AN $L^p$-APPROACH TO SINGULAR LINEAR PARABOLIC EQUATIONS IN BOUNDED DOMAINS

ANGELO FAVINI, ALFREDO LORENZI, HIROKI TANABE and ATSUSHI YAGI

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Abstract

Singular means here that the parabolic equation is not in normal form neither can it be reduced to such a form. For this class of problems, following the operator approach used in [1], we prove global in time existence and uniqueness theorems related to (spatial) $L^p$-spaces. Various improvements to [2], [3] are given.

1. Introduction

In this paper we will consider the following boundary value problem

\begin{align}
D_t [m(x)u(x,t)] + A(x, D_x) u(x, t) &= f(x, t), \quad \forall (x, t) \in \Omega \times [0, \tau), \\
u(x, t) &= 0, \quad \forall (x, t) \in \partial \Omega \times [0, \tau), \\
m(x)u(x,t) &\to m(x)u_0(x), \quad \text{for a.e. } x \in \Omega, \text{ as } t \to 0^+,
\end{align}

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a boundary of class $C^2$, while $A(x, D_x)$ is the following second-order uniformly elliptic operator in divergence form

\begin{equation}
A(x, D) = -\sum_{i,j=1}^n D_{x_j} [a_{ij}(x)D_{x_i}] + a_0(x).
\end{equation}

Moreover, $0 \neq m \in L^\infty(\Omega)$ is a non-negative function which need not to be bounded away from 0. Consequently, our parabolic equation is, in general, singular.

Particular cases of (1.1) are discussed in the monograph [3], pp.74–80. See also [2]. Note that in [3], p.80, the restriction $p \in (2, +\infty)$ should be made.

Using the theoretical results in [3] and the fundamental approach in [4] we can develop an $L^p$-theory, $p \in (1, +\infty)$, also in the present degenerate case. The keystone in order to apply the results in [1] and [3], Theorem 3.28, p.69, to (1.1)–(1.4)

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\item[We note that in this case the initial condition (1.3) should be more correctly meant as the following $L^p$-limit:] $\|m(x)u(\cdot, t) - m(x)u_0(\cdot)\|_{L^p(\Omega)} \to 0$ as $t \to 0^+$.
\end{itemize}
consists in showing an operator estimate of the form

\[ ||L(\lambda M + L)^{-1}||_{L^2(X)} \leq C(1 + |\lambda|)^{1-\beta}, \quad \forall \lambda \in \Sigma_\alpha, \]

where \( X = L^p(\Omega), 0 < \beta \leq \alpha \leq 1, \alpha + \beta > 1, \)

\[ \Sigma_\alpha = \{ \lambda \in \mathbb{C} : \Re \lambda \geq -c(1 + |\lambda|)^{\rho} \}, \quad (c > 0), \]

and

\[ \mathcal{D}(L) = \mathcal{D}(L_p) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \quad Lu(x) = A(x, D)u(x), \quad u \in \mathcal{D}(L), \]

\[ \mathcal{D}(M) = L^p(\Omega), \quad Mu(x) = m(x)u(x). \]

We in fact show that (1.5) holds with \( \alpha = 1, \beta = 1/p, \rho \in (1, +\infty). \)

Moreover, when \( m \) is \( \rho \)-regular, i.e.

\[ m \in C^1(\overline{\Omega}), \quad |\nabla m(x)| \leq C_1 m(x)^\rho, \quad \forall x \in \overline{\Omega}, \quad \text{for some } \rho \in (0, 1], \]

\( C_1 \) being a positive constant, we can improve the index \( \beta \) in estimate (1.5) from \( \beta = 1/p \) to

\[ \beta = \begin{cases} \frac{(2 - \rho)}{2}, & \text{if } \rho \in (1, 2), \\frac{(2 - \rho)}{2}, & \text{if } \rho \in [2, +\infty), \end{cases} \]

The result proved in this paper will be applied, in a subsequent paper, to identify the unknown kernel \( k \) in the integro-differential singular equation of parabolic type

\[ D_t[m(x)u(x, t)] + A(x, D_x)u(x, t) = \int_0^t k(t-s)B(x, D_x)u(x, s) \, ds + f(x, t), \quad \forall (x, t) \in \Omega \times [0, \tau], \]

\( B(x, D_x) \) being a linear second-order differential operator.

We stress that the present paper was originated by a requirement of additional smoothness of solution \( u \) of (1.11) needed to recover the unknown kernel \( k \). This occurrence is in accordance with the well-known fact that inverse problems usually force deeper, and sometimes, unexpected insights in direct problems.

2. Solving the spectral problem \( (\lambda M + L)u = f \)

The basic aim of this section consists in showing that estimate (1.5) holds when the linear operators \( M \) and \( L \) are defined by (1.7) and (1.8), respectively. To this aim we assume that the coefficients \( a_{i,j} \) and \( a_0 \) satisfy the properties

\[ a_{i,j} \in C^1(\overline{\Omega}), \quad a_0 \in C(\overline{\Omega}), \quad a_{i,j} = a_{j,i}, \quad i, j = 1, \ldots, n, \]
(2.2) \( c_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \leq c_1 |\xi|^2, \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n, \quad a_0(x) \geq \gamma, \quad \forall x \in \overline{\Omega}, \)

\( c_0, c_1 \) and \( \gamma \) being three positive constants.

A remarkable result by Okazawa [4, p.702] provides, for any \( u \in \mathcal{D}(L) \),

\[
\text{Re}((L - a_0)u, u|u|^{p-2}) \geq \begin{cases} 
    c_0 \int_{\Omega} |u|^{p-2} |\nabla u|^2 \, dx & \text{if } p \in [2, \infty), \\
    c_0(p - 1) \int_{\Omega} (|u|^2 + \delta)^{(p-2)/2} |\nabla u|^2 \, dx & \text{if } p \in (1, 2),
\end{cases}
\]

(2.3)

\[
|\text{Im}(Lu, u|u|^{p-2})| \leq \frac{|p-2|}{2\sqrt{p-1}} \text{Re}((L - a_0)u, u|u|^{p-2}),
\]

(2.4)

where the brackets denote

\[
(f, g) = \int_{\Omega} f(x) \overline{g(x)} \, dx, \quad f \in L^p(\Omega), \quad g \in L^{p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1,
\]

\( u|u|^{p-2} \) is assumed to vanish whenever \( u \) does, and \( \delta > 0 \) is arbitrary.

\textbf{Remark 2.1.} It is important to observe that bound (2.4) holds even in the degenerate elliptic case (cf. [4, p. 702] and the following Lemma 3.3).

From (2.4) we immediately deduce the estimate

\[
|\text{Im}(Lu, u|u|^{p-2})| + \frac{|p-2|}{2\sqrt{p-1}} \int_{\Omega} a_0(x)|u(x)|^p \, dx \\
\leq \frac{|p-2|}{2\sqrt{p-1}} \text{Re}(Lu, u|u|^{p-2}).
\]

(2.5)

Consider now the spectral problem

(2.6) \( u \in \mathcal{D}(L), \quad \lambda nu + Lu = f \in L^p(\Omega). \)

Taking the real and imaginary parts of the scalar product of both sides in (2.6) with \( u|u|^{p-2} \), we get

\[
\text{Re} \lambda \int_{\Omega} |u|^p \, dx + \text{Re}(Lu, u|u|^{p-2}) = \text{Re} \int_{\Omega} f \overline{u}|u|^{p-2} \, dx,
\]

(2.7)

\[
\text{Im} \lambda \int_{\Omega} |u|^p \, dx + \text{Im}(Lu, u|u|^{p-2}) = \text{Im} \int_{\Omega} f \overline{u}|u|^{p-2} \, dx.
\]

(2.8)
From (2.8) we deduce the inequalities

\[ |\text{Im} \lambda| \int_{\Omega} m|u|^p \, dx \leq |\text{Im}(Lu, u|u|^{p-2})| + |\text{Im} \int_{\Omega} f|u|^{p-2} \, dx|. \]

Multiply then both sides in (2.9) by a positive constant \( k \) and add the obtained inequality to equation (2.7). From (2.5) we get

\[ (\text{Re} \lambda + k|\text{Im} \lambda|) \int_{\Omega} m|u|^p \, dx + \left( 1 - \frac{k}{2} \frac{p-2}{2\sqrt{p-1}} \right) \text{Re}(Lu, u|u|^{p-2}) \]

\[ \leq \text{Re} \int_{\Omega} f|u|^{p-2} \, dx + k|\text{Im} \int_{\Omega} f|u|^{p-2} \, dx| \leq (1 + k)\|f\|_p \|u\|_p^{p-1}. \]

Choose now \( k = k_1(p) \) so small as to satisfy

\[ h_1(p) = 1 - k_1(p) \frac{p-2}{2\sqrt{p-1}} > 0, \quad \forall p \in (1, +\infty). \]

