Let $A$ be a finitely generated associative algebra over an algebraically closed field. We characterize the finite dimensional modules over $A$ whose orbit closures are regular varieties.

1. Introduction and the main result

Throughout the paper $k$ denotes a fixed algebraically closed field. By an algebra we mean an associative finitely generated $k$-algebra with identity, and by a module a finite dimensional left module. Let $d$ be a positive integer and denote by $\mathfrak{M}(d)$ the algebra of $d \times d$-matrices with coefficients in $k$. For an algebra $A$ the set $\text{mod}_A(d)$ of the $A$-module structures on the vector space $k^d$ has a natural structure of an affine variety. Indeed, if $A \cong k \langle X_1, \ldots, X_t \rangle / J$ for $t > 0$ and a two-sided ideal $J$, then $\text{mod}_A(d)$ can be identified with the closed subset of $(\mathfrak{M}(d))^t$ given by vanishing of the entries of all matrices $\rho(X_1, \ldots, X_t)$ for $\rho \in J$. Moreover, the general linear group $\text{GL}(d)$ acts on $\text{mod}_A(d)$ by conjugation and the $\text{GL}(d)$-orbits in $\text{mod}_A(d)$ correspond bijectively to the isomorphism classes of $d$-dimensional $A$-modules. We shall denote by $\mathcal{O}_M$ the $\text{GL}(d)$-orbit in $\text{mod}_A(d)$ corresponding to (the isomorphism class of) a $d$-dimensional $A$-module $M$. It is an interesting task to study geometric properties of the Zariski closure $\overline{\mathcal{O}_M}$ of $\mathcal{O}_M$. We note that using a geometric equivalence described in [4], this is closely related to a similar problem for representations of quivers. We refer to [2], [3], [4], [5], [6], [9], [10], [11], [12], [13] and [14] for results concerning geometric properties of orbit closures in module varieties or varieties of representations.

The main result of the paper concerns the global regularity of such varieties. Let $\text{Ann}(M)$ denote the annihilator of a module $M$. It is the kernel of the algebra homomorphism $A \rightarrow \text{End}_k(M)$ induced by the module $M$, and therefore the algebra $B = A / \text{Ann}(M)$ is finite dimensional. Obviously $M$ can be considered as a $B$-module.

**Theorem 1.1.** Let $M$ be an $A$-module and let $B = A / \text{Ann}(M)$. Then the orbit closure $\overline{\mathcal{O}_M}$ is a regular variety if and only if the algebra $B$ is hereditary and $\text{Ext}_B^1(M, M) = 0$. 

2000 Mathematics Subject Classification. Primary 14B05; Secondary 14L30, 16G20.
Let $d = \dim_k M$. Observe that $\text{mod}_B(d)$ is a closed $\text{GL}(d)$-subvariety of $\text{mod}_A(d)$ containing $\bar{O}_M$. Moreover, $M$ is faithful as a $B$-module. Hence we may reformulate Theorem 1.1 as follows:

**Theorem 1.2.** Let $M$ be a faithful module over a finite dimensional algebra $B$. Then the orbit closure $\bar{O}_M$ is a regular variety if and only if the algebra $B$ is hereditary and $\text{Ext}^1_B(M, M) = 0$.

The next section contains a reduction of the proof of Theorem 1.2 to Theorem 2.1 presented in terms of properties of regular orbit closures for representations of quivers. Sections 3 and 4 are devoted to the proof of Theorem 2.1. For basic background on the representation theory of algebras and quivers we refer to [1].

**2. Representations of quivers**

Let $Q = (Q_0, Q_1; s, t : Q_1 \to Q_0)$ be a finite quiver, i.e. $Q_0$ is a finite set of vertices, and $Q_1$ is a finite set of arrows $\alpha : s(\alpha) \to t(\alpha)$. By a representation of $Q$ we mean a collection $V = (V_i, V_j)$ of finite dimensional $k$-vector spaces $V_i$, $i \in Q_0$, together with linear maps $V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$, $\alpha \in Q_1$. The dimension vector of the representation $V$ is the vector

$$\dim V = (\dim_k V_i) \in \mathbb{N}^{Q_0}.$$

By a path of length $m \geq 1$ in $Q$ we mean a sequence of arrows in $Q_1$:

$$\omega = \alpha_m \alpha_{m-1} \cdots \alpha_2 \alpha_1,$$

such that $s(\alpha_{l+1}) = t(\alpha_l)$ for $l = 1, \ldots, m - 1$. In the above situation we write $s(\omega) = s(\alpha_1)$ and $t(\omega) = t(\alpha_m)$. We agree to associate to each $i \in Q_0$ a path $e_i$ in $Q$ of length zero with $s(e_i) = t(e_i) = i$. The paths of $Q$ form a $k$-linear basis of the path algebra $kQ$. We define

$$V_\omega = V_{\alpha_m} \circ V_{\alpha_{m-1}} \circ \cdots \circ V_{\alpha_2} \circ V_{\alpha_1} : V_{s(\omega)} \to V_{t(\omega)}$$

for a path $\omega = \alpha_m \cdots \alpha_1$ and extend easily this definition to $V_\rho : V_i \to V_j$ for any $\rho$ in $e_j \cdot kQ \cdot e_i$, where $i, j \in Q_0$, as $\rho$ is a $k$-linear combination of paths $\omega$ with $s(\omega) = i$ and $t(\omega) = j$. Finally, we set

$$\text{Ann}(V) = \{ \rho \in kQ \mid V_{e_j \rho e_i} = 0 \text{ for all } i, j \in Q_0 \},$$

which is a two-sided ideal in $kQ$. In fact, it is the annihilator of the $kQ$-module induced by $V$ with underlying $k$-vector space $\bigoplus_{i \in Q_0} V_i$. 


Let \( \mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0} \) be a dimension vector. Then the representations \( V = (V_i, V_\alpha) \) of \( Q \) with \( V_i = k^{d_i}, \ i \in Q_0, \) form a vector space

\[
\text{rep}_Q(\mathbf{d}) = \bigoplus_{a \in Q_1} \text{Hom}_k(V_{s(\alpha)}, V_{t(\alpha)}) = \bigoplus_{a \in Q_1} \mathbb{M}(d_{i(\alpha)} \times d_{i(\alpha)}),
\]

where \( \mathbb{M}(d' \times d'') \) stands for the space of \( d' \times d'' \)-matrices with coefficients in \( k \). For abbreviation, we denote the representations in \( \text{rep}_Q(\mathbf{d}) \) by \( V = (V_\alpha) \). The group \( \text{GL}(\mathbf{d}) = \bigoplus_{i \in Q_0} \text{GL}(d_i) \) acts regularly on \( \text{rep}_Q(\mathbf{d}) \) via

\[
(g_{i})_{i \in Q_0} \ast (V_\alpha)_{\alpha \in Q_1} = (g_{i(\alpha)} \cdot V_\alpha \cdot g_{s(\alpha)}^{-1})_{\alpha \in Q_1}.
\]

Given a representation \( W = (W_i, W_\alpha) \) of \( Q \) with \( \text{dim} \ W = \mathbf{d} \), we denote by \( \mathcal{O}_W \) the \( \text{GL}(\mathbf{d}) \)-orbit in \( \text{rep}_Q(\mathbf{d}) \) of representations isomorphic to \( W \).

Let \( M \) be a faithful module over a finite dimensional algebra \( B \). It is well known that the algebra \( B \) is Morita-equivalent to the quotient algebra \( kQ/I \), where \( Q \) is a finite quiver and \( I \) an admissible ideal in \( kQ \), i.e. \( I \) is a two-sided ideal such that \( (R_Q)^r \subseteq I \subseteq (R_Q)^2 \) for some positive integer \( r \), where \( R_Q \) denotes the two-sided ideal of \( kQ \) generated by the paths of length one (arrows) in \( Q \). Furthermore, the algebra \( B \) is hereditary if and only if \( I = [0] \) (in particular, the quiver \( Q \) has no oriented cycles, i.e. paths \( \omega \) of positive lengths with \( s(\omega) = t(\omega) \)). According to the above equivalence, the faithful \( B \)-module \( M \) corresponds to a representation \( N = (N_\alpha) \) in \( \text{rep}_Q(\mathbf{d}) \) for some \( \mathbf{d} \), such that \( \text{Ann}(N) = I \). Applying the geometric version of the Morita equivalence described by Bongartz in [4], \( \overline{\mathcal{O}}_M \) is isomorphic to an associated fibre bundle \( \text{GL}(d) \times_{\text{GL}(\mathbf{d})} \overline{\mathcal{O}}_N \). In particular, \( \overline{\mathcal{O}}_M \) is regular if and only if \( \overline{\mathcal{O}}_N \) is. By the Artin-Voigt formula (see [8]):

\[
\text{codim}_{\text{rep}_Q(\mathbf{d})} \overline{\mathcal{O}}_N = \text{dim}_k \text{Ext}^1_{Q}(N, N),
\]

the vanishing of \( \text{Ext}^1_{Q}(N, N) \) means that \( \overline{\mathcal{O}}_N = \text{rep}_Q(\mathbf{d}) \). Consequently, one implication in Theorem 1.2 is proved and it suffices to show the following fact:

**Theorem 2.1.** Let \( N \) be a representation in \( \text{rep}_Q(\mathbf{d}) \) such that \( \text{Ann}(N) \) is an admissible ideal in \( kQ \) and \( \overline{\mathcal{O}}_N \) is a regular variety. Then \( \text{Ann}(N) = [0] \) and \( \overline{\mathcal{O}}_N = \text{rep}_Q(\mathbf{d}) \).

3. **Tangent spaces of orbit closures and nilpotent representations**

From now on, \( N \) is a representation in \( \text{rep}_Q(\mathbf{d}) \) such that \( \text{Ann}(N) \) is an admissible ideal in \( kQ \) and \( \overline{\mathcal{O}}_N \) is a regular variety. The aim of the section is to prove that the quiver \( Q \) has no oriented cycles.

Let \( S[j] = (S[j], S[j]_\alpha) \) stand for the simple representation of \( Q \) such that \( S[j]_j = k \) is the only non-zero vector space and all linear maps \( S[j]_\alpha \) are zero, for any vertex.
$j \in Q_0$. Observe that the point 0 in $\text{rep}_Q(d)$ is the semisimple representation $\bigoplus_{i \in Q_0} S[i]^d$.

A representation $W = (W_i, W_o)$ of $Q$ is said to be nilpotent if one of the following equivalent conditions is satisfied:

1. The endomorphism $W_\omega \in \text{End}_k(W_{s(\omega)})$ is nilpotent for any oriented cycle $\omega$ in $Q$.
2. The ideal $\text{Ann}(W)$ contains $(R_Q)'$ for some positive integer $r$.
3. Any composition factor of $W$ is isomorphic to some $S[i], i \in Q_0$.
4. The orbit closure $\overline{O}_W$ in $\text{rep}_Q(\text{dim} W)$ contains 0.

Obviously the representation $N$ is nilpotent. Thus the set $N_Q(d)$ of nilpotent representations in $\text{rep}_Q(d)$ is a closed $GL(d)$-invariant subset which contains $\overline{O}_N$. Furthermore, $N_Q(d)$ is a cone, i.e. it is invariant under multiplication by scalars in the vector space $\text{rep}_Q(d)$.

We shall identify the tangent space $T_{\text{rep}_Q(d),0}$ of $\text{rep}_Q(d)$ at 0 with $\text{rep}_Q(d)$ itself. Thus the tangent space $T_{\overline{O}_N,0}$ is a subspace of $\text{rep}_Q(d)$ and is invariant under the action of $GL(d)$, i.e. it is a $GL(d)$-subrepresentation of $\text{rep}_Q(d)$. Since $\overline{O}_N$ is a regular variety, the tangent space $T_{\overline{O}_N,0}$ is contained in the tangent cone of $\overline{O}_N$ at 0. Therefore

\[(3.1) \quad T_{\overline{O}_N,0} \subseteq N_Q(d).\]

**Lemma 3.1.** Let $W = (W_\omega)$ be a tangent vector in $T_{\overline{O}_N,0}$. Then $W_\gamma = 0$ for any loop $\gamma \in Q_1$.

Proof. Suppose that the nilpotent matrix $W_\gamma \in M(d_j)$ is non-zero for some loop $\gamma: j \rightarrow j$ in $Q_1$. Then there are two linearly independent vectors $v_1, v_2 \in k^{d_j}$ such that $W_\gamma \cdot v_1 = v_2$ and $W_\gamma \cdot v_2 = 0$. We choose $g = (g_i)$ in $GL(d)$ such that $g_j \cdot v_1 = v_2$ and $g_j \cdot v_2 = v_1$. Then $U = W + g \ast W$ belongs to $T_{\overline{O}_N,0}$. Observe that $U_\gamma \cdot v_1 = v_2$ and $U_\gamma \cdot v_2 = v_1$. Hence the representation $U$ is not nilpotent, contrary to (3.1). \qed

