of the Bott generator \( x_{2m} \) over the sphere \( S^{2m} \). Section 3 contains the proof of Theorem 1.1.

**Acknowledgements.** The author expresses his deep gratitude to Ulrich Suter for several fruitful discussions, and to the referee for a careful reading and for suggesting
some improvements.

2. On the $K$-theory of 2-cell complexes

In this section, we recall some basic and well-known properties of the $K$-theory of 2-cell complexes, in order to establish a key ingredient (Proposition 2.2 below) for the proof of Theorem 1.1.

Let $f : S^{2m+2l-1} \to S^{2m}$ be a pointed map with $m, l \geq 1$, and let $X$ be the mapping cone of $f$, i.e. the 2-cell complex $S^{2m} \cup_f e^{2m+2l}$. Denote by $i$ the inclusion of $S^{2m}$ in $X$, and let $p : X \to X/S^{2m} \simeq S^{2m+2l}$ be the collapsing map; they fit in the cofibre sequence $S^{2m} \to X \to S^{2m+2l}$. For a sphere $S^d$, we designate the Bott generator of $K(S^d)$ by $x_d$. Taking $\xi \in (\epsilon^*)^{-1}(x_{2m})$ and $\eta := p^*(x_{2m+2l})$, we get

\[ K^0(X) \cong \mathbb{Z} \oplus \mathbb{Z} \cdot \xi \oplus \mathbb{Z} \cdot \eta \cong \mathbb{Z}^3. \]

Notice that $\xi$ is uniquely determined up to addition of an integral multiple of $\eta$. Similarly, the integral cohomology of $X$ is given by

\[ H^*(X; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \cdot y \oplus \mathbb{Z} \cdot z \cong \mathbb{Z}^3, \]

with $y$ corresponding via $i^*$ to a generator of $H^{2m}(S^{2m}; \mathbb{Z})$, and $z$ corresponding via $p^*$ to a generator of $H^{2m+2l}(S^{2m+2l}; \mathbb{Z})$; we use the same notation for the rational cohomology of $X$. The ring structure is given by $xy = 0$, $z^2 = 0$ and $y^2 = H(f) \cdot z$, where $H(f)$ denotes the Hopf invariant of $[f] \in \pi_{4m-1}(S^{2m})$ when $m = l$, and $H(f) := 0$ when $m \neq l$. The Chern character is given by $ch(\xi) = y + \lambda \cdot z$ and $ch(\eta) = z$, for some rational number $\lambda$. Because of the different possible choices for $\xi$, the rational number $\lambda$ is only determined modulo 1, i.e. it represents a unique element $e(f)$ in the group $\mathbb{Q}/\mathbb{Z}$, called the Adams e-invariant of $f$ (also denoted by $e_C(f)$). It only depends on the homotopy class of $f$. Without loss of generality, we can consider $e(f)$ as a uniquely determined element of $\mathbb{Q}/\{1/2, 1/2\}$. (See [1], pp. 321–323 for some more details on the e-invariant.) Since $ch$ is an injective ring homomorphism ($X$ being torsion-free), the product in $K(X)$ is given by $\xi^2 = H(f) \cdot \xi$, $\xi \eta = 0$ and $\eta^2 = 0$. We would like to compute the Chern classes of $\xi$ and $\eta$. They are closely related to the Chern character, as we now recall. For a connected finite CW-complex $Y$, we denote by $ch_{2k}$ the component of $ch$ in $H^{2k}(Y; \mathbb{Q})$. One has $ch_{2k} = (1/k!)s_k(c_1, \ldots, c_k)$ (for $k \geq 1$), where the $s_k$'s are the Newton polynomials. They are defined by the relation $s_k(\sigma_1, \ldots, \sigma_k) = t_1^k + \cdots + t_k^k$, with $\sigma_j$ the $j$-th elementary symmetric polynomial in $t_1, \ldots, t_k$ (see for example [5], p. 92). Newton’s formula reads

\[ s_k = c_1 s_{k-1} + c_2 s_{k-2} - \cdots + (-1)^{k-1} c_{k-1} s_1 + (-1)^k k \cdot c_k = 0 \]

(see loc. cit.). Coming back to $X$, it is straightforward to check that

\[ c_m(\xi) = (-1)^{m-1}(m-1)! \cdot y \quad \text{and} \quad c(\eta) = 1 + (-1)^{m+l-1}(m+l-1)! \cdot z. \]
Clearly, for \( j \not\in \{m, m+1\} \) and \( 1 \leq k \leq m-1 \), one has the equalities \( s_m(\xi) = m! \cdot y \) and \( c_j(\xi) = s_0(\xi) = 0 \). In Newton’s formula for \( s_{m+1} \), the only possible nonzero contributions are \((-1)^{m+1}(m+1)c_{m+1}\) and, if \( m = l \), the product \((-1)^n c_n s_m\). After a short computation, we get

\[
 c(\xi) = 1 + (-1)^{m-1}(m-1)! \cdot y + \left( \frac{(m-1)!^2}{2} \cdot H(f) + (-1)^{m+l-1}(m+l-1)! \cdot e(f) \right) \cdot z. 
\]

Now, for \( a, b \in \mathbb{Z} \), we find

\[
 c(a\xi + b\eta) = c(\xi)^a \cdot c(\eta)^b = 1 + (-1)^{m-1}(m-1)! \cdot a \cdot y + \left( \frac{(m-1)!^2}{2} \cdot a^2 \cdot H(f) + (-1)^{m+l-1}(m+l-1)! \cdot (a \cdot e(f) + b) \right) \cdot z. 
\]