Observe that

\[ \text{Re}(Lu, u|u|^{p-2}) = \text{Re}((L - a_0)u, u|u|^{p-2}) \]

\[ + \frac{1}{2} \text{Re}(a_0u, u|u|^{p-2}) + \frac{1}{2} \text{Re}(a_0u, u|u|^{p-2}) \]

\[ \geq \text{Re}((L - a_0)u, u|u|^{p-2}) + \frac{\gamma}{2} \|u\|_p^p + \frac{\gamma}{2\|m\|_{\infty}} \int_{\Omega} m|u|^p \, dx, \]

since \( m(x) \leq \|m\|_{\infty} \) implies

\[ \frac{m(x)}{\|m\|_{\infty}} \frac{a_0(x)}{2} \leq \frac{a_0(x)}{2}. \]

In view of (2.11), (2.12) and (2.3), we obtain from (2.10) that

\[ \left( \text{Re} \lambda + k_1(p)|\text{Im} \lambda| + \frac{\gamma h_1(p)}{2\|m\|_{\infty}} \right) \int_{\Omega} m|u|^p \, dx \]

\[ + \frac{\gamma h_1(p)}{2} \|u\|_p^p + h_1(p) \text{Re}((L - a_0)u, u|u|^{p-2}) \leq [k_1(p) + 1]\|f\|_p \|u\|_p^{p-1}. \]

Introduce now the sector

\[ \Sigma_1 = \left\{ \mu \in \mathbb{C} : \text{Re} \mu + \frac{k_1(p)}{2} |\text{Im} \mu| + \frac{\gamma h_1(p)}{4\|m\|_{\infty}} \geq 0 \right\}. \]

Then, for \( \lambda \in \Sigma_1 \),

\[ \text{Re}((L - a_0)u, u|u|^{p-2}) \leq \frac{k_1(p) + 1}{h_1(p)} \|f\|_p \|u\|_p^{p-1}, \]
(2.15) \[ \|u\|_p \leq \frac{2(k_1(p) + 1)}{\gamma h_1(p)} \|f\|_{p^*} \]

Consequently,

\[
\left( \text{Re} \lambda + k_1(p) |\text{Im} \lambda| + \frac{2\gamma h_1(p)}{2\|m\|_{\infty}} \right) \int_\Omega m |u|^p \, dx \leq C_1(p) \|f\|_{p^*} \|u\|_p^p,
\]

We now need a simple proposition. For the proof see Section 6.

**Proposition 2.1.** Let \( k > 0 \) and \( \varepsilon > 0 \) be two positive constants, and let \( \Sigma_{k,\varepsilon} \) be a sectorial domain given by

\[
\Sigma_{k,\varepsilon} = \left\{ \mu \in \mathbb{C} : \text{Re} \mu + \frac{k}{2} |\text{Im} \mu| + \frac{\varepsilon}{2} \geq 0 \right\}.
\]

Then it holds that

\[
|\lambda| + 1 \leq \left( \frac{2}{k} + \frac{2}{\varepsilon} + 1 \right) (\text{Re} \lambda + k |\text{Im} \lambda| + \varepsilon), \quad \lambda \in \Sigma_{k,\varepsilon},
\]

Since \( \Sigma_1 = \Sigma_{k_1(p), h_1(p) \|m\|_{\infty}} \), this proposition then yields

(2.16) \[ (|\lambda| + 1) \int_\Omega m |u|^p \, dx \leq C_2(p) \|f\|_p \|u\|_{p^*}^p \leq C_3(p) \|f\|_{p^*} \|u\|_p^p, \quad \lambda \in \Sigma_1. \]

To show that \((\lambda M + L)^{-1}\) is a bounded operator on \( L^p(\Omega) \) for \( \lambda \in \Sigma_1 \), it now suffices to verify that \( \mathcal{R}(\lambda M + L) = L^p(\Omega) \). But this is verified by the usual techniques without difficulty. In fact, for each \( \lambda \in \Sigma_1 \), we already know that \( \mathcal{R}(\Lambda + \lambda M + L) = L^p(\Omega) \) provided \( \Lambda > 0 \) is a sufficiently large number. Let \( 0 \leq \theta \leq 1 \) be a parameter, and consider the family of closed linear operators \( A(\theta) = \theta \Lambda + \lambda M + L, \quad 0 \leq \theta \leq 1 \). Then the desired result is obtained by the following proposition the proof of which will be given in the final section.

**Proposition 2.2.** Let \( A(\theta), \quad 0 \leq \theta \leq 1, \) be a family of closed linear operators acting on a Banach space \( X \) with constant domain \( \mathcal{D}(A(\theta)) \equiv \mathcal{D} \). Assume that the family satisfies the conditions

(2.17) \[ \delta \|u\| \leq \|A(\theta)u\|, \quad u \in \mathcal{D}, \]

(2.18) \[ \|\{A(\theta) - A(\theta')\}u\| \leq N |\theta - \theta'| \|u\|, \quad u \in \mathcal{D} \]

with some constants \( \delta > 0 \) and \( N > 0 \) independent of \( \theta, \theta' \in [0,1] \). Then, \( \mathcal{R}(A(1)) = X \) implies \( \mathcal{R}(A(\theta)) = X \) for every \( \theta \in [0,1] \).

We can now summarize the results proved in this section in Theorem 2.1.
Theorem 2.1. Let $L$ and $M$ be the linear operators defined by (1.7) and (1.8), the coefficients $a_{i,j}$, $i, j = 1, \ldots, n$, $a_0$ enjoying properties (2.1) and (2.2) and $m$ being a non-negative function in $L^\infty(\Omega)$. Then the spectral equation $\lambda M u + L u = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma_1 = \{\mu \in C : \Re \mu + (k_1(p)/2)|\Im \mu|^2 + \gamma h_1(p)/(4|m|_\infty) \geq 0\}$ and $p \in (1, +\infty)$, a unique solution $u \in W^2_p(\Omega) \cap W^{1,p}_0(\Omega)$ satisfying the estimates

$$
\|u\|_p \leq C_4(p)\|f\|_p, \quad \|Mu\|_p \leq C_5(p)|\lambda|^{-1/p}\|f\|_p, \quad \lambda \in \Sigma_1,
$$

$$
\|Lu\|_p \leq C_6(p)(1 + |\lambda|^{1/p^2})\|f\|_p, \quad \lambda \in \Sigma_1.
$$

3. The case when $m$ is $\rho$-regular and $p \in [2, +\infty)$

We will show that when the multiplier $m$ is more regular, i.e. it satisfies (1.9), our $\beta$ can be chosen larger than $1/p$. We recall that all the previous estimates (2.6)–(2.16) hold for any $p \in (1, +\infty)$.

First of all we need the following lemma concerning the computation of the gradient of the function $|\overline{u}|^{p-2} u$ when $p \in [2, +\infty)$. For this purpose we need some lemmata.

Lemma 3.1. Let $u \in W^{1,p}_0(\Omega)$ with $p \in [2, +\infty)$. Then the function $|\overline{u}|^{p-2} u$ belongs to $W^{1,p}_0(\Omega)$ and the following formulae hold

$$
D_{\bar{x}_j}|\overline{u}|^{p-2} u = \overline{u}^{p-2} D_{\bar{x}_j} \overline{u} + (p - 2) g_p(u) \Re(g_p(u) D_{\bar{x}_j} u),
$$

(a.e. in $\Omega$, $j = 1, \ldots, n$),

where

$$
g_p(u)(x) = \begin{cases} 
|\overline{u}(x)||u(x)|^{p-1}/2, & \text{if } u(x) \neq 0, \\
0, & \text{if } u(x) = 0.
\end{cases}
$$

Proof. Let $\phi$ be any function in $C_0^\infty(\Omega)$. Then the following equalities hold:

$$
\langle D_{\bar{x}_j} \phi, |\overline{u}|^{p-2} u \rangle = \lim_{\varepsilon \to 0^+} \langle D_{\bar{x}_j} \phi, |u|^2 + \varepsilon \rangle^{(p-2)/2}.
$$

We have used here the relation $\lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2} D_{\bar{x}_j} \overline{u} + \frac{p - 2}{2} |\overline{u}|^2 + \varepsilon \rangle^{(p-4)/2} \langle \overline{u} D_{\bar{x}_j} u + u D_{\bar{x}_j} \overline{u} \rangle = \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2} D_{\bar{x}_j} \overline{u} + (p - 2) |\overline{u}|^2 + \varepsilon \rangle^{(p-4)/2} \Re(\overline{u} D_{\bar{x}_j} u) \rangle = \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2} D_{\bar{x}_j} \overline{u} + (p - 2) |\overline{u}|^2 + \varepsilon \rangle^{(p-4)/2} \Re(\overline{u} |u|^2 + \varepsilon) \rangle^{(p-4)/4} D_{\bar{x}_j} u \rangle
$$

(3.3)

We have used here the relation $\lim_{\varepsilon \to 0^+} \langle \phi, (|u|^2 + \varepsilon)^{(p-2)/2} D_{\bar{x}_j} \overline{u} + (p - 2) g_p(u) \Re(g_p(u) D_{\bar{x}_j} u) \rangle$.
Remark 3.1. From definition (3.2) we easily deduce the identity
\[(3.4) \quad |g_p(u)(x)| = |u(x)|^{(p-2)/2}.\]

We can now prove the following Lemma 3.2.

