THE TAIL ESTIMATION OF THE QUADRATIC VARIATION OF A QUASI LEFT CONTINUOUS LOCAL MARTINGALE

SHUNSUKE KAJI

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Abstract

We discuss some estimates of the tail distributions of the supremum and the quadratic variation of a local martingale. The assumption made so far in the literature on exponential moments involving a quasi left continuous local martingale is improved.

1. Introduction and main result

There have been a number of works on tail distributions of the supremum and the quadratic variation of a local martingale. On the other hand, in the paper [7] Kotani gave a necessary and sufficient condition for one-dimensional diffusion processes to be martingales. In Azéma, Gundy, and Yor [1], the uniform integrability of a continuous martingale in terms of tails of its supremum and quadratic variation was first characterized. The existence of the limits of the tails was considered by Galtchouk and Novikov [5] (for a discrete time martingale), Novikov [10], Elworthy, Li, and Yor [2], [3], Madan and Yor [9] (for a continuous local martingale), Liptser and Novikov [8], and Kaji [6] (for a càdlàg local martingale) by using the Tauberian theorem. In the statements on the quadratic variation of a local martingale, the existence of some exponential moments involving a local martingale is assumed, but Takaoka [11] relaxed its assumption for a continuous local martingale. In this paper we also do so for a càdlàg local martingale.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+}, P)\) be a filtered probability space with usual conditions, where \(\mathbb{R}_+ = [0, \infty)\), and \(M = \{M_t\}_{t \in \mathbb{R}_+}\) is a càdlàg local martingale with \(M_0 = 0\) defined on it. We denote by \(\mu\) the random measure on \(\mathbb{R}_+ \times \mathbb{X}\) such that for all \(t \in \mathbb{R}_+\) and Borel subsets \(U\) of \(\mathbb{X}\)

\[
\mu(\cdot, (0, t] \times U) = \sum_{0 < s \leq t} \mathbb{1}_U(\Delta M_s),
\]

where \(\mathbb{X} = \mathbb{R} - \{0\}\) and \(\Delta M_t = M_t - M_{t^-}\), \(t > 0\). That is, \(\mu\) is the counting measure of jumps of \(M\). Then we denote by \(\hat{\mu}\) its predictable compensator. If \(M\) is a locally square integrable martingale, then it is well-known that we can define a predictable

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quadratic variation process \((M) = [(M)_t]_{t \in \mathbb{R}}\), and an optional quadratic variation process \([M] = [(M)_t]_{t \in \mathbb{R}}\), and the canonical decomposition

\[ M = M^c + M^d \]

holds, where \(M^c\) is a continuous local martingale with \(M^c_0 = 0\) and \(M^d\) is a stochastic integral process with respect to \(\mu^c\) defined as

\[ M^d_t = \int_{(0,t] \times X} x\{ \mu(\cdot, ds dx) - \mu^c(\cdot, ds dx) \}, \quad t \in \mathbb{R}_+. \]

Moreover recall that

\[ \langle M^d \rangle_t = \int_{(0,t] \times X} x^2 \mu^c(\cdot, ds dx), \quad t \in \mathbb{R}_+. \]

First, we recall the result by Liptser and Novikov [8].

**Theorem 1.1.** Assume that \(M\) is a locally square integrable martingale, \((M)_{\infty} = \lim_{t \to \infty} (M)_t < \infty\) a.s., and \([M^*_T]_{T \in \mathcal{T}}\) is uniformly integrable, where \(\mathcal{T}\) is the set of stopping times \(T\). Then

(i) \(0 \leq E[M_{\infty}] \leq E[M^*_T] < \infty\).

Besides,

(ii) if \([\Delta M_T]_{T \in \mathcal{T}}\) is uniformly integrable, then

\[ \lim_{\lambda \to \infty} \lambda P\left( \sup_{t \in \mathbb{R}_+} (M^*_t) > \lambda \right) = E[M_{\infty}]; \]

(iii) if \(|\Delta M| \leq K\) and \(E[e^{\epsilon M_{\infty}}] < \infty\) for some \(K > 0\), and \(\epsilon > 0\), then

\[ \lim_{\lambda \to \infty} \lambda P(\sqrt{(M)_{\infty}} > \lambda) = \lim_{\lambda \to \infty} \lambda P(\sqrt{[M]_{\infty}} > \lambda) = \frac{2}{\pi} E[M_{\infty}]. \]

Here we notice that the uniform boundedness for jumps is assumed in the above result. But Kaji [6] gave the following improvement.

