ON UNIQUENESS IN THE CAUCHY PROBLEM FOR SYSTEMS WITH PARTIAL ANALYTIC COEFFICIENTS

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Abstract

In this paper, we consider the uniqueness in the Cauchy problem for systems of differential operators with partial analytic coefficients. By proving Carleman’s estimate, we obtain the uniqueness theorem which contains the classical Holmgren’s theorem for systems of differential operators.

1. Introduction

The problem of the uniqueness in the Cauchy problem is a fundamental problem in a theory of partial differential equation. In this paper, we consider the uniqueness in the Cauchy problem for systems with partial analytic coefficients. In the case when coefficients are analytic, by Holmgren’s theorem, the uniqueness holds for any non-characteristic initial hypersurface. On the other hand, it is well known that, when coefficients are merely $C^\infty$ function, the uniqueness is false for some non-characteristic initial hypersurface. For example, Alinhac and Baouendi [1] showed that there exists some second order hyperbolic operator $P = \partial_t^2 - A(t, x, \partial_x)$, where $A$ is a second order elliptic operator with $C^\infty$ coefficients, and some time-like initial hypersurface for which the uniqueness result is false. Here time-like means that, if $C^2$-hypersurface $S$ is locally defined by $S = \{ x \mid \varphi(x) = 0 \}$, $\varphi$ satisfies that

$$\varphi'(0)^2 - A(0, 0, \varphi'(0)) < 0.$$  

In [10], [5], [8], Tataru, Hörmander, Robbiano and Zuily showed that under the assumption of partial analyticity of coefficients, uniqueness result holds for any non-characteristic initial hypersurface. But they considered the uniqueness only for scalar differential operators. In this paper, we consider uniqueness for systems of partial differential operators with partial analytic coefficients. The study of the uniqueness for systems is not so many as for scalar differential equation. For example, in [6], G. Hile and M. Protter considered the uniqueness for first order systems with $C^1$-coefficients. But their result holds for some restricted differential systems. The purpose of this paper is that, under the assumption of partial analyticity, we prove the uniqueness result for differential systems which can’t be proved in [6].

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We introduce some notation. Let \( n_a, n_b \) be non-negative integers with \( n = n_a + n_b \geq 1 \). We set \( \mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \) and, for \( x, \xi \in \mathbb{R}^n \), \( x = (x_a, x_b), \xi = (\xi_a, \xi_b) \). Let \( P(x, D_x) = (p_{ij}(x, D_x))_{1 \leq i, j \leq N} = \sum_{|\alpha| \leq m} A_\alpha(x)D_x^\alpha \) be a linear differential system with the principal part \( P_m(x, \xi) = \sum_{|\alpha| = m} A_\alpha(x)\xi^\alpha \). Let \( S \) be a \( C^2 \) hypersurface through \( 0 \) locally given by
\[
S = \{ x : \varphi(x) = 0 \},
\]
where \( \varphi \) satisfies
\[
\varphi(0) = 0, \quad \varphi'(0) = (\varphi'_a(0), \varphi'_b(0)) \neq 0.
\]
Our result is stated as follows;

**Theorem 1.1.** Let \( P(x, D_x) \) be a differential system of order \( m \) with \( C^\infty \) coefficients. We assume that all coefficients of \( P \) are analytic with respect to \( x_a \) in a neighborhood of \( 0 \), and that the principal part \( P_m(x, \xi) \) satisfies the following conditions:

1. For any \( \xi_b \in \mathbb{R}^{n_b} \setminus \{0\} \)
\[
\det P_m(0, 0, \xi_b) \neq 0.
\]
2. For any \( \xi_b \in \mathbb{R}^{n_b} \)
\[
\det P_m(0, i\varphi'_a(0), i\varphi'_b(0) + \xi_b) \neq 0.
\]

Let \( V \) be a neighborhood of \( 0 \) and \( u = (u_1, u_2, \ldots, u_N) \in C^\infty(V)^N \), which satisfy
\[
\begin{cases}
P(x, D_x)u(x) = 0, & x \in V, \\
\text{supp } u = \bigcup_{k=1}^N \text{ supp } u_k \subset \{ x \in V : \varphi(x) \leq 0 \}.
\end{cases}
\]
Then there exists a neighborhood \( W \) of \( 0 \) in which \( u \equiv 0 \).

We make some remarks on this result. This theorem contains Holmgren’s uniqueness theorem. In fact, when we set \( n_a = n, n_b = 0 \), the condition of \( P_m(x, \xi) \) in this theorem means that \( P(x, D_x) \) is non-characteristic with respect to the initial hypersurface \( S \). Moreover, by this theorem, we can show that the uniqueness holds in the following differential system.

**Example 1.** We set \( (t, x) \in \mathbb{R}^2 \). Let \( a(t, x), b(t, x), c(t, x) \in C^\infty(\mathbb{R}^2) \), and \( A(x) \in C^\infty(\mathbb{R}, M_2(\mathbb{C})) \). We assume that \( a = a(t, x) \), \( b = b(t, x) \), \( c = c(t, x) \) satisfy the following conditions:

1. \( a(t, x), b(t, x), c(t, x) \) is analytic with respect to \( t \) in some neighborhood of \( 0 \).
2. \( a_0, b_0, c_0 \in \mathbb{R} \) and \( a_0b_0 \neq 0 \), where \( a_0 = a(0, 0), b_0 = b(0, 0), c_0 = c(0, 0) \).
Then the equation
\[ \partial_t u = \left( \begin{array}{cc} c \partial_x & a \partial_x \\ b \partial_x & 0 \end{array} \right) u + A(x)u \]
has a unique continuation property with respect to the initial surface \( S = \{ x \mid \varphi(t, x) = 0 \} \), where \( \varphi \) satisfies
1. \( \varphi'_t(0)^2 - c_0 \varphi'_t(0) \varphi'_x(0) - a_0 b_0 \varphi'_x(0)^2 \neq 0 \);
2. \( c_0 \varphi'_t(0) + 2 a_0 b_0 \varphi'_x(0) \neq 0 \).

When \( \varphi = t \), this equation is not always strictly hyperbolic, so by the method of energy estimate, as in [11], we can’t prove the uniqueness of this equation. And because the coefficient of \( u \), is not normal, the result in [6] doesn’t contain this example.

Our proof is based on the Carleman method and the FBI transformation theory, as the proof given in [8]. By Sjöstrand’s theory of FBI transformation, we microlocalize the symbol of \( P(x, D_x) \) with respect to \( x \), and, by using semi-classical pseudo differential symbolic calculus and Gårding’s inequality, we establish the Carleman estimate of \( P(x, D_x) \).

2. Preliminaries

In this section, we prepare the tools for our proof taken from [2], [7], [8], [9].