**Lemma 3.2.** Let \((b_{i,j})_{i,j=1}^{n}\) be a matrix of functions in \(C^1(\overline{\Omega}, \mathbb{R})\) such that
\[(3.5) \quad b_{i,j} = b_{j,i} \quad i, j = 1, \ldots, n,\]
\[(3.6) \quad c_0 |\xi|^2 \mu(x) \leq \sum_{i,j=1}^{n} b_{i,j}(x) \xi_i \xi_j \leq c_1 |\xi|^2 \mu(x), \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n,\]
where \(\mu \in C(\overline{\Omega})\) is a non-negative function and \(c_0, c_1\) are two positive constants. Then for any \(p \in (2, +\infty)\), the linear operator \(K = -\sum_{i,j=1}^{n} D_{\xi_i}[b_{i,j}(x)D_{\xi_j}]\) with \(\mathcal{D}(K) = \mathcal{D}(L)\) (cf. (1.7)) satisfies the relations
\[(3.7) \quad \text{Re}(Ku, \overline{u}) |u|^{p-2} \leq c_1 \left( \int_{\Omega} \mu |u|^{p-2} |Du|^2 \, dx + \int_{\Omega} \mu \sum_{j=1}^{n} \text{Re}(g_p(u)D_{\xi_j}u)^2 \, dx \right),\]
\[(3.8) \quad \text{Im}(Ku, \overline{u}) |u|^{p-2} = (p-2) \int_{\Omega} \sum_{i,j=1}^{n} b_{i,j}[\text{Re}(g_p(u)D_{\xi_i}u)][\text{Im}(g_p(u)D_{\xi_j}u)] \, dx.\]

Proof. From Lemma 3.1 and an integration by parts we easily deduce the identity
\[(3.9) \quad (Ku, \overline{u}) |u|^{p-2} = \int_{\Omega} \sum_{i,j=1}^{n} b_{i,j} D_{\xi_j}u D_{\xi_i} \overline{u} |u|^{p-2} \, dx\]
\[= \int_{\Omega} \sum_{i,j=1}^{n} |u|^{p-2} b_{i,j} D_{\xi_j}u D_{\xi_i} \overline{u} \, dx\]
\[+ (p-2) \int_{\Omega} \sum_{i,j=1}^{n} b_{i,j} g_p(u) D_{\xi_j}u \text{Re}(g_p(u)D_{\xi_i}u) \, dx.\]
Relations (3.7) and (3.8) follow immediately from (3.9) taking the real and the imaginary parts. \(\square\)

**Lemma 3.3.** Under the assumptions in the statement of Lemma 3.2 operator \(K\) satisfies inequalities (2.3) and (2.4) with \(K\) in the place of \(L - a\).
Proof. This lemma has essentially been proved in [4], although a slight modification is needed in its proof. For any $\varepsilon > 0$ define $a_{i,j} = b_{i,j} + \varepsilon \delta_{i,j}$, $i, j = 1, \ldots, n$, and set $K_\varepsilon = K - \varepsilon \Delta$. Since the matrix $(a_{i,j})_{i,j=1,\ldots,n}$ is uniformly positive definite, from (2.3) and (2.4), with $u \in \mathcal{D}(L_0)$, we obtain the inequalities

\begin{align}
0 \leq \text{Re}(K_\varepsilon u, u|u|^{p-2}) &= \text{Re}(Ku, u|u|^{p-2}) + \varepsilon \text{Re}(-\Delta u, u|u|^{p-2}), \\
|\text{Im}(K_\varepsilon u, u|u|^{p-2})| &= |\text{Im}(Ku, u|u|^{p-2}) + \varepsilon \text{Im}(-\Delta u, u|u|^{p-2})| \leq \frac{|p - 2|}{2\sqrt{p - 1}} |\text{Re}(Ku, u|u|^{p-2}) + \varepsilon \text{Re}(-\Delta u, u|u|^{p-2})|.
\end{align}

Taking the limit as $\varepsilon \to 0^+$ in (3.10) and (3.11), we easily deduce that $K$ satisfies (2.3) and (2.4).

We shall use also the following identity

\begin{equation}
(Lu, m^{p-1}u|u|^{p-2}) = (m^{p-1}Lu, u|u|^{p-2})
\end{equation}

where

\begin{equation}
K_0 = -\sum_{i,j=1}^n D_{x_i} [m(x)^{p-1}a_{i,j}(x)D_{x_j} + m(x)^{p-1}a_0(x)].
\end{equation}

Let now $u$ be a solution to equation (2.6). Taking the scalar product of both sides in (2.6) with $m^{p-1}|u|^{p-2}$ and using (3.12), we easily get the equalities

\begin{equation}
(f, m^{p-1}u|u|^{p-2}) = (\lambda m u + Lu, m^{p-1}u|u|^{p-2})
\end{equation}

where

\begin{equation}
\lambda = ||Mu||_p^p + (K_0 u, u|u|^{p-2}) + (p - 1) \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right).
\end{equation}

Taking the real and imaginary parts in (3.13) and using (2.4) with $L - a_0$ replaced by $K = K_0 - m^{p-1}a_0$, we easily deduce the inequalities

\begin{align}
\text{Re} \lambda ||Mu||_p^p + \gamma \int_{\Omega} m^{p-1}|u|^p \, dx + \text{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) &
\leq \left| (f, m^{p-1}u|u|^{p-2}) + (p - 1) \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right) \right|,
\end{align}

\begin{align}
|\text{Im} \lambda ||Mu||_p^p | &\leq |\text{Im}((K_0 - m^{p-1}a_0)u, u|u|^{p-2})| \\
&+ \left| (f, m^{p-1}u|u|^{p-2}) + (p - 1) \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \right) \right|.
\end{align}
\[ \frac{|p - 2|}{2\sqrt{p - 1}} \text{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) + |(f, m^{p-1}u|u|^{p-2})| \]

\[ + (p - 1) \left| \sum_{i,j=1}^n a_{ij} D_{xi} m D_{xj} u, u|u|^{p-2} \right|. \]

Multiply now by \( k_1(p) \) (cf. (2.11)) the first and last sides in (3.15) and add to the first and last sides in (3.14). We get the estimate

\[ \text{Re} \lambda + k_1(p) \text{Im} \lambda + \gamma \|m\|_\infty^{-1} \|Mu\|_p^p \]

\[ + \left( 1 - k_1(p) \right) \frac{|p - 2|}{2\sqrt{p - 1}} \text{Re}((K_0 - m^{p-1}a_0)u, u|u|^{p-2}) \]

\[ \leq [1 + k_1(p)] \left\{ |(f, m^{p-1}u|u|^{p-2})| + (p - 1) \left| \sum_{i,j=1}^n a_{ij} D_{xi} m D_{xj} u, u|u|^{p-2} \right| \right\}, \]

where we have made use of the elementary inequality

\[ m(x)^p \leq \|m\|_\infty m(x)^{p-1}, \quad x \in \overline{\Omega}. \]

We now estimate the last term in (3.16) with the aid of (1.9). Using twice Hölder’s inequality, we get

\[ \left| \sum_{i,j=1}^n a_{ij} D_{xi} m D_{xj} u \right| \leq \int_\Omega m^{p-2} |u|^{p-1} \left| \sum_{i,j=1}^n a_{ij} D_{xi} m D_{xj} u \right| \ dx \]

\[ \leq \int_\Omega m^{p-2} |u|^{p-1} \left| \sum_{i,j=1}^n a_{ij} D_{xi} m D_{xj} u \right|^{1/2} \left| \sum_{i,j=1}^n a_{ij} D_{xi} u D_{xj} u \right|^{1/2} \ dx \]

\[ \leq C_7 \int_\Omega m^{p-2+\rho} |u|^{p-1} \|\nabla u\| \ dx = C_7 \int_\Omega m^{p\rho/2} |u|^{p/2} m^{(p-2)(2-\rho)/2} |u|^{-1+p/2} \|\nabla u\| \ dx \]

\[ \leq C_7 \left( \int_\Omega m^{p\rho/2} |u|^{p(1-\rho)} \ dx \right)^{1/2} \left( \int_\Omega m^{(p-2)(2-\rho)} |u|^{p-2} \|\nabla u\|^2 \ dx \right)^{1/2} \]

\[ \leq C_7 \|Mu\|_p^{\rho/2} \|u\|_p^{(1-\rho)p/2} \|m\|_\infty^{(p-2)(2-\rho)/2} \left( \int_\Omega |u|^{p-2} \|\nabla u\|^2 \ dx \right)^{1/2}. \]

(3.17)