**Theorem 1.2.** Assume the existence of the random variable \(M_{\infty}\) such that \(\lim_{t \to \infty} M_t = M_{\infty} < \infty\) a.s. and that \([M^*_T]_{T \in \mathcal{T}}\) is uniformly integrable. Then

(i) \(-\infty < -E[M_{\infty}] \leq E[M_{\infty}] \leq 0\)

holds. Besides, if \([\Delta M_T]_{T \in \mathcal{T}}\) is uniformly integrable, then

(ii) \(\lim_{\lambda \to \infty} \lambda P(\sup_{t \in \mathbb{R}_+} M_t > \lambda) = -E[M_{\infty}]. \)
Theorem 1.3. Assume that \( M \) is a locally square integrable martingale and that \( \langle M \rangle_\infty < \infty \) a.s., \( [M^-]_t \) is uniformly integrable, and there exists \( \lambda_0 > 0 \) such that

\[
E \left[ \exp \left\{ \lambda_0 M^-_\infty + \int_{R_+ \times \{ |x| > K \}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx) \right\} \right] < \infty
\]

for some \( K > 0 \), where \( \phi_\lambda(x) = e^{-\lambda x} - 1 + \lambda x - (\lambda^2/2)x^2 \). Then

(i) \( \lim_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{\langle M \rangle_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty] \).

(ii) \( \lim_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{[M]_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty] \).

As a remark, we note that the condition (1) refines the conditions “\( |\Delta M| \leq K \) and \( E[e^{\lambda_0 M^-}] < \infty \) for some \( \lambda_0, K > 0 \)”.

Finally, we introduce our main result:

Theorem 1.4. Assume that \( M \) is a locally square integrable martingale and quasi left continuous, \( \langle M \rangle_\infty < \infty \) a.s., \( [M^-]_t \) is uniformly integrable.

(i) Assume moreover that there exists \( \lambda_0 > 0 \) such that

\[
E \left[ \int_{R_+ \times \{ |x| > K \}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx) \right] < \infty
\]

for some \( K > 0 \). Then

\[
\lim_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{\langle M \rangle_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty].
\]

(ii) On the other hand, if we assume that there exists \( \lambda_0 > 0 \) such that

\[
E \left[ \left( \int_{R_+ \times \{ |x| > K \}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx) \right)^{2\alpha} \right] < \infty
\]

for some \( K > 0, \alpha > 0 \). Then

\[
\lim_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{[M]_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty].
\]

The proof of the above shall be divided in three steps. As a first step, we will relax the assumption involving the finiteness of some exponential moment of a local martingale in Theorem 1.3, but we assume its quasi left continuity:
Theorem 1.5. Assume that $M$ is a locally square integrable martingale and quasi left continuous, $\langle M \rangle_\infty < \infty$ a.s., $\{M_t^+\}_{t \in T}$ is uniformly integrable, and there exists $\lambda_0 > 0$ such that

$$E \left[ \exp \left\{ -\lambda_0 M_\infty + \int_{\mathbb{R} \times [x], K} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx) \right\} \right] < \infty$$

for some $K > 0$. Then

$$\lim_{\lambda \to \infty} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) = -\frac{2}{\pi} E[M_\infty].$$

As a second step, in Subsection 3.2 we will describe the proof of (i) from Theorem 1.5 by Takaoka’s method [10]. Finally, we can obtain (ii) from (i). This proof is the same as in Subsection 6.4 of Kaji [6] and is omitted.

2. Proof of Theorem 1.5

2.1. Two lemmas. First, it is known that

$$\int_{\mathbb{R} \times X} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx) < \infty \text{ a.s.}$$

and

$$\int_{\mathbb{R} \times X} |\psi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx) < \infty \text{ a.s.,}$$

where $\psi_{\lambda}(x) = e^{-\lambda x} - 1 + \lambda x$. See Subsection 5.1 in Kaji [6].

Lemma 2.1.

$$E \left[ e^{-\lambda M_\infty - (\lambda^2/2) |M_\infty| - \int_{\mathbb{R} \times X} \psi_{\lambda}(x) \hat{\mu}(\cdot, ds \, dx) } \right] = 1, \quad 0 < \forall \lambda < \lambda_0.$$

Proof. According to Lemma 5.2 of Kaji [6], the condition $E[e^{\lambda_0 M_\infty}] < \infty$ implies the desired conclusion. In fact, we can see

$$E[e^{\lambda M_\infty}] \leq E[e^{-\lambda_0 M_\infty}] + 1,$$

where the right hand side is $< \infty$ by the assumption (3).