Let $V_i = k^{d_i}$ and $R_{i,j}$ be the vector space of formal linear combinations of arrows $\alpha \in Q_1$ with $s(\alpha) = i$ and $t(\alpha) = j$, for any $i, j \in Q_0$. We shall identify:

\[
\text{rep}_Q(d) = \bigoplus_{i,j \in Q_0} \text{Hom}_k(R_{i,j}, \text{Hom}_k(V_i, V_j)) \quad \text{and} \quad GL(d) = \bigoplus_{i \in Q_0} GL(V_i).
\]

Applying Lemma 3.1 we get

\[
T_{\overline{O}_N,0} \subseteq \bigoplus_{i,j \in Q_0} \text{Hom}_k(R_{i,j}, \text{Hom}_k(V_i, V_j)).
\]
Since the GL(d)-representations \( \text{Hom}_k(V_i, V_j), \ i \neq j, \) are simple and pairwise non-isomorphic, we have

\[
T_{\mathcal{V}_i,0} = \bigoplus_{i,j \in Q_0, i \neq j} \{ \varphi: R_{i,j} \rightarrow \text{Hom}_k(V_i, V_j) \mid \varphi(U_{i,j}) = 0 \}
\]

for some subspaces \( U_{i,j} \) of \( R_{i,j}, \ i \neq j. \)

The spaces \( U_{i,j} \) are not necessarily spanned by arrows \( \alpha: i \rightarrow j \) in \( Q_1, \) and we are going to replace \( N \) by a “better” representation in \( \text{rep}_Q(d). \) The group \( \widetilde{G} = \bigoplus_{i,j \in Q_0} \text{GL}(R_{i,j}) \) can be identified naturally with a subgroup of automorphisms of the path algebra \( kQ \) which change linearly the paths of length 1 but do not change the paths of length 0. Let \( \tilde{g} = (\tilde{g}_{i,j}) \) be an element of \( \widetilde{G}. \) Then \( \tilde{g} \star (\mathcal{R}_Q)^p = (\mathcal{R}_Q)^p \) for any positive integer \( p, \) where \( \star \) denotes the action of \( \widetilde{G} \) on \( kQ. \)

For a representation \( W \) of \( Q \) presented in the form

\[
W = (W_i, W_{i,j}: R_{i,j} \rightarrow \text{Hom}_k(W_i, W_j))_{i,j \in Q_0},
\]

we define the representation

\[
\tilde{g} \star W = (W_i, W_{i,j} \circ (\tilde{g}_{i,j})^{-1})_{i,j \in Q_0}.
\]

Hence \( \widetilde{G} \) acts regularly on \( \text{rep}_Q(d) \) and this action commutes with the GL(d)-action. Therefore the orbit closure \( \overline{\mathcal{O}}_{\tilde{g} \star N} = \overline{\mathcal{O}}_N \) is a regular variety, \( T_{\mathcal{O}_{\tilde{g} \star N},0} = \tilde{g} \star T_{\mathcal{O}_{N},0} \) and the ideal \( \text{Ann}(\tilde{g} \star N) = \tilde{g} \star \text{Ann}(N) \) is admissible as

\[
(\mathcal{R}_Q)^p = \overline{\mathcal{O}}_N \subseteq \tilde{g} \star \text{Ann}(N) \subseteq (\tilde{g} \star (\mathcal{R}_Q)^p = (\mathcal{R}_Q)^2.
\]

Hence, replacing \( N \) by \( \tilde{g} \star N \) for an appropriate \( \tilde{g}, \) we may assume that the spaces \( U_{i,j}, \ i \neq j, \) are spanned by arrows in \( Q_1. \) Consequently,

\[
(3.2) \quad T_{\mathcal{O}_{N},0} = \text{rep}_Q(d) \subseteq \text{rep}_Q(d)
\]

for some subquiver \( Q' \) of \( Q \) such that \( Q'_0 = Q_0 \) and \( Q'_1 \) has no loops.

**Lemma 3.2.** The quiver \( Q' \) has no oriented cycles.

**Proof.** Suppose there is an oriented cycle \( \omega \) in \( Q'. \) Let \( W = (W_\alpha) \) be a tangent vector in \( T_{\mathcal{O}_{\omega},0} = \text{rep}_Q(d) \) such that each \( W_\alpha, \ \alpha \in (Q')_1, \) is the matrix whose \( (1,1)-\)entry is 1, while the other entries are 0. Then the matrix \( W_\omega \) has the same form, contrary to (3.1).