2.1. From theory of PDO.

**Definition 2.1.** 1. Let \( g \in C^\infty(\mathbb{R}^d; (0, \infty)) \). We say that \( g \) is an order function if, for any \( \alpha \in \mathbb{N}^d \), there exists some \( C_\alpha > 0 \) such that
\[
|\partial^\alpha g(x)| \leq C_\alpha g(x). \tag{3}
\]
2. Let \( g \) be an order function. For \( a(x, \lambda) \in C^\infty(\mathbb{R}^d \times (0, \infty)) \), we say that \( a(x, \lambda) \) is of the symbol class \( S_d(g) \) if, for any \( \alpha \in \mathbb{N}^d \), there exists some positive constant \( C_\alpha \) such that
\[
|\partial^\alpha a(x, \lambda)| \leq C_\alpha g(x). \tag{4}
\]

**Definition 2.2.** Let \( g \) be an order function. For \( \{a_j\} \subset S_d(g), \ a \in S_d(g) \), we write
\[
\sum_{j=0}^N \frac{1}{\lambda^j} a_j \sim a \quad \text{in} \quad S_d(g) \tag{5}
\]
if, for any \( \alpha \in \mathbb{N}^d \) and \( N \in \mathbb{N} \), there exist some positive constants \( C_{N, \alpha}, \lambda_{N, \alpha} \) such
that, for \((x, \lambda) \in \mathbb{R}^d \times [\lambda, N, a, \infty)\), \(a(x, \lambda)\) satisfies the following inequality:

\[
(6) \quad \left| \frac{\partial^a}{\partial x} \left( a(x, \lambda) - \sum_{j=0}^{N} \frac{1}{\lambda^j} a_j(x, \lambda) \right) \right| \leq \frac{C_{N, a}}{\lambda^{N+1}} g(x).
\]

In the following context, we set \(d = 2n\), \(\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^n\), \(\langle \xi \rangle = (1 + |\xi|^2)^{1/2}\).

**Definition 2.3.** For \(\sigma \in S_{2n}(\langle \xi \rangle^m)\), we define \(\text{Op}_{\lambda}(\sigma)(x, D_x) = \sigma_{\lambda}(x, D_x)\) by

\[
(7) \quad \sigma_{\lambda}(x, D_x) u(x) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\langle x-y \rangle \xi} \sigma \left( \frac{x+y}{2}, \xi \right) u(y) \, dy \, d\xi,
\]

where the integral means oscillatory integral. Similarly, we define \(\sigma(x, D_x)\) by

\[
(8) \quad \sigma(x, D_x) u(x) = \left(\frac{1}{2\pi}\right)^n \iint e^{i\langle x-y \rangle \xi} \sigma \left( \frac{x+y}{2}, \xi \right) u(y) \, dy \, d\xi.
\]

**Theorem 2.4.** 1. For \(\sigma \in S_{2n}(\langle \xi \rangle^m)\), we have

\[
(9) \quad \sigma_{\lambda}(x, D_x)^* = \overline{\sigma}_{\lambda}(x, D_x),
\]

where \(\sigma_{\lambda}(x, D_x)^*\) is the formal adjoint of \(\sigma_{\lambda}(x, D_x)\).

2. For \(\sigma_i \in S_{2n}(\langle \xi \rangle^m)\) (\(i = 1, 2\)), there exists some \(\sigma_3 \in S_{2n}(\langle \xi \rangle^{m_1 + m_2})\) such that

\[
(10) \quad \sigma_{1,\lambda}(x, D_x) \sigma_{2,\lambda}(x, D_x) = \sigma_{3,\lambda}(x, D_x).
\]

And we have the following expansion formula for \(\sigma_3:\)

\[
(11) \quad \sigma_3(x, \xi) \sim \sum_{k=0}^{\infty} \left( \frac{1}{k!} \right) \left( \frac{i}{2\lambda} \right)^k (D_\xi D_y - D_\eta D_x)^k \sigma_1(x, \eta) \sigma_2(y, \eta)|_{(y, \eta) = (x, \xi)}
\]

in \(S_{2n}(\langle \xi \rangle^{m_1 + m_2})\).

By a slight modification of the proof of Theorem 2.4 in [7], we have the following theorem.

**Theorem 2.5.** 1. For \(\sigma \in S_{2n}(\langle \xi b \rangle^m)\), we have

\[
(12) \quad \sigma_{\lambda}(x, D_x)^* = \overline{\sigma}_{\lambda}(x, D_x),
\]

where \(\sigma_{\lambda}(x, D_x)^*\) is the formal adjoint of \(\sigma_{\lambda}(x, D_x)\).

2. For \(\sigma_i \in S_{2n}(\langle \xi b \rangle^m)\) (\(i = 1, 2\)), there exists some \(\sigma_3 \in S_{2n}(\langle \xi b \rangle^{m_1 + m_2})\) such that

\[
(13) \quad \sigma_{1,\lambda}(x, D_x) \sigma_{2,\lambda}(x, D_x) = \sigma_{3,\lambda}(x, D_x).
\]
And we have the following expansion formula for $\sigma_3$:

$$\sigma_3(x, \xi) \sim \sum_{k=0}^{\infty} \left( \frac{1}{k!} \right) \left( \frac{i}{2\lambda} \right)^k (D_x D_y - D_y D_x)^k \sigma_1(x, \xi) \sigma_2(y, \eta)|_{(y, \eta) = (x, \xi)}$$

in $S_2n((\xi_b)\eta^{m_1+m_2})$.

**Theorem 2.6.** Let $\sigma \in C^\infty(\mathbb{R}^{2n})$ satisfy the following inequality:

$$\|\sigma\|_s = \sum_{|\alpha+\beta| \leq 2n+1} \|D^\alpha_x D^\beta_y \sigma\|_\infty < +\infty.$$ 

Then $\sigma\lambda(x, D_x)$ is $L^2$ bounded. Moreover we have

$$\|\sigma_\lambda(x, D_x)\| \leq C\|\sigma\|_s,$$

where $C$ is a positive constant independent of $\sigma, \lambda$.

**2.2. Review on the theory of FBI transformation.** In this section, we review on the theory of FBI transformation from [7], [8], [9].

**Definition 2.7.** We denote the partial FBI transformation by $T$, that is, for $u \in S(\mathbb{R}^n)$, we define $Tu$ by

$$Tu(z_a, x_b, \lambda) = \sum I1_{(\zeta)\eta} K(\lambda) \int e^{-(\lambda/2)(z_a - x_a)^2} u(y_a, x_b) dy_a,$$

where $z_a \in \mathbb{C}^{n_a}, x_b \in \mathbb{R}^{n_b}, \lambda \geq 1$, and $K(\lambda) = 2^{-n_a/2}(\lambda/\pi)^{3n_a/4}, z_a^2 = \sum_{j=1}^{n_a} z_{aj}^2$.

We introduce some notations about the FBI transformation.

$$\Phi(z_a) = \frac{1}{2}(\text{Im } z_a)^2,$$

$$\kappa_T(x_a, \xi_a) = (x_a - i\xi_a, \xi_a), \quad (x_a, \xi_a) \in T^*\mathbb{R}^{n_a}.$$ 

**Definition 2.8.** For $u \in S(\mathbb{R}^n)$ and $0 \leq \eta \leq 1$, we define the partial FBI transformation $T_\eta u$ by

$$T_\eta u(z_a, x_b, \lambda) = \sum I1_{(\zeta)\eta} K_\eta(\lambda) \int e^{-(\lambda/2)(1+\eta)(z_a - x_a)^2} u(y_a, x_b) dy_a,$$

where $z_a \in \mathbb{C}^{n_a}, x_b \in \mathbb{R}^{n_b}, \lambda \geq 1$, and $K_\eta(\lambda) = 2^{-n_a/2}(\lambda(1+\eta)/\pi)^{3n_a/4}, z_a^2 = \sum_{j=1}^{n_a} z_{aj}^2$.