Lemma 2.2.

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \left( E \left[ e^{-\lambda M_\infty - (\lambda^2/2) |M_\infty| - \int_{\mathbb{R} \times X} \psi_{\lambda}(x) \hat{\mu}(\cdot, ds \, dx) } \right] - E[e^{-\lambda M_\infty}] \right) = -E[M_\infty].$$
Proof. First, we will show

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} \left\{ e^{-\lambda M_{\infty}} - \frac{\psi(x) \hat{\mu}(\cdot, ds \, dx)}{\lambda} e^{-\lambda (2/2)/M_{\infty}} \right\} = -M_{\infty} \quad \text{a.s.}
\]

Observe the equality

\[
\frac{1}{\lambda} \left\{ e^{-\lambda M_{\infty}} - \frac{\psi(x) \hat{\mu}(\cdot, ds \, dx)}{\lambda} e^{-\lambda (2/2)/M_{\infty}} \right\}
\]

\[
= \frac{1}{\lambda} \left\{ e^{-\lambda M_{\infty}} - \frac{\psi(x) \hat{\mu}(\cdot, ds \, dx)}{\lambda} e^{-\lambda (2/2)/M_{\infty}} \right\}
\]

\[
+ \frac{1}{\lambda} \left\{ e^{-\lambda M_{\infty}} - \frac{\psi(x) \hat{\mu}(\cdot, ds \, dx)}{\lambda} - 1 \right\}
\]

\[
= e^{-\lambda M_{\infty}} (2/2)/M_{\infty} \frac{1}{\lambda} \left\{ e^{-\lambda M_{\infty}} \phi_{\lambda}(x) \hat{\mu}(\cdot, ds \, dx) - 1 \right\}
\]

where the last “=” holds by the fact \( M_{\infty} = M_{\infty} + \int_{\mathbb{R}^d} x^2 \hat{\mu}(\cdot, ds \, dx) \). Since it is clear that

\[
\lim_{\lambda \to 0} e^{-\lambda M_{\infty}} - 1 = -M_{\infty} \quad \text{a.s.}
\]

holds, the second term of the right-hand side of the observation converges to \(-M_{\infty}\) a.s. Therefore, to get (6), it is sufficient to show that the first term of the right-hand side of the observation converges to 0 a.s. According to the dominated convergence theorem with respect to \( \hat{\mu}(\cdot, ds \, dx) \), Lemma 4.1 of Kaji [6], (4), and the fact \( \lim_{\lambda \to 0} \phi_{\lambda} / \lambda = 0 \) imply

\[
\lim_{\lambda \to 0} \int_{\mathbb{R}^d} \frac{\phi_{\lambda}(x)}{\lambda} \hat{\mu}(\cdot, ds \, dx) = 0 \quad \text{a.s.}
\]

On the other hand, by using the inequality

\[
\left| e^{v x} - 1 \right| \leq |x|e^{v|x|}, \quad v > 0,
\]

we have

\[
\left| \frac{1}{\lambda} \left\{ e^{-\lambda \int_{\mathbb{R}^d} \phi_{\lambda}(x) \hat{\mu}(\cdot, ds \, dx)} - 1 \right\} \right|
\]

\[
\leq \left| \int_{\mathbb{R}^d} \phi_{\lambda}(x) \hat{\mu}(\cdot, ds \, dx) \right| \exp \left\{ \left| \int_{\mathbb{R}^d} \phi_{\lambda}(x) \hat{\mu}(\cdot, ds \, dx) \right| \right\}
\]

\[
\leq \left| \int_{\mathbb{R}^d} \frac{\phi_{\lambda}(x)}{\lambda} \hat{\mu}(\cdot, ds \, dx) \right| \exp \left\{ \int_{\mathbb{R}^d} \left| \phi_{\lambda}(x) \right| \hat{\mu}(\cdot, ds \, dx) \right\}. \]
where the last line holds, since $\lambda \to |\phi_2(x)|$ is increasing for each $x \in X$. By (7) and (8) the left-hand side of the last inequality converges to 0 a.s. as $\lambda \to 0$. Hence (6) holds.