Let \( W = (W_i, W_\alpha) \) be a representation of \( Q. \) We denote by \( \text{rad}(W) \) the radical of \( W. \) In case \( W \) is nilpotent, \( \text{rad}(W) = \sum_{\alpha \in Q_1} \text{Im}(W_\alpha). \) We write \( \langle w \rangle \) for the sub-representation of \( W \) generated by a vector \( w \in \bigoplus_{i \in Q_0} W_i. \)
Lemma 3.3. Let $\alpha : i \to j$ be an arrow in $Q_1$ such that $N_\alpha(v)$ does not belong to $\text{rad}^2(v)$ for some $v \in V_i$. Then $\alpha \in Q'_1$.

Proof. Let $d = \sum_{i \in Q_0} d_i$ and $c = \dim_k(v)$. Then $\dim_k \text{rad}(v) = c - 1$ and $d \geq c \geq 2$. Since $N_\alpha(v)$ does not belong to $\text{rad}(\text{rad}(v))$, there is a codimension one sub-representation $W$ of $\text{rad}(v)$ which does not contain $N_\alpha(v)$. We choose a basis $[\epsilon_1, \ldots, \epsilon_d]$ of the vector space $\bigoplus_{i \in Q_0} V_i$ such that:

- the vector $\epsilon_b$ belongs to $V_i$ for some vertex $i_b \in Q_0$, for any $b \leq d$;
- the vectors $\epsilon_1, \ldots, \epsilon_b$ span a sub-representation, say $N(b)$, of $N$ for any $b \leq d$;
- $N(c-2) = W$, $\epsilon_{c-1} = N_\alpha(v)$, $N(c-1) = \text{rad}(v)$, $\epsilon_c = v$ and $N(c) = (v)$.

In fact, $0 = N(0) \subset N(1) \subset N(2) \subset \cdots \subset N(d) = N$ is a composition series of $N$. In particular, $N_\beta(\epsilon_b)$ belongs to $N(b-1)$, for any $b \leq d$ and any arrow $\beta : i_b \to j$ in $Q_1$.

We take a decreasing sequence of integers

$$p_1 > p_2 > \cdots > p_d$$

and define a group homomorphism $\varphi : k^d \to \text{GL}(d) = \bigoplus_{i \in Q_0} \text{GL}(V_i)$ such that $\varphi(t)(\epsilon_b) = t^{p_b} \cdot \epsilon_b$ for any $b \leq d$. Observe that

$$N_\beta(\epsilon_b) = \sum_{i < b} \lambda_i \cdot \epsilon_i, \quad \lambda_i \in k, \quad \text{implies} \quad (\varphi(t) * N)_\beta(\epsilon_b) = \sum_{i < b} t^{p_i-p_b} \lambda_i \cdot \epsilon_i$$

for any $b \leq d$ and any arrow $\beta : i_b \to j$ in $Q_1$. This leads to a regular map $\psi : k \to \mathcal{O}_N$ such that $\psi(t) = \varphi(t) * N$ for $t \neq 0$ and $\psi(0) = 0$.

Assume now that $p_{c-1} - p_c = 1$. Applying the induced linear map $T_{\psi,0} : T_{\psi,0} \to T_{\mathcal{O}_N,0}$ and using the fact that $N\alpha(\epsilon_c) = \epsilon_{c-1}$, we obtain a tangent vector $W = (W_\alpha) \in T_{\mathcal{O}_N,0}$ such that $W_\alpha(\epsilon_c) = \epsilon_{c-1} \neq 0$. Thus $\alpha \in Q'_1$. \hfill $\square$

Lemma 3.4. For any arrow $\alpha : i \to j$ in $Q_1$, there exists a path $\omega$ in $Q'$ of positive length such that $s(\omega) = i$ and $t(\omega) = j$.