On account of (2.3), (2.14) and (2.15), we easily observe the estimate

\[ \int_\Omega |u|^{p-2} \|\nabla u\|^2 \ dx \leq C_8(p) \|f\|_p^p. \]

(3.18)
From (2.15), (3.17) and (3.18) we finally deduce the estimates

\[
(3.19) \quad \left( m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u, u |u|^{p-2} \right) \leq C_0(p) \|f\|_p^{\rho(2-\rho)/2} \|Mu\|_p^\rho/2.
\]

Moreover, we have

\[
(3.20) \quad \| (f, m^{p-1} u |u|^{p-2}) \| \leq \|f\|_p \|Mu\|_p^{p-1}.
\]

Finally, from (3.16), (3.19), (3.20) and Lemma 3.2 with \( K = K_0 - m^{p-1} a_0 \) (which makes use of the assumption \( p \in [2, +\infty) \)) we deduce the inequality

\[
\begin{align*}
[&\Re \lambda + k_1(p) \Im \lambda] + \gamma \|m\|_\infty \|Mu\|_p^\rho / (p-1) \\
+ &\left( 1 - k_1(p) \frac{|p-2|}{2 \sqrt{p-1}} \right) \Re((K_0 - m^{p-1} a_0) u, u |u|^{p-2}) \leq C_10(p) \|f\|_p \|Mu\|_p^{p-1} + \|f\|_p^{\rho(p-\rho)/2} \|Mu\|_p^{\rho/2}, \quad \lambda \in \Sigma_1.
\end{align*}
\]

We now introduce the sector

\[
\Sigma_2 = \left\{ \lambda \in \mathbb{C} : \Re \lambda + \frac{k_1(p)}{2} \Im \lambda + \frac{\gamma}{2 \|m\|_\infty} \geq 0 \right\}.
\]

Since \( h_1(p) \in (0, 1) \), (cf. (2.11)), we immediately deduce the inclusion \( \Sigma_2 \subset \Sigma_1 \) (see the definition of \( \Sigma_2 \)).

Then, recalling that \( \Re((K_0 - m^{p-1} a_0) u, u |u|^{p-2}) \) is non-negative (cf. Lemma 3.2) and applying Proposition 2.1, we obtain

\[
(\lambda | + 1) \|Mu\|_p^\rho \leq \gamma \|f\|_p \|Mu\|_p^{p-1} + \|f\|_p^{\rho(p-\rho)/2} \|Mu\|_p^{\rho/2}, \quad \lambda \in \Sigma_2.
\]

Consequently, since \( \|u\|_p \leq C_{12}(p) \|f\|_p \) (cf. (2.15)), (3.15) and (3.22) imply

\[
(\lambda | + 1) \|Mu\|_p^{\rho(2-\rho)/2} \leq C_{13}(p) \|f\|_p \|Mu\|_p^{p-1-p/2} + \|f\|_p^{\rho(p-\rho)/2}, \quad \lambda \in \Sigma_2.
\]

By Proposition 2.2, it is verified that \( \lambda M + L \) is surjective on \( L^p(\Omega) \). Hence, estimate (1.5) holds with \( \alpha = 1 \) and \( \beta = 2(\rho-\rho)/\rho \).

We can summarize the results in this section in Theorem 3.1.

**Theorem 3.1.** Let \( L \) and \( M \) be the linear operators defined by (1.7) and (1.8), the coefficients \( a_{i,j} \) \( i, j = 1, \ldots, n \), \( a_0 \) enjoying properties (2.1) and (2.2) and \( m \) being
a non-negative function satisfying (1.9). Then the spectral equation \( \lambda Mu + Lu = f \), with \( f \in L^p(\Omega) \), admits, for any \( \lambda \in \Sigma_2 = \{ \mu \in \mathbb{C} : \text{Re} \mu + (k_1(p)/2)[\text{Im} \mu] + (\gamma/2)[\|u\|_{\infty}] \geq 0 \} \) and \( p \in [2, +\infty) \), a unique solution \( u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \) satisfying the estimates

\[
\|u\|_p \leq C_{14}(p)\|f\|_p, \quad \|Mu\|_p \leq C_{15}(p)\|\lambda\|^{-2/(p^2-\rho)}\|f\|_p, \quad \lambda \in \Sigma_2, \\
\|Lu\|_p \leq C_{16}(p)(1 + \|\lambda\|^{(p^2-\rho)/2(p^2-\rho)})\|f\|_p, \quad \lambda \in \Sigma_2.
\]

**Example 3.1.** Let \( \Omega \) be a bounded domain and let \( x_0 \) be a fixed point in \( \partial \Omega \). Define then \( r = \max_{x \in \Omega} |x - x_0| \) and choose

\[
m(x) = [(|x - x_0|[r - |x - x_0| - r_1])^q, \quad q \in (1, +\infty).
\]

An elementary computation shows that

\[
|\nabla m(x)| = q(|x - x_0|[r - |x - x_0|])^{q-1}|2|x - x_0| - r| \leq qrm(x)^{(q-1)/q}, \quad x \in \Omega.
\]

Consequently, function \( m \) satisfies condition (1.9).

We notice that for any open interval \( \Omega \subset \mathbb{R} \) we have \( r = \text{length}(\Omega) \).

**4. The case when \( p \in (1, 2) \)**

In this section we are going to considering the case \( p \in (1, 2) \). From (2.4) we immediately deduce that the estimate

\[
|\text{Im}(Lu, u|u|^{p-2})| + \gamma\|u\|_p^p \leq |\text{Im}(Lu, u|u|^{p-2})| + \int_{\Omega} a_0(x)|u(x)|^p \, dx \\
\leq \frac{|p - 2|}{2\sqrt{p - 1}} \text{Re}(Lu, u|u|^{p-2}), \quad u \in \mathcal{D}(L)
\]

holds true for any \( p \in (1, +\infty) \).

Consider again the spectral problem

\[
u \in \mathcal{D}(L), \quad \lambda Mu + Lu = f \in L^p(\Omega).
\]

Multiplying both sides in (4.2) by \( u|u|^{p-2} \) and integrating over \( \Omega \), we get

\[
\lambda\|m^{1/p}u\|_p^p + (Lu, u|u|^{p-2}) = (f, u|u|^{p-2}).
\]

Taking the real and imaginary parts, from (4.3) we deduce

\[
\text{Re} \lambda\|m^{1/p}u\|_p^p + \text{Re}(Lu, u|u|^{p-2}) + (a_0u, u|u|^{p-2}) = \text{Re}(f, u|u|^{p-2}), \\
\text{Im} \lambda\|m^{1/p}u\|_p^p + \text{Im}(Lu, u|u|^{p-2}) = \text{Im}(f, u|u|^{p-2}),
\]
where we have set
\[(4.6) \quad L_0 = L - a_0.\]

Then from Okazawa [4, p.703] we get
\[(4.7) \quad (L_0 u, u|u|^{p-2}) = \lim_{\delta \to 0^+} I_p(u, \delta),\]

where \(\delta > 0\) and
\[(4.8) \quad I_p(u, \delta) = -\int_\Omega (|u(x)|^2 + \delta)^{(p-2)/2} \frac{u(x)^2}{u(x)} \sum_{j,k=1}^n D_{x^j} [a_{j,k}(x) D_{x^k} u(x)] \, dx.\]

As mentioned at the beginning of Section 2, we have
\[(4.9) \quad \text{Re}(L_0 u, u|u|^{p-2}) \geq c_0 \int_\Omega |u(x)|^{p-2} |\nabla u(x)|^2 \, dx, \quad \text{if} \quad p \in [2, +\infty),\]

\[(4.10) \quad \text{Re}(L_0 u, u|u|^{p-2}) \geq c_0 (p-1) \int_\Omega (|u(x)|^2 + \delta)^{(p-2)/2} |\nabla u(x)|^2 \, dx, \quad \text{if} \quad p \in (1, 2).\]

From (4.1) and (4.5) we deduce the inequalities
\[(4.11) \quad |\text{Im} \lambda| m^{1/p} u_p^p \leq |\text{Im}(Lu, u|u|^{p-2})| + \|f\|_p \|u\|_p^{p-1} \leq \frac{|p-2|}{2\sqrt{p-1}} \text{Re}(Lu, u|u|^{p-2}) + \|f\|_p \|u\|_p^{p-1}.\]

Multiply then both sides in (4.11) by a positive constant \(\nu\) and add the obtained inequality to equation (4.4) to get (cf. (2.2))
\[(\text{Re} \lambda + \nu |\text{Im} \lambda|) m^{1/p} u_p^p + \left(1 - \frac{|p-2|}{2\sqrt{p-1}}\right) \text{Re}(L_0 u, u|u|^{p-2}) + \frac{\nu}{2} \|u\|_p^p + \frac{\nu}{2} \|u\|_p^p \]
\[\leq (\text{Re} \lambda + \nu |\text{Im} \lambda|) m^{1/p} u_p^p + \left(1 - \frac{|p-2|}{2\sqrt{p-1}}\right) \text{Re}(L_0 u, u|u|^{p-2}) + (a_0 u, u|u|^{p-2})
\leq \text{Re}(f, u|u|^{p-2}) + \|f\|_p \|u\|_p^{p-1} \leq (1 + \nu) \|f\|_p \|u\|_p^{p-1}.\]