Next, we show that for all $0 < \lambda < \lambda_0 \land 1/(2c_0 K)$

$$
\begin{align*}
\left| \frac{1}{\lambda} \left( e^{-\lambda M_{\infty} - (\lambda/2) (M_{\infty} - f_{k_{\infty} + X}) \psi(x) \mu(\cdot, dx) - e^{-(\lambda/2) (M_{\infty} - f_{k_{\infty} + X}) \psi(x) \mu(\cdot, dx)} \right) \right|
\leq e^{-\lambda_0 M_{\infty} + \int_{\mathbb{R} \times [0, K]} \phi_0(x) \hat{\mu}(\cdot, ds dx) - \frac{1}{\lambda} \int_{\mathbb{R} \times [0, K]} \phi_0(x) \hat{\mu}(\cdot, ds dx) + e^{-\lambda M_{\infty} - f_{k_{\infty} + X} \phi_0(x) \mu(\cdot, dx)} - e^{-(\lambda / 2) (M_{\infty} - f_{k_{\infty} + X}) \phi_0(x) \mu(\cdot, dx)} - 1 \right| \\
= e^{-(\lambda / 2) (M_{\infty} - f_{k_{\infty} + X} \phi_0(x) \mu(\cdot, dx)} - e^{-\lambda M_{\infty} - f_{k_{\infty} + X} \phi_0(x) \mu(\cdot, dx)} - 1 \right| \\
= e^{-(\lambda / 2) (M_{\infty} - f_{k_{\infty} + X} \phi_0(x) \mu(\cdot, dx)} - e^{-\lambda M_{\infty} - f_{k_{\infty} + X} \phi_0(x) \mu(\cdot, dx)} - 1 \right| \\
= I_1 \times I_2 + I_3.
\end{align*}
$$

We will estimate $I_1$. By (10) we obtain

$$
I_1 \leq e^{-\lambda M_{\infty} - f_{k_{\infty} + X} \phi_0(x) \mu(\cdot, dx)} \\
 \leq e^{-\lambda M_{\infty} + c_0 K \lambda^3 f_{k_{\infty} + X} \mu(\cdot, dx)} \\
 \leq e^{-\lambda M_{\infty} + c_0 K \lambda^3 (M_{\infty})} \\
 \leq 1.
$$
We will estimate $I_2$. By using the inequality
\[ \left| e^{\alpha x} - 1 \right| \leq e^{\alpha x} \mathbb{1}_{\{x \geq 0\}} + x^{-1} \mathbb{1}_{\{x < 0\}}, \quad x > 0, \]
we have
\[
I_2 \leq e^{-\lambda_0 M_{\infty} - \int_{[0,1\cdot t]} \phi(x) \hat{\mu}(\cdot, ds dx)} \mathbb{1}_{\{M_{\infty} + \int_{[1\cdot t, \infty]} \phi(x) \hat{\mu}(\cdot, ds dx) \leq 0\}} \\
+ \left( -M_{\infty} - \int_{\mathbb{R} \times \{t > K\}} \frac{\phi(x)}{\lambda} \hat{\mu}(\cdot, ds dx) \right) \mathbb{1}_{\{M_{\infty} + \int_{[1\cdot t, \infty]} \phi(x) \hat{\mu}(\cdot, ds dx) > 0\}} \\
\leq e^{-\lambda_0 M_{\infty} + \int_{[0,1\cdot t]} \phi(x) \hat{\mu}(\cdot, ds dx)} + M_{\infty}^+ + \int_{\mathbb{R} \times \{t > K\}} \left| \frac{\phi(x)}{\lambda} \hat{\mu}(\cdot, ds dx) \right|.
\]
By Lemma 4.1 of Kaji [6], the right-hand side of the last inequality is
\[
\leq e^{-\lambda_0 M_{\infty} + \int_{[0,1\cdot t]} \phi(x) \hat{\mu}(\cdot, ds dx)} + M_{\infty}^+ + \frac{1}{\lambda_0} \int_{\mathbb{R} \times \{t > K\}} |\phi(x)| \hat{\mu}(\cdot, ds dx).
\]
We now estimate $I_3$. By using the inequality
\[ \left| e^{\alpha x} - 1 \right| \leq e^{\alpha x} \mathbb{1}_{\{x \geq 0\}} + x^{-1} \mathbb{1}_{\{x < 0\}}, \quad x > 0, \]
we have
\[
I_3 \leq e^{-\lambda_0^2/2} \left( M_{\infty} + \int_{[0,1\cdot t]} \phi(x) \hat{\mu}(\cdot, ds dx) \right) \mathbb{1}_{\{M_{\infty} \leq 1\}} \\
+ \left( -M_{\infty} - \int_{\mathbb{R} \times \{t > K\}} \frac{\phi(x)}{\lambda} \hat{\mu}(\cdot, ds dx) \right) \mathbb{1}_{\{M_{\infty} + \int_{[1\cdot t, \infty]} \phi(x) \hat{\mu}(\cdot, ds dx) > 0\}} \\
\leq e^{-\lambda_0^2/2} \left( M_{\infty} + \int_{[0,1\cdot t]} \phi(x) \hat{\mu}(\cdot, ds dx) \right) + \int_{\mathbb{R} \times \{t > K\}} \left| \frac{\phi(x)}{\lambda} \right| \hat{\mu}(\cdot, ds dx).
\]
Moreover, by (10) the right-hand side of the last inequality is
\[
\leq e^{-\lambda_0^2/2} \left( M_{\infty} + \int_{[0,1\cdot t]} \phi(x) \hat{\mu}(\cdot, ds dx) \right) + \int_{\mathbb{R} \times \{t > K\}} \left| \frac{\phi(x)}{\lambda} \right| \hat{\mu}(\cdot, ds dx) \\
\leq e^{-\lambda_0^2/2} \left( M_{\infty} + c_0 K \lambda^2 \right) \\
\leq 1 + 2 c_0 K e^{-1},
\]
where we can see \((\lambda^2/2)(M)_\infty e^{-\lambda^2/2}(M)_\infty \leq e^{-1}\) by using the inequality \(xe^{-x} \leq e^{-1}\). Hence, the above three estimations of \(I_1\), \(I_2\), and \(I_3\) imply (9).