Proof. Since $\text{Ann}(N)$ is an admissible ideal in $kQ$, there is a vector $v \in V_i$ such that $N_\alpha(v) \neq 0$. Let $\omega = \alpha_m \cdots \alpha_2 \alpha_1$ be a longest path from $i$ to $j$ with $N_\omega(v) \neq 0$. Hence $N_\omega(v) = 0$ for any $\rho \in \epsilon_j \cdot (\mathcal{R}_Q)^{m+1} \cdot \epsilon_i$. We show that the path $\omega$ satisfies the claim. Let $v_0 = v$ and $v_l = N_\alpha(v_{l-1})$ for $l = 1, \ldots, m$. According to Lemma 3.3, it is enough to show that $v_l \notin \text{rad}^2(v_{l-1})$ for any $1 \leq l \leq m$. Indeed, if $v_l \in \text{rad}^2(v_{l-1})$ for some $l$, then $v_m \in \text{rad}^{m+1}(v_0)$, or equivalently, $N_\omega(v) = N_\rho(v)$ for some $\rho \in \epsilon_j \cdot (\mathcal{R}_Q)^{m+1} \cdot \epsilon_i$, a contradiction. \hfill $\square$

Combining Lemmas 3.2 and 3.4, we get

Corollary 3.5. The quiver $Q$ does not contain oriented cycles.
4. Gradings of polynomials on $\text{rep}_Q(d)$

Let $\pi : \text{rep}_Q(d) \to \text{rep}_Q'(d)$ denote the obvious $\text{GL}(d)$-equivariant linear projection and let $N' = \pi(N)$. Then $\pi(O_N) = O_{N'}$ and we get a dominant morphism

$$\eta = \pi |_{O_N} : O_N \to O_{N'}.$$ 

**Lemma 4.1.** $O_{N'} = \text{rep}_Q'(d)$.

**Proof.** Since $\text{Ker}(\pi) \cap T_{O_{N'},0} = \{0\}$, the morphism $\eta$ is étale at 0. This implies that the variety $O_{N'}$ is regular at $\eta(0) = 0$ (see [7, III. 5] for basic information about étale morphisms). Since it is contained in $\text{rep}_Q'(d)$, it suffices to show that $T_{O_{N'},0} = \text{rep}_Q'(d)$. The latter can be concluded from the induced linear map $T_{\eta,0} : T_{O_N,0} \to T_{O_{N'},0}$, which is the restriction of $T_{\eta,0} = \pi$.

Let $R = k[X_{\alpha, p, q}]_{\alpha \in Q_1, p \leq d_{\mu}, q \leq d_{\nu}}$ denote the algebra of polynomial functions on the vector space $\text{rep}_Q(d)$ and $m = (X_{\alpha, p, q})$ be the maximal ideal in $R$ generated by variables. Here, $X_{\beta, p, q}$ maps a representation $W = (W_\alpha)$ to the $(p, q)$-entry of the matrix $W_\beta$. Using $\pi$, the polynomial functions on $\text{rep}_Q'(d)$ form the subalgebra $R' = k[X_{\alpha, p, q}]_{\alpha \in Q_1, p \leq d_{\mu}, q \leq d_{\nu}}$ of $R$. By Lemma 4.1,

$$I(O_N) \cap R' = \{0\},$$

where $I(O_N)$ stands for the ideal of the set $O_N$ in $R$.

Let $X_\alpha$ denote the $d_{(\alpha)} \times d_{(\alpha)}$-matrix whose $(p, q)$-entry is the variable $X_{\alpha, p, q}$ for any arrow $\alpha$ in $Q_1$. We define the $d_j \times d_i$-matrix $X_\rho$ for $\rho \in e_j \cdot kQ \cdot e_i$, with coefficients in $R$, in a similar way as for representations of $Q$.

The action of $\text{GL}(d)$ on $\text{rep}_Q(d)$ induces an action on the algebra $R$ by $(g \ast f)(W) = f(g^{-1} \ast W)$ for $g \in \text{GL}(d)$, $f \in R$ and $W \in \text{rep}_Q(d)$. We choose a standard maximal torus $T$ in $\text{GL}(d)$ consisting of $g = (g_i)$, where all $g_i \in \text{GL}(d_i)$ are diagonal matrices. Let $Q_0$ denote the set of pairs $(i, p)$ with $i \in Q_0$ and $1 \leq p \leq d_i$. Then the action of $T$ on $R$ leads to a $Z^{|Q_0|}$-grading on $R$ with

$$\deg(X_{\alpha, p, q}) = e_{s(\alpha), q} - e_{t(\alpha), p},$$

where $\{e_{i, p}\}_{(i, p) \in Q_0}$ is the standard basis of $Z^{|Q_0|}$.