(4.12)

Choose now \(\nu = \nu(p)\) so small as to satisfy
\[(4.13) \quad \nu_1(p) := 1 - \nu(p) \frac{|p-2|}{2\sqrt{p-1}} > 0, \quad \forall p \in (1, +\infty).\]

On the other hand, since \(m \in L^\infty(\Omega), \|u\|_p \geq \|m\|_\infty^{-1/p}\|m^{1/p} u\|_p.\) Then (4.12) and (4.13) imply
\[
\left(\frac{\nu}{2\|m\|_\infty} + \text{Re} \lambda + \nu |\text{Im} \lambda|\right) \|m^{1/p} u\|_p^p + \nu_1(p) \text{Re}(L_0 u, u|u|^{p-2}) + \frac{\nu}{2} \|u\|_p^p
\]
\begin{equation}
(4.14) \quad \leq [1 + \nu(p)] \|f\|_p \|u\|_p^{\beta - 1}.
\end{equation}

In other words, there exist two positive constants $C_{18}$ and $C_{19}$ such that

\begin{equation}
(4.15) \quad \left( \frac{\nu}{2 \|m\|_{\infty}} + \text{Re} \lambda + \nu |\text{Im} \lambda| \right) \|m^{1/p} u\|_p^p + C_{18} \text{Re}(L_0 u, u|u|^{p-2}) + \frac{\nu}{2} \|u\|_p^p \leq C_{19} \|f\|_p \|u\|_p^{\beta - 1}, \quad \lambda \in \Sigma,
\end{equation}

the sector $\Sigma$ being defined by

$$
\Sigma = \left\{ \lambda \in \mathbb{C} : \text{Re} \lambda + \nu |\text{Im} \lambda| + \frac{\nu}{2 \|m\|_{\infty}} \geq \varepsilon_0 > 0 \right\}.
$$

Notice that (4.7), (4.8), (4.10), (4.15) yield, in particular, the basic bounds

\begin{equation}
(4.16) \quad \|u\|_p \leq \frac{2}{\nu} C_{19} \|f\|_p, \quad \text{Re}(L_0 u, u|u|^{p-2}) \leq C_{20} \|f\|_p^p,
\end{equation}

and

\begin{equation}
(4.17) \quad \left( p - 1 \right) C_0 \lim_{\delta \to 0^+} \int_{\Omega} \left( |(Lu(x)|^2 + \delta)^{(p-2)/2} |\nabla u(x)|^2 \right) dx \leq \lim_{\delta \to 0^+} \text{Re} I_p(u, \delta) \leq C_{20} \|f\|_p \|u\|_p^{\beta - 1}, \quad \lambda \in \Sigma.
\end{equation}

From (4.3) we deduce the estimates

\begin{equation}
(4.18) \quad |\lambda| \|m^{1/p} u\|_p^p \leq \|(Lu, u|u|^{p-2})\| + \|f\|_p \|u\|_p^{\beta - 1}
\end{equation}

\begin{equation}
\leq \left( 1 + \frac{|p - 2|}{2 \sqrt{p - 1}} \right) \text{Re}(L_0 u, u|u|^{p-2}) + \|f\|_p \|u\|_p^{\beta - 1}
\end{equation}

\begin{equation}
\leq C_{21} \|f\|_p \|u\|_p^{\beta - 1} \leq C_{22} \|f\|_p^p.
\end{equation}

Consequently, (4.18) immediately yields

\begin{equation}
(4.19) \quad |\lambda|^{1/p} \|Mu\|_p \leq C_{23} \|f\|_p.
\end{equation}

This, in turn, implies that (1.5) holds with $\alpha = 1$ and $\beta = 1/p$ and provides a different proof to (1.5).

Now we focus our attention to the case when $m \in C^1(\overline{\Omega})$ satisfies inequality (1.9) with

\begin{equation}
(4.20) \quad \rho \in (2 - p, 1).
\end{equation}

Multiplying both sides in (4.2) by $m(x)^{p-1} |u(x)|^{1/p} |u(x)|^{1/p-2}$ and integrating over $\Omega$, we
easily get
\[
\lambda||Mu||_p \leq \lim_{\delta \to 0^+} \int_{\Omega} m(x)^{p-1} |u(x)|^2 + \delta^{(p-2)/2} \sum_{j,k=1}^n D_{ij} [a_{j,k}(x) D_{xj} u(x)] dx + \int_{\Omega} \alpha_0(x)m(x)^{p-2} |u(x)|^p dx = \int_{\Omega} f(x) \alpha_0(x) m(x)^{p-2} |u(x)|^p dx.
\]

(4.21)

An integration by parts in the integral appearing in the limit, which takes into account (4.20) and (4.21), easily yields
\[
- \int_{\Omega} m(x)^{p-1} |u(x)|^2 + \delta^{(p-2)/2} \sum_{j,k=1}^n D_{ij} [a_{j,k}(x) D_{xj} u(x)] dx
\]
\[
= \int_{\Omega} (|u(x)|^2 + \delta^{(p-2)/2} m(x)^{p-1} \sum_{j,k=1}^n a_{j,k}(x) D_{xj} u(x) D_{xj} u(x) dx
\]
\[
+ (p-1) \int_{\Omega} \alpha_0(x)m(x)^{p-2} D_{ij} m(x) a_{j,k}(x) D_{xk} u(x) dx
\]
\[
+ (p-2) \int_{\Omega} m(x)^{p-2} |u(x)|^2 + \delta^{(p-2)/2} \sum_{j,k=1}^n a_{j,k}(x) \text{Re} \left( \overline{u(x)} D_{xj} u(x) \right) D_{xj} u(x) dx
\]
\[
= : I_1(\delta) + (p-1)I_2(\delta) + (2-p)I_3(\delta).
\]

(4.22)

We have made use here of the following Proposition 4.1 whose proof is postponed to Section 6.

**Proposition 4.1.** Let $m$ satisfy property (1.9). Then for any $\beta \in (1 - \rho, 1)$, the function $m(x)\beta$ belongs to $C^1(\overline{\Omega})$ and $\nabla [m(x)\beta](x) = m_1(x)$ for any $x \in \overline{\Omega}$, where

\[
m_1(x) = \begin{cases} 0, & x \in Z(m), \\
\beta m(x)^{\beta-1} \nabla m(x), & x \notin Z(m),
\end{cases}
\]

and $Z(m)$ denotes the zero-set of $m$. Moreover,

\[
|\nabla [m(x)\beta](x)| \leq C m(x)^{\beta-1+\rho}, \quad x \in \overline{\Omega}.
\]

Since the matrix $(a_{j,k}(x))_{j,k=1,...,n}$ is real-valued and positive definite, from (4.22) we immediately deduce that

(4.24) $I_1(\delta)$ and $\text{Re} I_3(\delta)$ are positive for any $\delta \in \mathbb{R}_+.$
Then we observe that \( I_2(\delta) \) has a limit as \( \delta \to 0+ \) and

\[
(4.25) \quad \lim_{\delta \to 0+} I_2(\delta) = \int_{\Omega} \frac{u(x)}{\Omega} |u(x)|^{p-2} \sum_{j,k=1}^n m(x)^{p-2} D_{x_j}m(x)a_{jk}(x)D_{x_k}u(x) \, dx.
\]

Note that the integral in the right-hand side is well-defined on the whole of \( W^{1,p}(\Omega) \) since \( \overline{u} \|u\|^{p-2} \in L^p(\Omega), m^{p-2} D_{x_j}m \in L^\infty(\Omega) \) and \( D_{x_j}u \in L^p(\Omega) \).

Further, (4.25) implies that there exists also \( \lim_{\delta \to 0+} [I_1(\delta) - (2-p)I_3(\delta)] \).

From (4.24) we deduce that there exist the limits

\[
\lim_{\delta \to 0+} \text{Im} I_3(\delta) \quad \text{and} \quad \lim_{\delta \to 0+} [I_1(\delta) - (2-p)\text{Re} I_3(\delta)].
\]

We can now prove the following Lemma 4.1.