Finally, according to the dominated convergence theorem, (6), (9), \(E[M^+_\infty] < \infty\), and the assumption (3) imply the desired conclusion.

### 2.2. A Tauberian theorem.

**Theorem 2.1** ([4]). Let \(X\) be an \(\mathbb{R}_+\)-valued random variable such that
\[
\lim_{\lambda \to 0} \frac{1}{\lambda} (1 - E[e^{-\lambda^2/2}X]) \text{ exists in } \mathbb{R},
\]
then
\[
\sqrt{\frac{2}{\pi}} \lim_{\lambda \to 0} \frac{1}{\lambda} (1 - E[e^{-\lambda^2/2}X]) = \lim_{\lambda \to \infty} \lambda P(\sqrt{X} > \lambda).
\]

### 2.3. Proof of Theorem 1.5.

According to Lemmas 2.1 and 2.2, we have
\[
\lim_{\lambda \to 0} \frac{1}{\lambda} (1 - E[e^{-\lambda^2/2}(M)_\infty]) = -E[M_\infty]
\]
holds. Then, by using the Tauberian theorem the last result implies
\[
\lim_{\lambda \to \infty} \lambda P(\sqrt{M}_\infty > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_\infty].
\]

### 3. Proof of Theorem 1.4

#### 3.1. The lemma.

**Lemma 3.1.** Let \(\rho\) be a stopping time. Then it follows that for any \(0 < a < 1\)
\[
\limsup_{\lambda \to \infty} \lambda P(\sqrt{\langle M \rangle}_\infty > \lambda) \leq \frac{1}{a} \limsup_{\lambda \to \infty} \lambda P(\sqrt{\langle M \rangle}_\rho > \lambda) + \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathbb{F}} E[(M_{\rho+\tau} - M_\rho)_\tau; \rho < \infty],
\]
where \(C\) is a positive constant which does not depend on \(M\), \(a\), and \(\rho\).

**Proof.** Fix \(0 < a < 1\). We have
\[
P(\langle M \rangle_\infty > \lambda^2) \leq P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2) + P(\langle M \rangle_\rho > a^2 \lambda^2),
\]
and so
\[
\limsup_{\lambda \to \infty} \lambda P(\langle M \rangle_\infty > \lambda^2) \leq \frac{1}{a} \limsup_{\lambda \to \infty} \lambda P(\langle M \rangle_\rho > \lambda^2) + \sup_{\lambda} \lambda P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2).
\]
On the other hand, define the process $N = \{N_t\}_{t \in \mathbb{R}}$ and the filtration $\{\mathcal{G}_t\}_{t \in \mathbb{R}}$ as

$$N_t = M_{t+\rho} - M_t, \quad \mathcal{G}_t = \mathcal{F}_{t+\rho}, \quad \forall t \in \mathbb{R}.$$ 