**Proposition 4.2.** $Q' = Q$.

**Proof.** Suppose the contrary, which means there is an arrow $\beta$ in $Q_1 \setminus Q_1$. Since the quiver $Q$ has no oriented cycles, we can choose $\beta$ minimal in the sense that any path $\omega$ in $Q$ of length greater than 1 with $s(\omega) = s(\beta)$ and $t(\omega) = t(\beta)$ is in fact a path
in $Q'$. We conclude from (3.2) that $X_{\beta, u, v} \in m^2 + I(\mathcal{O}_N)$ for $u \leq d_i(\beta)$ and $v \leq d_i(\beta)$.

Since the polynomials $X_{\beta, u, v}$ as well as the ideals $m^2$ and $I(\mathcal{O}_N)$ are homogeneous with respect to the above grading, there are homogeneous polynomials $f_{\beta, u, v}$ in the ideal $m^2$ such that

$$X_{\beta, u, v} - f_{\beta, u, v} \in I(\mathcal{O}_N) \quad \text{and} \quad \deg(f_{\beta, u, v}) = e(s(\beta), v) - e(t(\beta), u).$$

Let $\prod_{l \leq n} X_{\alpha_l, p_l, q_l}$ be a monomial in $R$ of degree $e(s(\beta), v) - e(t(\beta), u)$. Then

$$\#\{1 \leq l \leq n \mid s(\alpha_l) = i, \ q_l = r\} - \#\{1 \leq l \leq n \mid t(\alpha_l) = i, \ p_l = r\}$$

$$= \begin{cases} 1 & (i, r) = (s(\beta), v), \\ -1 & (i, r) = (t(\beta), u), \\ 0 & \text{otherwise}. \end{cases}$$

Thus by (4.2), up to a permutation of the above variables, we get that $\omega = \alpha_m \cdots \alpha_1$ is a path in $Q$ for some $m \leq n$ such that $(s(\alpha), q_1) = (s(\beta), v)$, $(t(\alpha), p_m) = (t(\beta), u)$ and $q_l = p_{l-1}$ for $l = 2, \ldots, m$. Consequently, $\deg(X_{\alpha_m, p_m, q_m} \cdots X_{\alpha_2, p_2, q_2} X_{\alpha_1, p_1, q_1}) = 0$. Since $Q$ has no oriented cycles, the only monomial in $R$ with degree zero is the constant function 1. Hence $m = n$ and the homogenous polynomial $f_{\beta, u, v}$ is the following linear combination:

$$f_{\beta, u, v} = \sum \lambda(u, \alpha_m, p_{m-1}, \alpha_{m-1}, \ldots, p_1, \alpha_1, v) \cdot X_{\alpha_m, p_m, q_m} \cdots X_{\alpha_2, p_2, p_1} X_{\alpha_1, p_1, q_1},$$

where the sum runs over all paths $\omega = \alpha_m \cdots \alpha_1$ in $Q$ with $s(\omega) = s(\beta)$, $t(\omega) = t(\beta)$ and positive integers $p_i \leq d_i(\alpha_i)$ for $l = 1, \ldots, m - 1$. Since $f_{\beta, u, v}$ belongs to the ideal $m^2$, we may assume that $m \geq 2$. Then the arrows $\alpha_1, \ldots, \alpha_m$ belong to $Q'$, by the minimality of $\beta$. In particular, $f_{\beta, u, v}$ belongs to $R'$.

We claim that the scalars $\lambda(u, \alpha_m, p_{m-1}, \alpha_{m-1}, \ldots, p_1, \alpha_1, v)$ do not depend on the integers $u$, $p_{m-1}$, $\ldots$, $p_1$ and $v$. Indeed, take $u' \leq d_i(\beta)$, $v' \leq d_i(\beta)$ and $p'_l \leq d_i(\alpha_l)$ for $l = 1, \ldots, m - 1$. We choose $g = (g_l)$ in GL($d$) with each $g_l$ being the permutation matrix associated to a specific permutation $\sigma_l \in S_d$. Then the multiplication by $g$ in the algebra $R$ permutes the monomials in $R$. We assume that

$$\sigma_{s(\beta)}(u') = u', \quad \sigma_{s(\beta)}(v') = v, \quad \sigma_{t(\beta)}(u) = u', \quad \sigma_{t(\beta)}(v) = v,$$

$$\sigma_{t(\alpha_l)}(p_l) = p'_l \quad \text{and} \quad \sigma_{t(\alpha_l)}(q_l) = q'_l, \quad \text{for} \quad l = 1, \ldots, m - 1.$$