Recall that for a connected finite CW-complex \( Y \), the geometric dimension of a stable bundle \( \vartheta \in \tilde{K}(Y) = [Y, BU] \) is the smallest integer \( n \geq 0 \) such that \( \vartheta \) lifts, up to homotopy, to a map \( Y \rightarrow BU(n) \), in other words, such that \( n + y \in \mathbb{Z} \oplus \tilde{K}(Y) \) can be represented by a complex \( n \)-bundle over \( Y \); we denote it by \( n = g \cdot \dim(\vartheta) \). We also define \( c \cdot \dim(\vartheta) \) as the smallest positive integer \( i \) such that \( c_j(\vartheta) = 0 \) in \( H^2(Y; \mathbb{Z}) \) for all \( j > i \). Clearly, \( c \cdot \dim(\vartheta) \leq g \cdot \dim(\vartheta) \). (The reader may refer to [7] for details on the functions \( g \cdot \dim \) and \( c \cdot \dim \).)

Now, suppose that \( l < m \) (as a consequence of which \( H(f) = 0 \) holds). Fix an integer \( a \) and let \( a\xi + b\eta \in \tilde{K}(X) \), where \( b \) is considered as an unknown integral parameter; let \( s \) satisfy \( m \geq s \leq m + l - 1 \). Denote by \( i_s \) the inclusion of \( U(s) \) in \( U \), and consider the following diagram representing a lifting and extension problem:

\[
\begin{array}{c}
\text{S}^{2m+2l-1} \quad \xrightarrow{f} \\
\text{S}^{2m} \quad \xrightarrow{\alpha \xi} \quad \xrightarrow{\alpha x_m} \quad BU(s) \\
X \quad \xrightarrow{a\xi + b\eta} \quad \xrightarrow{\exists \gamma_a} \quad BU \\
\end{array}
\]

Clearly, there exists, up to homotopy, an extension of \( \alpha x_{2m} \) to \( X \) if and only if the composition \( (\alpha x_{2m}) \circ f \) is zero in \( \pi_{2m+2l-1}(BU(s)) \). In this case, the composition \( Bi_s \circ \alpha \in \tilde{K}(X) \) is a stable vector bundle \( \zeta \) over \( X \) such that \( c^*(\zeta) = \alpha x_{2m} \) and with \( g \cdot \dim(\zeta) \leq s \). It follows that there exists an integer \( b \) (our parameter!) such that \( \zeta = a\xi + b\eta \) and \( c \cdot \dim(\zeta) \leq s \leq m + l - 1 \), and therefore \( c_{m+l}(\zeta) = 0 \). We have thus proved that

\[
(\alpha x_{2m}) \circ f = 0 \in \pi_{2m+2l-1}(BU(s)) \implies \exists b \in \mathbb{Z} \text{ s.t. } c_{m+l}(a\xi + b\eta) = 0.
\]
We call this condition (♣). Since $H(f) = 0$, the above computation of the Chern classes for $X$ shows that

$$c_{m+l}(a\xi + b\eta) = 0 \iff a \cdot e(f) + b = 0.$$  

This means that the denominator of $e(f) \in \mathbb{Q}[-1/2, 1/2]$, expressed in lowest terms, must divide $a$. By Theorem 41.5 in Steenrod [11], we have

$$\text{g-dim}(a\xi + b\eta) < m + l \iff c_{m+l}(a\xi + b\eta) = 0.$$  

So, for $s = m + l - 1$, condition (♣) is an equivalence. Now, the following lemma provides the necessary control, with respect to $a$, of the element $(ax_{2m}) \circ f$.

**Lemma 2.1.** For $l < m$ and for $a \in \mathbb{Z}$, we have

$$(ax_{2m}) \circ f = a \cdot (x_{2m} \circ f) \in \pi_{2m+2l-1}(BU(s)).$$  

Proof. For $l < m$, the Freudenthal Suspension Theorem (see [4], Theorem VI.2.10) implies that $f$ is a suspension and the lemma follows directly from Theorem VI.2.3 in [4].

The group $\pi_{2m+2l-1}(BU(s))$ is finite for $1 \leq s \leq m + l - 1$, as is well-known (see for example Lemma 4.2 in [7] for a proof). We now collect the results obtained so far in a proposition.

**Proposition 2.2.** For $1 \leq l \leq m - 1$, let $f: S^{2m+2l-1} \to S^2m$ be a pointed map; let $x_{2m}$ be the Bott generator of $K(S^{2m}) \cong [S^{2m}, BU(s)]$, $m \leq s \leq m + l - 1$. Then, the composition $x_{2m} \circ f$ represents a class in $\pi_{2m+2l-1}(BU(s))$, whose order is a multiple of $\text{denom}(e(f))$, the denominator of the Adams $e$-invariant $e(f)$ expressed in lowest terms. For $s = m + l - 1$, the order of $x_{2m} \circ f$ is precisely $\text{denom}(e(f))$.

3. The proof of Theorem 1.1

We apply Proposition 2.2 with $f = j_{k-1}: S^{2m+4k-1} \to S^{2m}$ and with $l = 2k$. By Adams [1] and Quillen [10], the image of $J$ is a direct summand in $\pi_{2m+4k-1}(S^{2m})$ and is of order exactly $M_k := \text{denom}(B_k/4k)$ (see also Switzer [12], p. 488). This means that $j_{k-1}$ is of order $M_k$ and generates a direct summand. On the other hand, by Theorem 1 of Dyer [3], the Adams $e$-invariant $e(j_{k-1})$ (expressed in lowest terms) has denominator $M_k/b_k$, where $b_k$ is equal to 1 (resp. 2) for $k$ even (resp. odd). (This result is also a consequence of Adams [1], Proposition 7.14 and Theorem 7.16.) The proof is complete. 

□
References


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