Then $N$ is a local martingale with respect to $\{\mathcal{G}_t\}_{t \in \mathbb{R}}$, and

$$\langle N \rangle_\infty = \langle M \rangle_\infty - \langle M \rangle_\rho$$

holds. Also, observe

$$\sup_{\lambda} \lambda P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2) \leq \sup_{\lambda} \lambda P(\langle N \rangle_\infty > \lambda^2 - a^2 \lambda^2)$$

$$= \frac{1}{\sqrt{1 - a^2}} \sup_{\lambda} \lambda P(\langle N \rangle_\infty > \lambda^2).$$

Then, by using the appendix the right-hand side of the last inequality is

(12) $$\leq \frac{C}{\sqrt{1 - a^2}} \sup_{\lambda} \lambda P \left( \sup_{t \in \mathbb{R}} |N_t| > \lambda \right),$$

where $C$ is a positive constant which does not depend on $M$, $a$, and $\rho$. If we let $\lambda > 0$ and

$$\tau_\lambda = \begin{cases} \inf \{t \in \mathbb{R} : |N_t| > \lambda \} & \text{if } \{\} \neq \emptyset \\ \infty & \text{if } \{\} = \emptyset, \end{cases}$$

then $|N_{\tau_\lambda}| \geq \lambda$ on $\{\tau_\lambda < \infty\} = \{\sup_{t \in \mathbb{R}} |N_t| > \lambda\}$, and so

$$\lambda P \left( \sup_{t \in \mathbb{R}} |N_t| > \lambda \right) \leq E[|N_{\tau_\lambda}|].$$

Therefore by the last result we have

(12) $$\leq \frac{C}{\sqrt{1 - a^2}} \sup_{\tau \in T(N)} E[|N_\tau|]$$

$$\leq \frac{C}{\sqrt{1 - a^2}} \sup_{\tau \in T(N)} 2E[N^-_\tau]$$

$$= \frac{2C}{\sqrt{1 - a^2}} \sup_{\tau \in T(N)} \{E[(M_{\rho+\tau} - M_\rho)^-; \rho = \infty] + E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty]\}$$

$$\leq \frac{2C}{\sqrt{1 - a^2}} \sup_{\tau \in T} E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty],$$

where $T(N) = \{\tau : \text{stopping time } |[N_{\tau\wedge t}]_{t \in \mathbb{R}} \text{ is uniformly integrable}\}$. That is,

$$\sup_{\lambda} \lambda P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2) \leq \frac{2C}{\sqrt{1 - a^2}} \sup_{\tau \in T} E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty].$$
Hence, by the last inequality and (11) we get the desired conclusion.

3.2. Proof of (i). For any $u > 0$, introduce the stopping time

$$
\tau_u = \begin{cases} 
\inf\{t \in \mathbb{R}_+ \mid -\lambda_0 M_t + A_t > u\} & \text{if } \{\} \neq \emptyset \\
\infty & \text{if } \{\} = \emptyset,
\end{cases}
$$

where

$$
A_t = \int_{(0,t] \times \{t > J\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx), \quad t \in \mathbb{R}_+.
$$

Fix $u > 0$. We consider the process $M^{(u)} = \{M^{(u)}_t\}_{t \in \mathbb{R}_+}$ defined as $M^{(u)}_t = M_{\tau_u \wedge t}$, $t \in \mathbb{R}_+$. Then it follows from the assumptions with respect to $M$ that $M^{(u)}$ is also a quasi left continuous and locally square integrable martingale which satisfying $M^{(u)}_0 = 0$, $\langle M^{(u)} \rangle = \langle M \rangle$ a.s., and the uniform integrability of $\{M^{(u)}_t^2\}_{t \in T}$. Moreover, if we pick the random measure $\mu^{(u)}$ on $\Omega \times \mathbb{R}_+ \times \mathbb{X}$ such that for all $t \in \mathbb{R}_+$ and Borel subsets $U$ of $\mathbb{X}$

$$
\mu^{(u)}(\cdot, (0, t] \times U) = \sum_{0 < s \leq t} 1_U(\Delta M^u_s)
$$

and its compensator $\hat{\mu}^{(u)}$, then it follows that for all $t \in \mathbb{R}_+$ and Borel subsets $U$ of $\mathbb{X}$

$$
\mu^{(u)}(\cdot, (0, t] \times U) = \sum_{0 < s \leq \tau_u \wedge t} 1_U(\Delta M_s) = \mu^{(u)}(\cdot, (0, \tau_u \wedge t] \times U) \quad \text{a.s.,}
$$