Since $g \ast X_{\beta, u', v'} = X_{\beta, u, v}$, the polynomial

$$f_{\beta, u, v} - g \ast f_{\beta, u', v'} = g \ast (X_{\beta, u', v'} - f_{\beta, u', v'}) - (X_{\beta, u, v} - f_{\beta, u, v})$$
belongs to the ideal \( I(\Omega_N) \), as the latter is \( \text{GL}(d) \)-invariant. Thus \( f_{\beta,u,v} = g \cdot f_{\beta',u',v'} \) by (4.1). Hence the claim follows from the fact that the monomial

\[
X_{\alpha_n,u,p_{n-1}} \cdot X_{\alpha_{n-1},p_{n-1},p_{n-2}} \cdots \cdot X_{\alpha_2,p_2,p_1} \cdot X_{\alpha_1,p_1,v}
\]

appears in \( g \cdot f_{\beta,u,v} \) with coefficient \( \lambda(u', \alpha_m, p'_{m-1}, \alpha_{m-1}, \ldots, p'_1, \alpha_1, v') \).

Let \( \Xi \) denote the set of all paths \( \xi \) in \( Q' \) of length greater than 1 with \( s(\xi) = s(\beta) \) and \( t(\xi) = t(\beta) \). Then there are scalars \( \lambda(\xi), \xi \in \Xi \), such that

\[
f_{\beta,u,v} = \sum_{\xi = \alpha_m \ldots \alpha_1 \in \Xi} \lambda(\xi) \cdot \sum_{p_1 \leq d_{i(\beta)}} \cdots \sum_{p_{n-1} \leq d_{i(n-1)}} X_{\alpha_n,u,p_{n-1}} \cdots \cdot X_{\alpha_1,p_1,v}
\]

for any \( u \leq d_{i(\beta)} \) and \( v \leq d_{s(\beta)} \). This equality means that \( f_{\beta,u,v} \) is the \((u, v)\)-entry of the matrix \( X_\rho \), where \( \rho = \sum_{\xi \in \Xi} \lambda(\xi) \cdot \xi \in kQ' \). Consequently, the entries of the matrix \( X_{\beta - \rho} \) belong to the ideal \( I(\Omega_N) \). This implies that \( \beta - \rho \) belongs to \( \text{Ann}(N) \). Since \( \beta - \rho \) does not belong to \( (R_\Omega)^2 \), the ideal \( \text{Ann}(N) \) is not admissible, a contradiction. 

Combining Lemma 4.1 and Proposition 4.2 we get

\[
(4.3) \quad \Omega_N = \text{rep}_Q(d).
\]

Hence the following lemma finishes the proof of Theorem 2.1.

**Lemma 4.3.** \( \text{Ann}(N) = \{0\} \).

Proof. Suppose the contrary, that there is a non-zero element \( \rho \) in \( e_j \cdot \text{Ann}(N) \cdot e_i \) for some vertices \( i \) and \( j \). Observe that the set of representations \( W = (W_\alpha) \) in \( \text{rep}_Q(d) \) such that \( W_\rho = 0 \) is closed and \( \text{GL}(d) \)-invariant. Hence \( W_\rho = 0 \) for any representation \( W = (W_\alpha) \) in \( \text{rep}_Q(d) \), by (4.3). Of course, \( \rho \) is a linear combination of paths in \( Q \) of length greater than 1 with \( s(\omega) = i \) and \( t(\omega) = j \). Let \( \omega_0 \) be a path appearing in \( \rho \) with coefficient \( \lambda \neq 0 \). We choose a representation \( W = (W_\alpha) \) in \( \text{rep}_Q(d) \) such that \( W_\alpha \) is the matrix whose \((1, 1)\)-entry is 1 and the other entries are 0 if the arrow \( \alpha \) appears in the path \( \omega_0 \), and \( W_\alpha = 0 \) otherwise. Then the \((1, 1)\)-entry of \( W_\rho \) equals \( \lambda \), a contradiction.

**Acknowledgments.** The second author gratefully acknowledges support from the Polish Scientific Grant KBN No. 1 P03A 018 27.
References


Nguyen Quang Loc
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18, 87–100 Toruń
Poland
e-mail: loc@mat.uni.torun.pl

Grzegorz Zwara
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18, 87–100 Toruń
Poland
e-mail: gzwara@mat.uni.torun.pl