**Lemma 4.1.** The following estimates hold for any \( \delta \in \mathbb{R}_+, \ p \in (1,2) \) and \( \eta \in (0,2(p-1)(2-p)^{-1}) \):

\[
(4.26) \quad I_1(\delta) - (2-p)\text{Re} I_3(\delta) - \eta(2-p)|\text{Im} I_3(\delta)| \geq 0,
\]

\[
I_1(\delta) + (p-1)\text{Re} I_2(\delta) - (2-p)\text{Re} I_3(\delta)
\]

\[
(4.27) \quad - \eta(p-1)\text{Im} I_2(\delta) - (2-p)\text{Im} I_3(\delta) \geq -(p-1)(1 + \eta^2)^{1/2}|I_2(\delta)|,
\]

\[
\lim_{\delta \to 0+} \left[ I_1(\delta) + (p-1)\text{Re} I_2(\delta) - (2-p)\text{Re} I_3(\delta) \right]
\]

\[
- \eta \lim_{\delta \to 0+} \left[ (p-1)\text{Im} I_2(\delta) - (2-p)\text{Im} I_3(\delta) \right]
\]

\[
(4.28) \quad \geq -C_2\|f\|^{p/2}_p \|M\|^{p-2+p}_p \|u\|^{2-p}_p p/2,
\]

\( C_2 \) being a suitable positive constant.

Proof. Since the matrix \((a_{jk}(x))_{j,k=1,...,n}\) is real-valued and positive definite, we immediately deduce the equality

\[
\sum_{j,k=1}^n a_{jk}(x)\xi_j \xi_k = \sum_{j,k=1}^n a_{jk}(x)\left[\text{Re}(\xi_j)\text{Re}(\xi_k) + \text{Im}(\xi_j)\text{Im}(\xi_k)\right], \quad \forall \xi \in \mathbb{C}^n.
\]

Consider now the formulae

\[
I_1(\delta) = \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^n a_{jk}(x)\overline{u(x)}D_{x_j}u(x) \overline{D_{x_k}u(x)} \, dx
\]

\[
+ \delta \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^n a_{jk}(x)D_{x_j}u(x) \overline{D_{x_k}u(x)} \, dx
\]

\[
= \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \left\{ \sum_{j,k=1}^n a_{jk}(x) \text{Re} [\overline{u(x)}D_{x_j}u(x)] \text{Re} [\overline{D_{x_k}u(x)}] \right\}
\]

\[
+ \delta \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \left\{ \sum_{j,k=1}^n a_{jk}(x) \text{Im} [\overline{u(x)}D_{x_j}u(x)] \text{Im} [\overline{D_{x_k}u(x)}] \right\}
\]

\[
+ \delta \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \left\{ \sum_{j,k=1}^n a_{jk}(x) \text{Re} [\overline{u(x)}D_{x_j}u(x)] \text{Im} [\overline{D_{x_k}u(x)}] \right\}
\]

\[
+ \delta \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \left\{ \sum_{j,k=1}^n a_{jk}(x) \text{Im} [\overline{u(x)}D_{x_j}u(x)] \text{Re} [\overline{D_{x_k}u(x)}] \right\}
\]

\[
+ \delta \int_{\Omega} (|u(x)|^2 + \delta)^{(p-4)/2} m(x)^{p-1} \left\{ \sum_{j,k=1}^n a_{jk}(x) \text{Im} [\overline{u(x)}D_{x_j}u(x)] \text{Im} [\overline{D_{x_k}u(x)}] \right\}.
\]
\[
\sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \frac{u(x)}{D_{jx,\alpha}} u(x) \right] \text{Im} \left[ \frac{\overline{u(x)}}{D_{\alpha x,\alpha}} u(x) \right] \mathrm{d}x
\]

(4.29) \quad + \delta \int_{\Omega} \left( |\mu(x)|^2 + \delta \right)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{jx,\alpha} u(x) \overline{D_{\alpha x,\alpha} u(x)} \mathrm{d}x, \quad \forall \delta \in \mathbb{R}_+,

I_1(\delta) = (2 - p) \text{Re} I_3(\delta) = \int_{\Omega} \left( |\mu(x)|^2 + \delta \right)^{(p-4)/2} m(x)^{p-1} \times \left( (p - 1) \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \frac{u(x)}{D_{jx,\alpha}} u(x) \right] \text{Re} \left[ \frac{\overline{u(x)}}{D_{\alpha x,\alpha}} u(x) \right] \right.

+ \sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \frac{u(x)}{D_{jx,\alpha}} u(x) \right] \text{Im} \left[ \frac{\overline{u(x)}}{D_{\alpha x,\alpha}} u(x) \right] \mathrm{d}x

(4.30) \quad + \delta \int_{\Omega} \left( |\mu(x)|^2 + \delta \right)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) D_{jx,\alpha} u(x) \overline{D_{\alpha x,\alpha} u(x)} \mathrm{d}x, \quad \forall \delta \in \mathbb{R}_+,

\text{Im} I_3(\delta)

= \left| \int_{\Omega} \left( |\mu(x)|^2 + \delta \right)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \frac{u(x)}{D_{jx,\alpha}} u(x) \right] \text{Im} \left[ \frac{\overline{u(x)}}{D_{\alpha x,\alpha}} u(x) \right] \mathrm{d}x \right|

\leq \frac{1}{2} \int_{\Omega} \left( |\mu(x)|^2 + \delta \right)^{(p-4)/2} m(x)^{p-1} \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \frac{u(x)}{D_{jx,\alpha}} u(x) \right] \text{Re} \left[ \frac{\overline{u(x)}}{D_{\alpha x,\alpha}} u(x) \right]

(4.31) \quad + \text{Im} \left[ \frac{u(x)}{D_{jx,\alpha}} u(x) \right] \text{Im} \left[ \frac{\overline{u(x)}}{D_{\alpha x,\alpha}} u(x) \right] \mathrm{d}x, \quad \forall \delta \in \mathbb{R}_+.

We have here used the Cauchy–Schwarz inequality and the geometric-arithmetic mean, i.e.

\[
\left| \sum_{j,k=1}^{n} a_{j,k}(x) \xi_j \eta_k \right| \leq \left( \sum_{j,k=1}^{n} a_{j,k}(x) \xi_j \xi_k \right)^{1/2} \left( \sum_{j,k=1}^{n} a_{j,k}(x) \eta_j \eta_k \right)^{1/2}
\]

\[
\leq \frac{1}{2} \left( \sum_{j,k=1}^{n} a_{j,k}(x) \xi_j \xi_k + \sum_{j,k=1}^{n} a_{j,k}(x) \eta_j \eta_k \right) = \frac{1}{2} \sum_{j,k=1}^{n} a_{j,k}(x) [\xi_j \xi_k + \eta_j \eta_k], \quad \forall \xi, \eta \in \mathbb{R}^n.
\]

From (4.24) and (4.31) we deduce the following inequality, where we take advantage of the membership \( \eta \in (0, 2(p - 1)(2 - p)^{-1}) \):

\[
I_1(\delta) = (2 - p) \text{Re} I_3(\delta) - \eta(2 - p) |\text{Im} I_3(\delta)| = \int_{\Omega} \left( |\mu(x)|^2 + \delta \right)^{(p-4)/2} m(x)^{p-1} \times \left[ \left( p - 1 - \frac{1}{2} \eta(2 - p) \right) \sum_{j,k=1}^{n} a_{j,k}(x) \text{Re} \left[ \frac{u(x)}{D_{jx,\alpha}} u(x) \right] \text{Re} \left[ \frac{\overline{u(x)}}{D_{\alpha x,\alpha}} u(x) \right] \right.
\]

\[
\left. + \sum_{j,k=1}^{n} a_{j,k}(x) \text{Im} \left[ \frac{u(x)}{D_{jx,\alpha}} u(x) \right] \text{Im} \left[ \frac{\overline{u(x)}}{D_{\alpha x,\alpha}} u(x) \right] \mathrm{d}x \right]
\]

\[
= \frac{1}{2} \sum_{j,k=1}^{n} a_{j,k}(x) [\xi_j \xi_k + \eta_j \eta_k], \quad \forall \xi, \eta \in \mathbb{R}^n.
\]
\[ + \left[ 1 - \frac{1}{2} \eta(2-p) \right] \sum_{j,k=1}^n a_{j,k}(x) \text{Im} \left[ \overline{u(x)} D_{x_j} u(x) \right] \text{Im} \left[ \overline{u(x)} D_{x_k} u(x) \right] \text{d}x \]

\[ (4.32) \quad + \delta \int_{\Omega} \left| |u(x)|^2 + \delta \right|^{(p-1)/2} m(x)^{p-1} \sum_{j,k=1}^n a_{j,k}(x) D_{x_j} u(x) D_{x_k} u(x) \text{d}x \geq 0, \quad \forall \delta \in \mathbb{R}_+. \]

We have thus proved (4.26).