and so $\hat{\mu}^{(u)}$ is the random measure on $\Omega \times \mathbb{R}_+ \times \mathbb{X}$ such that for all $t \in \mathbb{R}_+$ and Borel subsets $U$ of $\mathbb{X}$

$$
\hat{\mu}^{(u)}(\cdot, (0, t] \times U) = \hat{\mu}(\cdot, (0, \tau_u \wedge t] \times U) \quad \text{a.s.,}
$$

and therefore we can have that

$$
E\left[e^{-\lambda_0 M^{(u)}_\infty} \cdot \mathbb{1}_{\mathbb{R}_+ \times [t, \infty]} \cdot \hat{\mu}^{(u)}(\cdot, ds \, dx)\right]
= E[e^{-\lambda_0 M_u + A_u}]
= E[e^{-\lambda_0 M_u + A_u} ; \tau_u < \infty] + E[e^{-\lambda_0 M_u + A_u} ; \tau_u = \infty]
\leq E[e^{u - \lambda_0 \Delta M_u} ; \tau_u < \infty] + e^u P(\tau_u = \infty)
= E[e^{u - \lambda_0 \times \infty} ; \tau_u < \infty] + e^u P(\tau_u = \infty) \quad (= e^u),
$$

where the fourth line of the above holds by the definition of $\tau_u$ and the last line does by the quasi left continuity of $M$. By applying Theorem 1.5 to the process $M^{(u)}$, we have

$$
-\infty < E[M^{(u)}_\infty] \leq 0, \quad \lim_{\lambda \to \infty} \lambda P(\sqrt{\langle M^{(u)} \rangle_\infty} > \lambda) = -\sqrt{\frac{\gamma}{\pi}} E[M^{(u)}_\infty],
$$
that is, $-\infty < E[M_{\tau_u}] \leq 0$ and

$\lim_{\lambda \to \infty} \lambda P(\sqrt{\langle M \rangle_{\tau_u}} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_{\tau_u}]$. \hspace{1cm} (13)

Now we show

$\liminf_{\lambda \to \infty} \lambda P(\sqrt{\langle M \rangle_{\infty}} > \lambda) \geq -\sqrt{\frac{2}{\pi}} E[M_{\infty}]$. \hspace{1cm} (14)

Indeed, the left-hand side of (13) is

$$\leq \liminf_{\lambda \to \infty} \lambda P(\sqrt{\langle M \rangle_{\infty}} > \lambda)$$

and the right-hand side of (13) is

$$= -\sqrt{\frac{2}{\pi}} E[M_{\infty}; \tau_u = \infty] - \sqrt{\frac{2}{\pi}} E[M_{\tau_u}; \tau_u < \infty]$$

$$\geq -\sqrt{\frac{2}{\pi}} E[M_{\infty}; \tau_u = \infty] + \sqrt{\frac{2}{\pi}} E \left[ \frac{\mu}{\lambda_0} - \frac{1}{\lambda_0} A_{\tau_u}; \tau_u < \infty \right]$$

$$\geq -\sqrt{\frac{2}{\pi}} E[M_{\infty}; \tau_u = \infty] - \frac{1}{\lambda_0} \sqrt{\frac{2}{\pi}} E[A_{\tau_u}; \tau_u < \infty],$$

where the second line of the above holds by the definition of $\tau_u$. Also, the right-hand side of the above converges to $-\sqrt{2/\pi} E[M_{\infty}]$ as $u \to \infty$, because by the dominated convergence theorem, the fact $E[|M_{\infty}|] < \infty$ we have known and the assumption (2) imply

$$\lim_{u \to \infty} E[M_{\infty}; \tau_u = \infty] = E[M_{\infty}], \quad \lim_{u \to \infty} E[A_{\tau_u}; \tau_u < \infty] = 0.$$

Therefore we can get (14).

On the other hand, we will show

$\limsup_{\lambda \to \infty} \lambda P(\sqrt{\langle M \rangle_{\infty}} > \lambda) \leq -\sqrt{\frac{2}{\pi}} E[M_{\infty}]$. \hspace{1cm} (15)