We define $\Phi_\eta(z_a)$ and $\kappa_{T_\eta}$ by

$$\Phi_{(1+\eta)}(z_a) = \frac{1}{2}(1+\eta)(\text{Im } z_a)^2,$$
Next we introduce some function spaces where \( T \) and \( T_\eta \) operate. For \( k \in \mathbb{Z} \) and \( 0 \leq \eta \leq 1 \), we define \( L^2_{(1+\eta)\phi}(\mathbb{C}^{n_x}, H^k(\mathbb{R}^{m_b})) \) by
\[
L^2_{(1+\eta)\phi}(\mathbb{C}^{n_x}, H^k(\mathbb{R}^{m_b})) = L^2\left(\left(\mathbb{C}^{n_x}, e^{-2(1+\eta)\phi(z_a)}L(dz_a)\right), H^k(\mathbb{R}^{m_b})\right),
\]
where \( L(dz_a) \) is Lebesgue measure on \( \mathbb{C}^{n_x} \), and \( H^k(\mathbb{R}^{m_b}) \) is the Sobolev space of order \( k \). In particular, when \( k = 0 \), we denote \( L^2_{(1+\eta)\phi}(\mathbb{C}^{n_x}, L^2(\mathbb{R}^{m_b})) \) by \( L^2_{(1+\eta)\phi} \). And we define \( \mathcal{L}^2_{(1+\eta)\phi} \) by
\[
\mathcal{L}^2_{(1+\eta)\phi} = L^2_{(1+\eta)\phi} \cap \mathcal{H}(\mathbb{C}^{n_x}),
\]
where \( \mathcal{H}(\mathbb{C}^{n_x}) \) is the space of all entire functions in \( \mathbb{C}^{n_x} \). Then we have the following proposition.

**Proposition 2.9.** For \( \eta \in [0, 1] \), we have
1. \( T_\eta \) is an isometry mapping from \( L^2(\mathbb{R}^{n_x}, H^k(\mathbb{R}^{m_b})) \) into \( L^2_{(1+\eta)\phi}(\mathbb{C}^{n_x}, H^k(\mathbb{R}^{m_b})) \).
2. \( T_\eta^* T_\eta \) is an identity on \( L^2(\mathbb{R}^{n_x}), \) where \( T_\eta^* \) is the formal adjoint of \( T_\eta \).
3. \( T_\eta^* T_\eta^* \) is the projection from \( L^2_{(1+\eta)\phi} \) to \( L^2_{(1+\eta)\phi} \), in particular, if \( \widetilde{v} = T_\eta v, v \in \mathcal{S}(\mathbb{R}^{n_x}), \) we have \( T_\eta^* \widetilde{v} = \widetilde{v} \).

### 2.3. Microlocalization of PDO whose coefficients are analytic.

In this section we prepare some theorems taken from [8]. Let \( p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha \) be a polynomial in \( \xi \) with coefficients in \( C^\infty_0(\mathbb{R}^{m_b}) \). Assume moreover that there exists some positive constant \( c_0 \) such that all \( a_\alpha(x) \) are holomorphic with respect to \( x_a \) in \( \omega \), where \( \omega = \{z_a \in \mathbb{C}^{n_x} : |z_a| < c_0\} \).

**Theorem 2.10.** For \( v \in C^\infty_0(\mathbb{R}^{n_x}), \) and \( \eta \in [0, 1] \), we have \( T_\eta p(x, D_x) v = \widetilde{p}_\eta T_\eta v \) where
\[
\widetilde{p}_\eta T_\eta v(x, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{2n} \int \int e^{i\lambda(x_b-y_b)\xi_b} \left(\int_{\xi_a = -(1+\eta)\text{Im}(x+y)/2} \omega_\eta\right) dy_b d\xi_b
\]
with \( \omega_\eta \) defined by
\[
\omega_\eta = e^{i\lambda(x_a-y_a)\xi_a} p\left(\frac{x_a + y_a}{2} + i \frac{\xi_a}{1+\eta}, \frac{x_b + y_b}{2}, \lambda \xi_b\right) T_\eta u(y_a, y_b, \lambda) dy_a \wedge d\xi_a.
\]

Let \( d \) be a positive number such that \( 13d < c_0 \), and let \( \chi(z_a, \xi_a) \in C^\infty_0(\mathbb{C}^{2n_x}) \) satisfy
\[
\chi(z_a, \xi_a) = \begin{cases} 1 & |z_a| + |\xi_a| < 12d, \\ 0 & |z_a| + |\xi_a| > 13d. \end{cases}
\]
Moreover let $\chi$ be almost analytic in $\Lambda_{\Phi_k} = \kappa(\mathbb{R}^{2n})$, which means that, for any positive integer $N$, there exists some positive number $C_N$ such that
\begin{equation}
|\tilde{\phi}(\zeta_\alpha, \zeta_\beta)| \leq C_N |\zeta_\alpha| + (1 + \eta) \text{Im} \ z_\alpha^N.
\end{equation}

Then we have the following theorem.

**Theorem 2.11.** Let $v \in C_0^\infty(\mathbb{R}^n)$ with supp $v \subset \{|x| < d\}$, and let us set, for $\eta \in (0, 1)$,
\begin{equation}
\tilde{Q}_\lambda T v(x, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{2n} \iint e^{i\lambda(x_\alpha - y_\alpha)\xi_\alpha} \left(\iint_{\xi_\alpha = -(1 + \eta)\text{Im}(x_\beta)/2} \chi \left(\frac{x_\alpha + y_\alpha}{2}, \xi_\beta\right) \omega \right) dy_\beta d\xi_\beta,
\end{equation}
where $\omega = \omega_0$ is defined in Theorem 2.10. Then we have
\begin{equation}
\tilde{p}(x, \lambda) T v = \tilde{Q}_\lambda T v + \tilde{R}_\lambda T v + \tilde{g}_\lambda
\end{equation}
with $\tilde{p}(x, \lambda) = \tilde{p}(x, D_x)$ defined in Theorem 2.10, and $\tilde{R}_\lambda$ and $\tilde{g}_\lambda$ satisfy the following properties: For any $N \in \mathbb{N}$, there exists some positive constant $C_N$ such that
\begin{align}
\|\tilde{R}_\lambda T v\|_{L_{2i}^1(\mathbb{R}^n)} &\leq C_N \lambda^{-N} \|T v\|_{L_{2i}^1(\mathbb{R}^n)}, \\
\|\tilde{g}(\cdot, \lambda)\|_{L_{2i}^1(\mathbb{R}^n)} &= O(e^{-|\lambda/3|\eta d^2} \|v\|_{H^0(\mathbb{R}^n)}), \quad \lambda \to +\infty,
\end{align}
where $n_0$ depends only on the dimension $n$ and the order of $p$.

**2.4. Extension to the system case.** Let us introduce some notation for system. We denote $\mathcal{S}(\mathbb{R}^n)^N$ by
\begin{equation}
\mathcal{S}(\mathbb{R}^n)^N = \mathcal{S}(\mathbb{R}^n)^N = \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)
\end{equation}
and define $L^2(\mathbb{R}^n)^N$, $L_{(1+\eta)^p}(\mathbb{R}^n)^N$, $L^2(\mathbb{R}^n)^N$ similarly. Let $u = (u_1, u_2, \ldots, u_N)$, $v = (v_1, v_2, \ldots, v_N)$ be in $L^2(\mathbb{R}^n)^N$, we denote $(u, v)_{L^2}$ by
\begin{equation}
(u, v)_{L^2} = (u_1, v_1)_{L^2} + \cdots + (u_N, v_N)_{L^2}
\end{equation}
and denote $(u, v)_{L_{(1+\eta)^p}}$ similarly. For $u \in \mathcal{S}(\mathbb{R}^n)^N$ and $\eta \in [0, 1]$, $T_\eta u$ is denoted by
\begin{equation}
T_\eta u = (T_\eta u_1, \ldots, T_\eta u_N).
\end{equation}

Let a matrix valued function $A(x, \xi) = (a_{ij}(x, \xi))_{1 \leq i, j \leq N}$ on $\mathbb{R}^{2n}$ be in $S_{2n}(\mathbb{R}^n)$ with $m \in \mathbb{R}$, which means that, for any $i, j, a_{i, j} \in S_{2n}(\mathbb{R}^n)$. For $u \in \mathcal{S}(\mathbb{R}^n)^N$, we define $A_\lambda(x, D_x)u(x)$ by
\begin{equation}
A_\lambda(x, D_x)u(x) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\xi} A \left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi
\end{equation}
and define \( A(x, D_x)u \) similarly. For a differential system \( P(x, D_x) = \sum_{|\alpha| \leq m} A_\alpha(x)D_x^\alpha \), \( \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha \) is called the principal part of \( P(x, D_x) \), and denoted by \( P_m(x, \xi) \). Then similarly to the scalar case, we have the following theorems.