Then we note that (4.27) is a consequence of (4.26):

\[ I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta) - \eta(\delta - (2-p) \text{Re} I_3(\delta)) \]
\[ \geq I_1(\delta) - (2-p) \text{Re} I_3(\delta) - \eta(\delta - (2-p) \text{Re} I_3(\delta)) + (p-1) \text{Re} I_2(\delta) - \eta \text{Re} I_3(\delta) \]
\[ \geq -(p-1)(1 + \eta^2)^{1/2} |I_2(\delta)|, \quad \forall \delta \in \mathbb{R}_+. \]

(4.33)

To conclude the proof of the lemma we take into account the relations

\[ \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta) \right] \]
\[ - \eta \lim_{\delta \to 0^+} [(p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta)] \]
\[ \geq \lim_{\delta \to 0^+} \left\{ \text{Re} I_1(\delta) + (p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta) \right\} \]
\[ - \eta [(p-1) \text{Re} I_2(\delta) - (2-p) \text{Re} I_3(\delta)] \]
\[ \geq -(p-1)(1 + \eta^2)^{1/2} \lim_{\delta \to 0^+} |I_2(\delta)|, \quad \forall \delta \in \mathbb{R}_+. \]

(4.34)

Next, consider the following chain of inequalities, which holds for any \( \delta \in \mathbb{R}_+ \):

\[ \lim_{\delta \to 0^+} |I_2(\delta)| \]
\[ \leq \limsup_{\delta \to 0^+} \int_\Omega \left( |u(x)|^2 + \delta \right)^{(p-1)/2} \sum_{j,k=1}^n m(x)^{p-2} |D_{x_j} m(x)| |a_{j,k}(x) D_{x_k} u(x)| \text{d}x \]
\[ \leq \limsup_{\delta \to 0^+} \int_\Omega \left( |u(x)|^2 + \delta \right)^{p/4} \]
\[ \times \left( |u(x)|^2 + \delta \right)^{(p-2)/4} \sum_{j,k=1}^n m(x)^{p-2} |D_{x_j} m(x)| |a_{j,k}(x) D_{x_k} u(x)| \text{d}x \]
\[ \leq C_1 \limsup_{\delta \to 0^+} \left[ \int_\Omega m(x)^{2(p-2+p)/2} (|u(x)|^2 + \delta)^{p/2} \text{d}x \right]^{1/2} \]
\[ \times \limsup_{\delta \to 0^+} \left[ \int_\Omega \sum_{j,k=1}^n |a_{j,k}(x) D_{x_k} u(x)|^2 (|u(x)|^2 + \delta)^{(p-2)/2} \text{d}x \right]^{1/2} \]

(cf. (4.16), (4.17))
\[ \leq C_{25} \lim_{\delta \to 0^+} \left\{ \left[ \int_{\Omega} m(x)^{3(\rho - 2 - \rho)}(|u(x)|^2 + \delta)^{\rho/2} \, dx \right]^{1/2} \times \left[ \int_{\Omega} (|u(x)|^2 + \delta)^{(\rho - 2)/2} |\nabla u(x)|^2 \, dx \right]^{1/2} \right\} \]
\[ \leq C_{26} \left[ \int_{\Omega} m(x)^{2(\rho - 2 - \rho)} |u(x)|^p \, dx \right]^{1/2} \|f\|_p^{\rho/2} \]

(4.35) \[ \leq C_{27} \|f\|_p^{\rho/2} \|Mu\|_p^{2-\rho(\rho+1)/2} \|u\|_p^{2-\rho(\rho+1)/2}. \]

To derive the last inequality we have applied Hölder’s inequality with index \( q = p[2(\rho - 2 + \rho)]^{-1} \) to the integral
\[ \int_{\Omega} |m(x)||u(x)|^{2(\rho - 2 + \rho)} |u(x)|^{-p+4-2\rho} \, dx. \]

From (4.34) and (4.35) we immediately conclude (4.28).

Taking now the real part and the modulus of the imaginary part in (4.21) and using (4.22), we easily derive the relations

\[ \text{Re} \lambda \|Mu\|_p^{\rho} + \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta) \right] \]
\[ + \int_{\Omega} a_0(x)m(x)^{p-1}|u(x)|^p \, dx = \text{Re} \int_{\Omega} m(x)^{p-1}f(x)\overline{u(x)}|u(x)|^{p-2} \, dx, \]
\[ \|\text{Im} \lambda\|_p^{\rho} \leq \lim_{\delta \to 0^+} \|(p - 1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_3(\delta)\| \]
\[ + \left| \text{Im} \int_{\Omega} m(x)^{p-1}f(x)\overline{u(x)}|u(x)|^{p-2} \, dx \right|, \quad \forall \lambda \in \mathbb{C}. \]

Add now member by member (4.36) and (4.37) multiplied by \( \eta \in (0, 2\sqrt{p - 1}(2 - p)^{-1}) \) and use (4.28) and (2.2). We easily deduce the following estimate for any \( \lambda \in \Sigma =: \{\mu \in \mathbb{C} : \text{Re} \mu + \eta |\text{Im} \mu| \geq 0\} \):

\[ \left[ \text{Re} \lambda + \eta |\text{Im} \lambda| + \frac{\gamma}{|m|_\infty} \right]\|Mu\|_p^{\rho} \]
\[ \leq - \left[ \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta) \right] \right. \]
\[ - \eta \lim_{\delta \to 0^+} \left[ (p - 1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_3(\delta) \right] \]
\[ + \text{Re} \int_{\Omega} f(x)m(x)^{p-1}\overline{u(x)}|u(x)|^{p-2} \, dx + \eta \left| \text{Im} \int_{\Omega} f(x)m(x)^{p-1}\overline{u(x)}|u(x)|^{p-2} \, dx \right| \]
\[ \leq - \lim_{\delta \to 0^+} \left[ I_1(\delta) + (p - 1) \text{Re} I_2(\delta) - (2 - p) \text{Re} I_3(\delta) \right] \]
\[ - \eta \left[ (p - 1) \text{Im} I_2(\delta) - (2 - p) \text{Im} I_3(\delta) \right]. \]
$$+ \text{Re} \int_{\Omega} f(x)m(x)r^{1-\alpha(x)}u(x)dx + \eta \left| \int_{\Omega} f(x)m(x)r^{1-\alpha(x)}u(x)dx \right| \leq C_28 \|f\|_{p/2}^2 \|Mu\|_p^{2-\rho} \|u\|_p^{2-\rho/p} + (1 + \eta^2)^{1/2} \|f\|_p \|Mu\|_p^{-\rho}. \tag{4.38}$$

Take $\lambda$ in the sector
$$\Sigma_3 = \left\{ \mu \in C : \text{Re} \mu + \eta/2 \left| \text{Im} \mu \right| + \frac{\gamma}{2 \|m\|_\infty} \geq 0 \right\}. \tag{4.39}$$

Then, since $\|u\|_p \leq C_{19} \|f\|_p$ (cf. (2.11), (2.12) and our definition of $\eta$) and $2 - \rho - p/2 > 0$ (cf. (4.20)), by Proposition 2.1 we immediately derive the inequality
$$\left( |\lambda| + 1 \right) \|Mu\|_p^{2-\rho} \leq C_{24} \left[ \|f\|_p^{2-\rho} + \|f\|_p \|Mu\|_p^{1-\rho} \right], \quad \text{if} \quad \lambda \in \Sigma_3. \tag{4.40}$$

Finally, $\|Mu\|_p \leq \|m\|_\infty \|u\|_p \leq C_{19} \|m\|_\infty \|f\|_p$ implies
$$\left( |\lambda| + 1 \right) \|Mu\|_p^{2-\rho} \leq C_{30} \|f\|_p^{2-\rho}, \quad \text{if} \quad \lambda \in \Sigma_3. \tag{4.41}$$

We can now collect the result in this section in the following Theorem 4.1.

**Theorem 4.1.** Let $L$ and $M$ be the linear operators defined by (1.7) and (1.8), the coefficients $a_{i,j}, i, j = 1, \ldots, n,$ $a_0$ enjoying properties (2.1) and (2.2) and $m$ being a non-negative function satisfying (1.9). Then the spectral equation $\lambda Mu + Lu = f,$ with $f \in L^p(\Omega),$ admits, for any $\lambda \in \Sigma_3$ and $p \in (1, 2), \rho \in [2 - p, 1],$ a unique solution $u \in W^{2-p}(\Omega) \cap W^{p-\rho}_0(\Omega)$ satisfying the estimates
$$\|u\|_p \leq C_{30} \|f\|_p, \quad \|Mu\|_p \leq C_{31}(p) |\lambda|^{2-\rho} \|f\|_p, \quad \lambda \in \Sigma_3, \tag{4.42}$$
$$\|Lu\|_p \leq C_{32}(1 + |\lambda|^{(1-\rho, 2-\rho)_{-1}}) \|f\|_p, \quad \lambda \in \Sigma_3.$$

**Example 4.1.** Let $n = 1, m(x) = x^{q}(1 - x)^{q},$ $q \in (1, +\infty),$ $\Omega = (0, 1).$ Then
$$m'(x) = q(1 - 2x)m(x)^{(q-1)/q}, \quad x \in (0, 1).$$

Hence (4.25) holds true for any $q \in (1, +\infty).$ If we have to deal with $L^p(0, 1)$ with $p \in (1, 2),$ to satisfy (4.20) we are forced to assume $q > (p - 1)^{-1}.$

5. **Solving problem (1.1)–(1.3)**

Taking the spectral Theorems 2.1, 3.1, 4.1 into account, from Theorem 3.26 in [3] we can easily derive our existence and uniqueness result. For this purpose we need to introduce the following interpolation space

$$L^p_{\delta_1, \infty} = \left\{ g \in L^p(\Omega) : \sup_{t \geq 1} t^\theta \|L(tM + L)^{-1}\|_{L^p(\Omega)} < +\infty \right\}. \tag{5.1}$$
In particular, any $g = mh$ belongs to $L^p_{\theta, \infty}$, whenever $m \in L^\infty(\Omega)$ and $h \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. Notice that $L^p_{\theta, \infty} \subset (X; D(LM^{-1}))_{\theta, \infty}$.