**Theorem 2.12.** For \( A(x, \xi) \in S_{2n}(\langle \xi_b \rangle^m_1) \), and \( B(x, \xi) \in S_{2n}(\langle \xi_b \rangle^m_2) \), with \( m_1, m_2 \in \mathbb{R} \), we have

1. \( A_\lambda(x, D_x)^* = A_\lambda^*(x, D_x) \), where \( A^*(x, \xi) = \overline{A(x, \xi)} \).
2. There exists some constant \( C \) such that

\[
(35) \quad A_\lambda(x, D_x)B_\lambda(x, D_x) = C_\lambda(x, D_x).
\]

Moreover we have

\[
(36) \quad C(x, \xi) \sim \sum_{k=0}^{\infty} \left( \frac{1}{k!} \right)^k (D_{\xi} D_{\eta} - D_\eta D_{\xi})^k A(x, \xi)B(y, \eta)_{|y, \eta = (x, \xi)}
\]

in \( S_{2n}(\langle \xi_b \rangle^{m_1+m_2}) \).

**Theorem 2.13.** Let \( A(x, \xi) \in S_{2n}(1) \). Then \( A_\lambda(x, D_x) \) is \( L^2 \)-bounded. Furthermore, we set \( \| A \|_\star = \left( \sum_{i,j} \| a_{i,j} \|^2_2 \right)^{1/2} \). Then, for \( u \in \mathcal{S}(\mathbb{R}^n)^N \), we have

\[
(37) \quad \| A_\lambda(x, D_x)u \| \leq C \| A \|_\star \| u \|
\]

where \( C \) is independent of \( \lambda, A \).

**Theorem 2.14.** If \( A(x, \xi) = (a_{ij}(x, \xi))_{1 \leq i, j \leq N}, a_{ij} \in S_{2n}(\langle \xi_b \rangle^m) \) satisfy, for all \( (x, \xi) \in \mathbb{R}^{2n} \),

\[
(38) \quad A(x, \xi) \succeq 0
\]

then, there exists some constant \( C > 0 \) such that, for all \( u \in \mathcal{S}(\mathbb{R}^n)^N \) and \( \lambda > 0 \), we have

\[
(39) \quad (A_\lambda(x, D_x)u, u) \geq -\frac{C}{\lambda} \| (D_{\xi_b}/\lambda)^{m/2} u \|^2,
\]

where \( (D_{\xi_b}/\lambda) = \text{Op}_\lambda(\langle \xi_b \rangle)(x, D_x) \).

**Proposition 2.15.** For \( \eta \in [0, 1] \), \( T_\eta \) has the following properties.

1. \( T_\eta \) is an isometry mapping from \( L^2(\mathbb{R}^{n_a}, H^m(\mathbb{R}^{n_b}))^N \) into \( L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m(\mathbb{R}^{n_b}))^N \).
2. \( T_\eta^* T_\eta \) is an identity on \( L^2(\mathbb{R}^n)^N \).
3. \( T_\eta T_\eta^* \) is the projection from \( L^2_{(1+\eta)\Phi} \) to \( L^2_{(1+\eta)\Phi} \). In particular \( T_\eta T_\eta^* \vec{v} = \vec{v} \) if \( \vec{v} = T\vec{v}, \vec{v} \in \mathcal{S}(\mathbb{R}^n)^N \).
Let \( P(x, \xi) = \sum_{|\alpha| \leq m} A_\alpha(x)\xi^\alpha \), be a polynomial in \( \xi \) whose coefficients are matrix valued function in \( C_0^\infty(\mathbb{R}^n; M_N(\mathbb{C})) \). Assume moreover that there exists \( c_0 > 0 \) such that, for all \( \alpha \), \( A_\alpha(x) \) is analytic with respect to \( x_\alpha \) in \( \omega_\alpha \), where \( \omega_\alpha = \{ z_\alpha \in \mathbb{C}^n \mid |z_\alpha| < c_0 \} \).

**Proposition 2.16.** Let \( \psi(x) \) be a real quadratic polynomial. For \( u \in C_0^\infty(\mathbb{R}^n)^N \), we have
\[
(40) \quad e^{\lambda \psi} P(x, D_x) e^{-\lambda \psi} u(x) = P_{\lambda, \psi}(x, D_x) u(x),
\]
where \( P_{\lambda, \psi}(x, D_x) \) is a differential operator defined by
\[
(41) \quad P_{\lambda, \psi}(x, D_x) u(x) = \left( \frac{\lambda}{2\pi} \right)^n \int \int e^{i\lambda (x-y) \cdot \xi} P \left( \frac{x+y}{2}, \lambda \left( \xi + i \psi'(X) \right) \right) u(y) dy \, d\xi.
\]

**Theorem 2.17.** For \( \eta \in [0, 1] \) and \( v \in C_0^\infty(\mathbb{R}^n)^N \), we have
\[
(42) \quad \tilde{P}_{\lambda, \psi, \eta} T v(x, \lambda) = \left( \frac{\lambda}{2\pi} \right)^{2n} \int \int e^{i\lambda (x-y) \cdot \xi_b} \left( \int \int_{\xi_a = -(1+\eta) \Im((x+y)/2)} \omega_{\eta} \right) dy_b \, d\xi_b.
\]
Here \( \omega_{\eta} \) is defined by
\[
(43) \quad \omega_{\eta} = e^{i\lambda (x_a - y_a) \cdot \xi_a} P(X, \lambda (\xi + i \psi'(X))) T u(y_a, y_b, \lambda) \, dy_a \wedge \, d\xi_a
\]
with \( X = ((x_a + y_a)/2 + i(\xi_a/(1+\eta)), (x_b + y_b)/2) \).

In particular, we denote \( \tilde{P}_{\lambda, \psi, 0} \) by \( \tilde{P}_{\lambda, \psi} \). Let \( \chi \in C_0^\infty \) and \( d > 0 \) be the same as in Theorem 2.11, then we have the following theorem similar to the scalar case.

**Theorem 2.18.** Let \( v(x) \) be in \( C_0^\infty(\mathbb{R}^n)^N \) with \( \text{supp } v \subset \{|x| < d\} \). For \( \eta \in (0, 1] \), we set
\[
(44) \quad \tilde{Q}_\lambda T v(x, \lambda) = \left( \frac{\lambda}{2\pi} \right)^{2n} \int \int e^{i\lambda (x_a - y_a) \cdot \xi_a} \left( \int \int_{\xi_a = -(1+\eta) \Im((x+y)/2)} \chi \left( \frac{x_a + y_a}{2}, \xi_a \right) \omega \right) dy_a \, d\xi_a,
\]
where \( \omega = \omega_{\eta} \) with \( \omega_{\eta} \) defined in Theorem 2.17. Then we have
\[
(45) \quad \tilde{P}_{\lambda, \psi} T v = \tilde{Q}_\lambda T v + \tilde{R}_\lambda T v + \tilde{g}_\lambda.
\]
Here \( \tilde{g}_\lambda \) and \( \tilde{R}_\lambda T v \) satisfy the following properties. For any \( N \in \mathbb{N} \), there exists some positive number \( C_N \) such that \( \tilde{R} T v \) and \( \tilde{g} \) satisfy
\[
(46) \quad \| \tilde{R}_\lambda T v \|_{L^2_{(1+\eta)^0}} \leq C_N \lambda^{-N} \| T v \|_{L^2_{(1+\eta)^0}(C^\infty(\mathbb{R}^n))},
\]
\[ \left\| g(\cdot, \lambda) \right\|_{L^1_{(\xi, \eta) = 0}} = O \left( e^{-\lambda/3} \| u \|_{H^0(\mathbb{R}^n)} \right), \quad \lambda \to +\infty, \]

where \( n_0 \) depends only on the dimension \( n \) and on the order of \( P \).

### 3. Main estimate

Let \( v \) be in \( C_0^\infty(\mathbb{R}^n)^N \) and set \( w = T_\eta^* T v \). It follows from Proposition 2.15 that

\[ T_\eta w(x) = T v(x). \]

Let us set

\[ Q_\lambda(x, \xi) = \chi \left( x_a - \frac{i \xi}{\eta}, \xi \right) P(Z, \xi \eta^i Z) \]

with \( Z = (x_a + (i \eta/(1 + \eta))\xi_a, x_b) \), then we deduce from Theorem 2.17

\[ \widetilde{Q}_\lambda T v(x) = \widetilde{Q}_\lambda T_\eta w(x) \]

\[ = T_\eta Q_\lambda w(x), \]

where \( Q_\lambda \) is the operator defined by

\[ Q_\lambda w(x) = \left( \frac{\lambda}{2\pi} \right)^n \int e^{i\lambda(x-y)\xi} Q_\lambda \left( \frac{x+y}{2}, \xi \right) w(y) \, dy \, d\xi. \]

Moreover setting

\[ Q_k(x, \xi) = \sum_{|\alpha| = k} \chi \left( x_a - \frac{i \xi}{\eta}, \xi \right) A_\alpha(Z)(\xi + i \psi'(Z))^\alpha, \]

we have

\[ Q_\lambda(x, \xi) = \sum_{k=0}^{m} \lambda^k Q_k(x, \xi). \]

#### 3.1. The estimate of \( Q_\lambda \)

In this section, we prove Carleman estimate for \( Q_\lambda \). First of all, we precise our choice of \( \psi \). Let \( \varphi \) and \( P_m \) satisfy the assumption in Theorem 1.1.

**Proposition 3.1.** There exists a quadratic polynomial \( \psi(x) \) such that

1. \( \psi(0) = \varphi(0) = 0, \ \psi'(0) = \varphi'(0) \).
2. There exists some neighborhood \( W \) of 0 such that

\[ x \in W \setminus \{0\} \cap \{x \mid \psi(x) = 0\} \implies \varphi(x) > 0. \]
Proof. We set
\[
\psi(x) = \varphi'(0)x + \frac{1}{2}(\varphi''(0)x, x) + (\varphi'(0)x)^2 - |x|^2.
\]
Then \(\psi\) obviously satisfies
\[
\psi(0) = \varphi(0) = 0, \quad \psi'(0) = \varphi'(0).
\]
And, when \(\psi(x) = 0\), there exists some \(C > 0\) such that
\[
\varphi'(0)x = -\frac{1}{2}(\varphi''(0)x, x) - (\varphi'(0)x)^2 + |x|^2 \\
\leq C|x|^2.
\]
Then, by Taylor’s formula, we have
\[
\varphi(x) = \varphi'(0)x + \frac{1}{2}(\varphi''(0)x, x) + O(|x|^3) \\
= -(\varphi'(0)x)^2 + |x|^2 + O(|x|^3) \\
\geq -(C|x|^2)^2 + |x|^2 + O(|x|^3).
\]
Therefore \(\varphi(x) > 0\) when \(x\) is small enough and \(x \neq 0\). The proposition is proved. \(\square\)

**Lemma 3.2.** There exist positive numbers \(\varepsilon_0, \delta\) such that if \(|x| + |\xi_a| < \varepsilon_0\), for any \(\xi_b \in \mathbb{R}^n\), we have
\[
Q_m(x, \xi)^*Q_m(x, \xi) - \delta(\xi_b)^{2m}I \geq 0.
\]

Proof. Because \(|x| + |\xi_a|\) is small enough, we may assume \(\chi(x_a + i\xi_a/(1 + \eta), \xi_a) = 1\). Then it follows that
\[
Q_m(x, \xi)^*Q_m(x, \xi) = P_m(X, \xi + i\psi'(X))^*P_m(X, \xi + i\psi'(X)),
\]
where \(X = (x_a + i\eta\xi_a/(1 + \eta), x_b)\).

(1) The case of large \(\xi_b\).

By the definition of \(P_m\), we have
\[
P_m(X, \xi + i\psi'(X))^*P_m(X, \xi + i\psi'(X)) \\
= P_m(X, 0, \xi_b)^*P_m(X, 0, \xi_b) + \sum_{|\beta| \leq 2m-1} B_\beta(X, \xi_a)\xi_b^\beta,
\]
where, for any \(\beta\), \(B_\beta(x, \xi_a)\) is an hermitian matrix whose components are in \(C^\infty(\mathbb{R}_x^n \times \mathbb{R}_{\xi})\).
\( \mathbb{R}_q^N \). Then by homogeneity of \( P_m \), for any \( w \in \mathbb{R}^N \) with \( |w| = 1 \), we have

\[
(Q_m(x, \xi)^* Q_m(x, \xi) w, w) = (P_m(X, 0, \xi) P_m(X, 0, \xi) w, w) + \sum_{|\beta| \leq 2m-1} \xi^\beta (B_\beta(X, \xi) w, w)
\]

\[
= |\xi|^2m \left\{ \left( P_m \left( X, 0, \frac{\xi}{|\xi|} \right) P_m \left( X, 0, \frac{\xi}{|\xi|} \right) w, w \right) + \frac{1}{|\xi|} \sum_{|\beta| \leq 2m-1} \frac{\xi^\beta}{|\xi|^{2m-1}} (B_\beta(X, \xi) w, w) \right\}.
\]

On the other hand, it follows from the ellipticity of \( P_m(0, 0, \xi) \) and the continuity of \( P_m(X, 0, \xi) \) that there exist positive constants \( \delta_1, \varepsilon_1 \) such that, if \( |\xi| = 1 \) and \( |x| + |\xi| < \varepsilon_1 \), we have

\[
(60) \quad (P_m(X, 0, \xi) P_m(X, 0, \xi) w, w) > 2\delta_1.
\]

And, by the continuity of \( B_\beta \), there exists some \( C_\beta > 0 \) which satisfies that, for any \( (x, \xi) \) with \( |x| + |\xi| < \varepsilon_1 \), we have

\[
(61) \quad |B_\beta(X, \xi)| < C_\beta.
\]

It follows that if we take \( C > 0 \) such that \( C^{-1} \sum_\beta C_\beta < \delta_1 \), for any \( \xi \) with \( |\xi| > C \), we have

\[
(Q_m(x, \xi)^* Q_m(x, \xi) w, w) \geq |\xi|^2m \left( 2\delta_1 - \frac{1}{|\xi|} \sum_{|\beta| \leq 2m-1} |B_\beta(X, \xi) w| \right)
\]

\[
\geq \delta_1 |\xi|^2m.
\]

Therefore if \( |\xi| > C \) and \( |x| + |\xi| < \varepsilon_1 \),

\[
(62) \quad Q_m(x, \xi)^* Q_m(x, \xi) - \delta_1 |\xi|^2m I
\]

is positive definite.