**Theorem 5.1.** Let $p \in (1, +\infty)$, let $m \in L^\infty(\Omega)$ be a non-negative function and let the coefficients $a_{ij}, j = 1, \ldots, n$, $\alpha_0$ enjoy properties (2.1) and (2.2). Then for any

$$u_0 \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \quad f \in C^\theta([0, T]; L^p(\Omega)), \quad \theta \in (1 - \beta, 1),$$

with $\beta = 1/p$ and

$$-A(x, D_x)u + f(0, \cdot) = g_0, \quad g_0 \in L^p_{\theta, \infty},$$

problem (1.1)–(1.3) admits a unique solution

$$mu \in C^{\theta+\beta}([0, T]; L^p(\Omega)), \quad u \in C^{\theta+\beta-1}([0, T]; W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)).$$

Moreover, if $m$ is a non-negative function satisfying (1.9) and $\beta$ is defined by (1.10), the same result holds under assumptions (5.1) and (5.2) on $(u_0, f)$.

6. **Proofs of the propositions**

Proof of Proposition 2.1. Let $\lambda \in \Sigma_{k, \varepsilon}$ and $Re \lambda \geq 0$. Then it is clear that $|Re \lambda| \leq Re \lambda + k|Im \lambda| + \varepsilon$. On the other hand, if $\lambda \in \Sigma_{k, \varepsilon}$ and $Re \lambda < 0$, then $|Re \lambda| = -Re \lambda \leq (k/2)|Im \lambda| + (\varepsilon/2) \leq Re \lambda + k|Im \lambda| + \varepsilon$. Therefore, $|Re \lambda| \leq Re \lambda + k|Im \lambda| + \varepsilon$ for any $\lambda \in \Sigma_{k, \varepsilon}$. In the meantime it is obvious that $|Im \lambda| + 1 \leq 2[(1/k)+(1/\varepsilon)](Re \lambda+k|Im \lambda|+\varepsilon)$ for any $\lambda \in \Sigma_{k, \varepsilon}$. Hence we conclude that $|\lambda| + 1 \leq |Re \lambda| + |Im \lambda| + 1 \leq [2/k + (2/\varepsilon) + 1](Re \lambda + k|Im \lambda| + \varepsilon)$, $\lambda \in \Sigma_{k, \varepsilon}$.

Proof of Proposition 2.2. We consider the set $J = \{\theta \in [0, 1]; R(A(\theta)) = X\}$ and shall prove that this set is an open and closed subset of the interval $[0, 1]$ under (2.17) and (2.18). In fact, let $\theta \in J$; then, it follows from (2.17) that $A(\theta)^{-1} \in \mathcal{L}(X)$ with $\|A(\theta)^{-1}\| \leq \delta^{-1}$. Moreover, for any $\theta' \in [0, 1]$, we have

$$A(\theta') = [1 + \{A(\theta') - A(\theta)\}A(\theta)^{-1}]A(\theta).$$

Since $\|\{A(\theta') - A(\theta)\}A(\theta)^{-1}\| \leq N^{-1}\theta' - \theta$, the operator $1 + \{A(\theta') - A(\theta)\}A(\theta)^{-1}$ is a linear isomorphism of $X$ provided $|\theta' - \theta| < N^{-1}\delta$. This then shows that $\theta' \in J$ for any $\theta'$ such that $|\theta' - \theta| < N^{-1}\delta$; hence, $J$ is an open set. Consider now a sequence $\theta_n \in J$ and assume that $\theta_n \rightarrow \theta$ as $n \rightarrow +\infty$. Let $f \in X$ be any vector; then, there exists a sequence $u_n \in \mathcal{D}$ such that $A(\theta_n)u_n = f$. From (2.17) it follows that $\|u_n\| \leq \delta^{-1}\|f\|$. Furthermore we observe that $\|A(\theta)u_n - f\| \leq \|A(\theta) - A(\theta_n)\|u_n\| \leq N\delta^{-1}|\theta - \theta_n|\|f\|$. 


therefore, $A(\tilde{\theta})u_n \to f$ as $n \to +\infty$. In the meantime, $\delta \|u_m - u_n\| \leq \|A(\tilde{\theta})(u_m - u_n)\| \leq \|A(\tilde{\theta})u_m - f\| + \|f - A(\tilde{\theta})u_m\| \to 0$ as $m, n \to +\infty$. So, $u_n$ has a limit $u \in X$ as $n \to +\infty$. Since $A(\tilde{\theta})$ is a closed operator, $u \in D$ and $A(\tilde{\theta})u = f$; hence, $\tilde{\theta} \in J$. That is, $J$ is a closed set. As $1 \in J \neq 0$, we conclude that $J = [0, 1]$. \hfill \Box

Proof of Proposition 4.1. According to (1.9), we have the inclusion $Z(m) \subset Z(\nabla m)$. Moreover, formula (4.23) is trivial if $x \notin Z(m)$. This therefore shows that we have to deal with the case $x \in Z(m)$ only.

First we will consider the one-dimensional case ($n = 1$). For this purpose assume $x_0 \in Z(m)$. Our starting point is the following formula:

$$
\lim_{x \to x_0^+} \frac{|m(x)\beta - m(x_0)\beta|}{x - x_0} = \lim_{x \to x_0^+} \frac{|m(x) + \varepsilon|^{\beta - \varepsilon^{\beta}}}{x - x_0} = \lim_{x \to x_0^+} \left| \beta \int_{x_0}^x [m(t) + \varepsilon]^{\beta - 1} m'(t) \, dt \right|.
$$

(6.1)

We next notice that $\lim_{|x| \to 0^+} [m(t) + \varepsilon]^{\beta - 1} m'(t) = m_1(t)$ for any $t \in \Omega$ and that

$$
[m(t) + \varepsilon]^{\beta - 1} m'(t) \leq C[m(t) + \varepsilon]^{\beta - 1} m(t)^\rho
$$

$$
= C \left[ \frac{m(t)}{m(t) + \varepsilon} \right]^{1 - \beta} m(t)^{\beta - 1 + \rho} \leq C m(t)^{\beta - 1 + \rho}, \quad \forall t \in \Omega.
$$

By virtue of the dominated convergence theorem and by the bound $|m_1(t)| \leq C m(t)^{\beta - 1 + \rho}$ for any $t \in \overline{\Omega}$, we deduce the following relations:

$$
\lim_{x \to x_0^+} \frac{|m(x)\beta - m(x_0)\beta|}{x - x_0} = \lim_{x \to x_0^+} \left| \frac{1}{x - x_0} \int_{x_0}^x m_1(t) \, dt \right|
$$

(6.2)

$$
\leq \lim_{x \to x_0^+} \frac{1}{x - x_0} \int_{x_0}^x |m_1(t)| \, dt \leq \lim_{x \to x_0^+} \frac{C}{x - x_0} \int_{x_0}^x m(t)^{\beta - 1 + \rho} \, dt = 0.
$$

Note here that $m(\cdot)^{\beta - 1 + \rho}$ is continuous in $\Omega$ and $x_0 \in Z(m)$. An analogous argument holds for $\lim_{x \to x_0^-} \frac{|m(x)^\beta - m(x_0)^\beta|}{(x - x_0)}$ also.

We have thus shown that there exists $D_\varepsilon[1]m(\cdot)^\beta](x_0)$ and coincides with $0 = m_1(x_0)$. Therefore the formula $D_\varepsilon[1]m(\cdot)^\beta](x) = m_1(x)$ holds for any $x \in \Omega$. Since $\beta \in (1 - \rho, 1)$, bound (1.9) and (4.23) immediately imply that $m_1 \in C(\Omega)$. Consequently, $m(\cdot)^\beta \in C(\Omega)$.

Finally, the multi-dimensional case is an immediate consequence of the case $n = 1$. \hfill \Box
References


Angelo Favini
Dipartimento di Matematica
Università degli Studi di Bologna
Piazza di Porta S. Donato 5, 40126 Bologna, Italia

Alfredo Lorenzi
Dipartimento di Matematica
Università degli Studi
via Saldini 50, 20133 Milano, Italia

Hiroki Tanabe
Department of Economics
Otemon Gakuin University
Ibaraki, Osaka 567-8502, Japan

Atsushi Yagi
Department of Applied Physics
Osaka University
Suita, Osaka 565-0871, Japan