(2) The case that \( \xi \) is bounded.

We argue by contradiction. Assume that there exist sequences \( \{x_k\} \subset \mathbb{R}^n, \{\xi_k\} \subset \mathbb{R}^{n_x}, \{\xi_{bk}\} \subset \mathbb{R}^{n_b}, \{w_k\} \subset \mathbb{R}^N \) which satisfy

\[
|x_k| + |\xi_k| < \frac{1}{k}, \quad |\xi_{bk}| \leq C, \quad |w_k| = 1,
\]

\[
(Q_m(x_k, \xi_k, \xi_{bk})^* Q_m(x_k, \xi_k, \xi_{bk}) w_k, w_k) < \frac{1}{k} |\xi_{bk}|^{2m}.
\]
Because \( \{\xi_{bk}\}, \{w_k\} \) are bounded, we can take convergent sequences \( \{\xi_{bk}\}, \{w_k\} \) with \( \lim_{j \to \infty} \xi_{bk_j} = \xi_{b0}, \lim_{j \to \infty} w_{k_j} = w_0 \). By the fact that \( \lim_{j \to \infty} x_{k_j}, \lim_{j \to \infty} \xi_{a,k_j} \) are equal to 0, and by the continuity of \( Q_m \), we have

\[
(Q_m(0, 0, \xi_{b0})^* Q_m(0, 0, \xi_{b0})w_0, w_0) = 0,
\]

which contradicts that \( \det Q_m(0, 0, \xi_{b0}) = \det P_m(0, \varphi_0'(0), \xi_{b0} + i \varphi_0'(0)) \neq 0 \). Thus the lemma is proved.

Let \( h(x_b) \in C_0^\infty(\mathbb{R}^n_b: [0, 1]) \) satisfy

\[
h(x_b) = \begin{cases} 
1 & |x_b| < \frac{\varepsilon_1}{6}, \\
0 & |x_b| > \frac{\varepsilon_1}{3}, 
\end{cases}
\]

where \( \varepsilon_1 \) is a small parameter, and let \( \tilde{\vartheta}_0 \in C_0^\infty(\mathbb{C}^n \times \mathbb{C}^n: [0, 1]) \) be almost analytic in \( \Lambda_{(1+\eta)\Phi} \). Moreover we can take \( \tilde{\vartheta}_0 \) satisfies

\[
\tilde{\vartheta}_0(z_a, \xi_a) = \begin{cases} 
1 & |z_a| + |\xi_a| < \frac{\eta \varepsilon_1}{1 + \eta}, \\
0 & |z_a| + |\xi_a| > \frac{2\varepsilon_1}{3}.
\end{cases}
\]

When we define \( \theta_1(x, \xi_a) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) by

\[
\theta_1(x, \xi_a) = \tilde{\vartheta}_0 \circ \kappa_\eta(x_a, \xi_a) h(x_b),
\]

then we have

\[
\theta_1(x, \xi_a) = \begin{cases} 
1 & |x| + |\xi_a| < \frac{\varepsilon_1}{3}, \\
0 & |x| + |\xi_a| > \varepsilon_1.
\end{cases}
\]

**Lemma 3.3.** There exist positive numbers \( C_1, C_2, \lambda_0 \) such that, for any \( \lambda > \lambda_0 \) and any \( u \in \mathcal{S}(\mathbb{R}^n)^N \), we have

\[
C_1 \lambda^{2m} (\text{Op}_\lambda((1 - \theta_1)(\xi_{b})^{2m})(x, D_x)u, u) + C_2 \| Q_\lambda u \|^2 \geq \frac{\lambda^{2m} \delta}{2} \| (D_x / \lambda)^m u \|^2.
\]

**Proof.** By Lemma 3.2, we can take \( C_1 > 0 \) large enough such that

\[
C_1 (1 - \theta_1(x, \xi_a))(\xi_{b})^{2m} + Q_m(x, \xi)^* Q_m(x, \xi) - \delta (\xi_{b})^{2m} I \geq 0.
\]
By this inequality and Theorem 2.14, there exists some positive constant $C_3$ such that, for any $u \in S(\mathbb{R}^n)^N$,
\[
C_1(\text{Op}_\lambda((1 - \theta_1)(\xi_b)^{2m})u, u) + (Q_m^* Q_m)_{\lambda}(x, D_x)u, u) - \delta(|D_{x_b}/\lambda|^{2m}u, u)
\geq -\frac{C_3}{\lambda} \| |D_{x_b}/\lambda|^m u \| ^2.
\]

On the other hand, by the definition of $Q_m$, we have
\[
(Q_m^* Q_m)_{\lambda}(x, D_x) = Q_{m_{\lambda}}(x, D_x)^* Q_{m_{\lambda}}(x, D_x) + \frac{1}{\lambda} R_{\lambda}(x, D_x),
\]
where $R_{\lambda}(x, D_x)$ is a pseudodifferential system whose symbol $R(x, \xi, \lambda)$ be an hermitian matrix in $S_{2m}(\xi_b^{2m})$. Therefore we have
\[
C_1(\text{Op}_\lambda((1 - \theta_1)(\xi_b)^{2m})(x, D_x)u, u) + \| Q_{m_{\lambda}}(x, D_x)u \| ^2
\geq \left( \delta - \frac{C_3}{\lambda} \right) \| |D_{x_b}/\lambda|^m u \| ^2 - \frac{1}{\lambda} (R_{\lambda}(x, D_x)u, u).
\]

Moreover, by Theorem 2.13, there exists some positive number $C_4$ such that
\[
\| (R_{\lambda}(x, D_x)u, u) \| \leq \| (|D_{x_b}/\lambda|^{-m} R_{\lambda}(x, D_x)u) \| \| |D_{x_b}/\lambda|^m u \| 
\leq C_4 \| |D_{x_b}/\lambda|^m u \| ^2.
\]
Then when we take $\lambda_1 > 0$ with $2C_3/\lambda_1 < \delta/3$, it follows that, for any $\lambda > \lambda_1$,
\[
C_1(\text{Op}_\lambda((1 - \theta_1)(\xi_b)^{2m})(x, D_x)u, u) + \| Q_{m_{\lambda}}(x, D_x)u \| ^2 \geq \frac{2}{3} \| |D_{x_b}/\lambda|^m u \| ^2.
\]

On the other hand, by the definition of $Q_{\lambda}$, we have
\[
\| Q_{m_{\lambda}}(x, D_x)u \| ^2 = \| \lambda^{-m} Q_{\lambda}u - \sum_{k=0}^{m-1} \lambda^{k-m} Q_{\lambda_k}(x, D_x)u \| ^2 
\leq 2^m \left( \lambda^{-2m} \| Q_{\lambda}u \| ^2 + \sum_{k=0}^{m-1} \lambda^{2k-2m} \| Q_{\lambda_k}(x, D_x)u \| ^2 \right)
\leq 2^m \lambda^{-2m} \| Q_{\lambda}u \| ^2 + \frac{m C_4}{\lambda^2} \| |D_{x_b}/\lambda|^m u \| ^2,
\]
where $C_5$ is a positive constant independent of $u$ and $\lambda$. Then if we take $\lambda_2$ with $mC_4/\lambda_2^2 < \delta/6$, for any $\lambda > \lambda_2$, we have
\[
\| Q_{m, \lambda}(x, D_x)u \| ^2 \leq 2^m \lambda^{-2m} \| Q_{\lambda}u \| ^2 + \frac{\delta}{6} \| |D_{x_b}/\lambda|^m u \| ^2.
\]
It follows that, if $\lambda > \lambda_0 = \max(\lambda_1, \lambda_2)$, we have

$$C_1((\text{Op}_\lambda((1 - \theta_1)(\xi_b)^m)(x, D_x)u, u) + 2^m \lambda^{-2m} \|Q_\lambda u\|^2 \geq \frac{\delta}{2} \|D_{s_\lambda/\lambda} u\|^2,$$

which completes the proof.

**Lemma 3.4.** We can take positive real numbers $\varepsilon_2, \sigma_0$ which have the following properties: Let $v \in C_0^\infty(\mathbb{R}^n)^N$ satisfy supp $v \subset \{x : |x| < \varepsilon_2\}$. Then, for any $k \in \mathbb{N}$, there exist positive constants $C_{k,1}, C_{k,2}$ such that

$$\|\text{Op}_\lambda((1 - \theta_1)(\xi_b)^{2m})(x, D_x)T^*_\eta T v, T^*_\eta T v\| \leq \frac{C_{k,1}}{\lambda^k} \|D_{s_\lambda/\lambda} u\|_{L^2_{(1-m)\sigma}} + C_{2,k} e^{-\lambda \sigma_0} \|v\|^2_{H^0},$$

where $n_0$ is an integer depending only on the dimension $n$.

**Proof.** Using Proposition 2.15, we have

$$\text{Op}_\lambda((1 - \theta_1)(\xi_b)^{2m})(x, D_x)T^*_\eta T v, T^*_\eta T v) = (T_\eta \text{Op}_\lambda((1 - \theta_1)(\xi_b)^{2m})(x, D_x)T^*_\eta T v, T^*_\eta T v)_{L^2_{(1-m)\sigma}}.$$

And, by Theorem 2.17, we have

$$T_\eta \text{Op}_\lambda((1 - \theta_1)(\xi_b)^{2m})(x, D_x)T^*_\eta T v(x) = \left(\frac{\lambda}{2\pi}\right)^n \int \int e^{i\lambda(x_\sigma - y_\sigma)\xi_\sigma} ((\xi_b)^{2m}) \int_{-\xi_\sigma = (1+\eta) \text{Im}(x_\sigma + y_\sigma)/2} w_\eta d y_\sigma d \xi_\sigma,$$

where $w_\eta$ is defined by

$$w_\eta = e^{i\lambda(x_\sigma - y_\sigma)\xi_\sigma} \left(1 - \theta_1 \left(\frac{x_\sigma + y_\sigma}{2} + \frac{i\xi_\sigma}{1 + \eta}, \frac{x_\sigma + y_\sigma}{2}, \xi_\sigma\right)\right) T_{\eta} T^*_\eta T v(y_\sigma, y_\sigma) d y_\sigma \wedge d \xi_\sigma.$$

By the definition of $\theta_1$, we have

$$\theta_1 \left(\frac{x_\sigma + y_\sigma}{2} + \frac{i\xi_\sigma}{1 + \eta}, \frac{x_\sigma + y_\sigma}{2}, \xi_\sigma\right) = \theta_0 \left(\frac{x_\sigma + y_\sigma}{2}, \xi_\sigma\right) h \left(\frac{x_\sigma + y_\sigma}{2}\right)$$

and by Proposition 2.15,

$$T_\eta T^*_\eta T v = T v.$$

Therefore

$$w_\eta = e^{i\lambda(x_\sigma - y_\sigma)\xi_\sigma} \left(1 - \tilde{\theta}_0 \left(\frac{x_\sigma + y_\sigma}{2}, \xi_\sigma\right) h \left(\frac{x_\sigma + y_\sigma}{2}\right)\right) T v(y_\sigma, y_\sigma) d y_\sigma \wedge d \xi_\sigma.$$
We define \( \tilde{S}Tu \) by
\[
(83) \quad \tilde{S}Tu = \left( \frac{\lambda}{2\pi} \right)^n \int_A \int_{\xi_b} e^{i\lambda(x_y-y)b} \xi_b^{2m} \int_{\xi_b} w_{\eta} d\eta d\xi_b.
\]

By the same method in [8], we can show the following inequality: There exists some \( \varepsilon_2 > 0 \) such that if \( v \in C_0^{\infty}(\mathbb{R}^n)^N \) satisfy \( \text{supp} v \subset \{ x : |x| < \varepsilon_2 \} \), then, for any positive integer \( k \), we can take \( \sigma_0, C_{k,1}, C_{k,2} > 0 \) which satisfy
\[
(84) \quad \|\tilde{S}Tu\|_{L^2_{\{1+\varepsilon_0}\}} \leq \frac{C_{k,1}}{\lambda^k} \| (D_{xy}/\lambda)^m Tu \|_{L^2_{\{1+\varepsilon_0\}}} + C_{k,2} e^{-\lambda\varepsilon_2}\|v\|_{H^\varepsilon_0},
\]
where \( n_0 \) is a positive integer dependent only on the dimension \( n \). From this inequality, for any positive integer \( k \), we have
\[
\| (\text{Op}_2((1-\theta_1)(\xi_b)^{2m})(x, D_x)T_\eta^*Tv, T_\eta^*Tv) \|
\leq \left\| \tilde{S}Tu, Tu \right\|_{L^2_{\{1+\varepsilon_0\}}} \]
\[
\leq \frac{C_{k,1}}{\lambda^k} \| (D_{xy}/\lambda)^m Tu \|_{L^2_{\{1+\varepsilon_0\}}} + C_{k,2} e^{-\lambda\varepsilon_2}\|v\|_{H^\varepsilon_0} \|Tv\|_{L^2_{\{1+\varepsilon_0\}}}.
\]
which completes the proof because \( \|Tv\|_{L^2_{\{1+\varepsilon_0\}}} \leq \|Tv\|_{L^2_{\varepsilon_0}} = \|v\|_{L^2} \leq \|v\|_{H^\varepsilon_0}. \]

\[ \Box \]

**Theorem 3.5.** For \( \bar{P}_{\lambda, \psi} \) defined in Theorem 2.17, there exist some positive constants \( C_1, C_2, n_0, \lambda_0, \varepsilon, \sigma \) such that, for any \( v \in C_0^{\infty}(\mathbb{R}^n)^N \) with \( \text{supp} v \subset \{ x : |x| < \varepsilon \} \) and \( \lambda \geq \lambda_0 \), we have
\[
C_1 \lambda \bar{P}_{\lambda, \psi}Tu \|_{L^2_{\{1+\varepsilon_0\}}}^2 + C_2 e^{-\lambda\varepsilon_2} \|v\|_{H^\varepsilon_0}^2 \geq \|Tv\|_{L^2_{\{1+\varepsilon_0\}}}^2 \quad (\text{for} \; \partial u, H^1(\mathbb{R}^n)).
\]

**Proof.** By Proposition 2.15, we have
\[
\|Q_\lambda T_\eta^*Tv\|_{L^2_{\{1+\varepsilon_0\}}} = \|T_\eta Q_\lambda T_\eta^*Tv\|_{L^2_{\{1+\varepsilon_0\}}} = \|Q_\lambda T_\eta^*Tv\|_{L^2_{\{1+\varepsilon_0\}}} = \|\bar{Q}_\lambda T_\eta^*Tv\|_{L^2_{\{1+\varepsilon_0\}}}.
\]
Moreover, by choosing \( \varepsilon_1 \) with \( \varepsilon_1 < d \), it follows from Theorem 2.18 that there exist some constants \( C_{m,1}, C_{m,2} > 0 \) such that
\[
\|\bar{Q}_\lambda T_\eta^*Tv\|_{L^2_{\{1+\varepsilon_0\}}} = \|\bar{P}_{\lambda, \psi} Tv - \tilde{R}_\lambda Tv - \tilde{g}_\lambda\|_{L^2_{\{1+\varepsilon_0\}}} \leq \|\bar{P}_{\lambda, \psi} Tv\|_{L^2_{\{1+\varepsilon_0\}}} + \|\tilde{R}_\lambda Tv\|_{L^2_{\{1+\varepsilon_0\}}} + \|\tilde{g}_\lambda\|_{L^2_{\{1+\varepsilon_0\}}} \leq \|\bar{P}_{\lambda, \psi} Tv\|_{L^2_{\{1+\varepsilon_0\}}} + \frac{C_{m,1}}{\lambda^m} \|Tv\|_{L^2_{\{1+\varepsilon_0\}}} + C_{m,2} e^{-(1/3)\eta d^2\sigma_0} \|v\|_{H^\varepsilon_0}.
\]
On the other hand, if we take \( \varepsilon > 0 \) satisfying \( \varepsilon < \varepsilon_2 \), by Lemma 3.4, there exist some positive constants \( C_{m,3}, C_{m,4} \) such that
\[
\lambda^{2m} \left| \langle \text{Op}_h ((1 - \theta_1) \langle \xi \rangle^{2m}) (x, D_x) T_{\eta}^T T v, T_{\eta}^T T v \rangle \right| \\
\leq C_{m,3} \left\| (D_{x_b} / \lambda)^m T v \right\|_{L^2_{(1+m)\theta}}^2 + C_{m,4} \lambda^{2m} e^{-\lambda \sigma_0} \|v\|_{H^0}^2.
\]

Then, because \( \lambda^{2m} e^{-\lambda \sigma_0 / 2} < 1 \) with \( \lambda \) large enough, we have
\[
\lambda^{2m} \left| \langle \text{Op}_h ((1 - \theta_1) \langle \xi \rangle^{2m}) (x, D_x) T_{\eta}^T T v, T_{\eta}^T T v \rangle \right| \\
\leq C_{m,3} \left\| (D_{x_b} / \lambda)^m T v \right\|_{L^2_{(1+m)\theta}}^2 + C_{m,4} \lambda e^{-\lambda \sigma_0 / 2} \|v\|_{H^0}^2.
\]

It follows from these inequalities and Lemma 3.3 that, if \( \lambda \) is large enough, there exist positive constants \( C_5, C_6, C_7, C_8 \) such that
\[
C_5 \left| \tilde{P}_\lambda T v \right|_{L^2_{(1+m)\theta}}^2 + (C_6 e^{-\lambda / 2 \gamma_0} + C_7 e^{-(2/3)\eta d^2}) \|v\|_{H^0} \\
\geq \left( \frac{\delta \lambda^{2m} - C_8} {2} \right) \left\| (D_{x_b} / \lambda)^m T v \right\|_{L^2_{(1+m)\theta}}^2.
\]

Therefore setting \( \sigma = \min(\sigma_0 / 2, (2/3)\eta d^2) \), we have
\[
C_5 \left| \tilde{P}_\lambda T v \right|_{L^2_{(1+m)\theta}}^2 + (C_6 + C_7) e^{-\sigma \lambda} \|v\|_{H^0} \geq \frac{\delta} {4} \lambda^{2m} \left\| (D_{x_b} / \lambda)^m T v \right\|_{L^2_{(1+m)\theta}}^2,
\]
which completes the proof. \( \square \)

4. Proof of Main Theorem

In this section, using the estimate proved in the previous section, we prove our main theorem. Let us assume that \( P \) satisfies the condition of Theorem 1.1, that \( u \) is a \( C^\infty \)-solution near the origin of the equation \( Pu = 0 \) with \( \operatorname{supp} u \subset \{ x \in \mathbb{R}^n : \varphi(x) \leq 0 \} \), and that \( \psi \) is introduced in Proposition 3.1. Let \( \chi \in C_0^\infty (\mathbb{R} : [0, 1]) \) satisfy
\[
\chi(t) = \begin{cases} 
1 & t \geq -\frac{\varepsilon_2} {2}, \\
0 & t \leq -\varepsilon_2,
\end{cases}
\]
where \( \varepsilon_2 \) is a positive number small enough. We set
\[
u_1(x) = \chi(\psi(x))u(x).
\]

Then if \( \varepsilon_2 \) is small enough, it follows that there exists some neighborhood \( V_0 \) of 0, and some positive number \( C \) such that
\[
\operatorname{supp} u_1 \cap V_0 \subset \{ x : |x|^2 < C \varepsilon_2 \}.
\]
If \( \varepsilon_2 \) is small enough so that \( \varepsilon \geq \sqrt{C \varepsilon_2} \), we have
\[
\text{supp } u_1 \cap V_0 \subset \{ x : |x| < \varepsilon \}.
\]
Moreover we set \( Pu_1 = f \). Because \( Pu = 0 \), \( f \) satisfies
\[
\text{supp } f \subset \{ x : -\varepsilon \leq \psi(x) \leq -\varepsilon/2 \}.
\]
Let us set
\[
v = e^{\lambda \rho \psi} u_1,
\]
where \( \rho \) is a positive parameter. Then we have
\[
\widetilde{P}_{\lambda, \rho, \psi} T v = T(e^{\lambda \rho \psi} f).
\]
By Theorem 3.5, there exist positive constants \( \lambda_0, \sigma, C_1, C_2 \) such that, for \( \lambda > \lambda_0 \), we have
\[
\| T v \|_{L^2((\mathbb{R}^n, H^m(\mathbb{R}^n)))} \leq C_1 \lambda \| T(e^{\lambda \rho \psi} f) \|_{L^2((\mathbb{R}^n, H^m))} + C_2 e^{-\lambda \sigma} \| v \|_{H^m}.
\]
By using (95), we can show the following estimate:
\[
\| T(e^{\lambda \rho \psi} f) \|_{L^2((\mathbb{R}^n, H^m))} = O(e^{-\lambda \rho / 3}), \quad \lambda \to +\infty.
\]
Therefore, when \( \lambda \) is large enough, there exists some \( C_{\rho, \varepsilon} > 0 \) such that
\[
\| T v \|_{L^2((\mathbb{R}^n, H^m))} \leq C_{\rho, \varepsilon} e^{-\delta \lambda},
\]
where \( \delta = \min(\varepsilon \rho / 3, \sigma) \). By the same argument of the last section in [8], if \( \rho \) satisfies the inequality:
\[
\rho \| \psi''(0) \| < \frac{1}{4},
\]
there exists some neighborhood \( V \) of 0 where \( u = 0 \), which completes the proof of the main theorem.

References


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