According to Lemma 3.1, we have for all $0 < a < 1$

$$\limsup_{\lambda \to \infty} \lambda P(\langle M \rangle_{\infty} > \lambda^2) \leq \frac{1}{a} \liminf_{\lambda \to \infty} \lambda P(\langle M \rangle_{\tau_u} > \lambda^2)$$

$$+ \frac{C}{\sqrt{1 - a^2}} \sup_{\tau \in \mathcal{T}} E[(M_{\tau_u + \tau} - M_{\tau_u})^2; \tau_u < \infty],$$
where $C$ is a positive constant which does not depend on $a$ and $u$. Fix $0 < a < 1$. By (13) the first term on the right-hand side of the last inequality is

$$= \frac{1}{a} \left( -\sqrt{\frac{2}{\pi}} E[M_{\infty}^\lambda] \right).$$

Therefore

$$\limsup_{\lambda \to \infty} \lambda P(\langle M \rangle_\infty > \lambda^2) \leq \frac{1}{a} \left( -\sqrt{\frac{2}{\pi}} E[M_{\infty}^\lambda] \right) + \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E[(M_{\tau + \tau} - M_{\tau})^{-}; \tau_u < \infty].$$

By the definition of $\tau_u$ the second term on the right-hand side of the last inequality is

$$\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E \left[ (M_{\tau + \tau} + \frac{u}{\lambda_0} - \frac{1}{\lambda_0} A_{\tau})^{-}; \tau_u < \infty \right]$$

$$\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E \left[ M_{\tau + \tau}^{-} + \frac{1}{\lambda_0} A_{\tau}; \tau_u < \infty \right]$$

$$\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E[A_{\tau}; \tau_u < \infty] + \frac{C}{\sqrt{1-a^2}} \frac{1}{\lambda_0} E[A_{\infty}; \tau_u < \infty].$$

By the uniform integrability of $[M_{\tau}]_{\tau \in \mathcal{T}}$ the first term on the right-hand side of the last inequality converges to 0 a.s. as $u \to \infty$ and from the dominated convergence theorem the assumption (2) implies that the second term of it does so, too. Therefore

$$\limsup_{\lambda \to \infty} \lambda P(\langle M \rangle_\infty > \lambda^2) \leq \limsup_{u \to \infty} \frac{1}{a} \left( -\sqrt{\frac{2}{\pi}} E[M_{\infty}^{(u)}] \right).$$

Moreover, the right-hand side of the last inequality is

$$(1/a)(-\sqrt{2/\pi} E[M_{\infty}])$$

since

$$\liminf_{u \to \infty} E[M_{\infty}^{(u)}] \geq E[M_{\infty}^{(1)}]$$

holds by the Fatou lemma and since

$$E[M_{\infty}^{(1)}] = E[M_{\infty}^{-}]$$

holds by the uniform integrability of $[M_{\tau}]_{\tau \in \mathcal{T}}$. Therefore we can get (15).

Hence (14) and (15) imply the desired conclusion.

4. Appendix

Proposition 4.1. Assume that $M$ is a quasi left continuous and locally square integrable martingale. Then

$$\sup_{\lambda} \lambda P(\langle M \rangle_\infty > \lambda) \leq C \sup_{\lambda} \lambda P\left( \sup_{\tau \in \infty} |M_{\tau}| > \lambda \right),$$

where $C$ is a universal positive constant.
Proof. Pick any stopping times $\rho$ and $\tau$ with $\rho \leq \tau$. First, it is clear that we can get

\[
E[(\sqrt{\langle M \rangle_{\tau^-}} - \sqrt{\langle M \rangle_{\rho^-}})^2] \leq E[(\langle M \rangle_{\tau} - \langle M \rangle_{\rho})].
\]

(16)

In fact, $\langle M \rangle_t$ is continuous, since $M$ is quasi left continuous, and the inequality $(\sqrt{a} - \sqrt{b})^2 \leq a - b$ for $0 \leq b \leq a$ holds. Introduce the local martingale $N_t = M_{(\rho + t) \wedge \tau} - M_{\rho}$, $t < \infty$, and then we can see $\langle M \rangle_{\tau} - \langle M \rangle_{\rho} = \langle N \rangle_{\infty}$. Therefore, (16) and the last result imply

\[
E[(\sqrt{\langle M \rangle_{\tau^-}} - \sqrt{\langle M \rangle_{\rho^-}})^2] \leq E[\langle N \rangle_{\infty}]
\]

(17)

\[
\leq E\left[ \left( \sup_{t<\infty} |N_t| \right)^2 \right],
\]

where the last line of the last inequality holds by the property of a local martingale. By the definition of $N$ we have

\[
E\left[ \left( \sup_{t<\infty} |N_t| \right)^2 \right] = E\left[ \left( \sup_{t<\infty} |N_t| \right)^2 ; \rho < \tau \right]
\]

\[
\leq 2E\left[ \left( \sup_{t<\infty} |M_{t \wedge \tau}| \right)^2 + M_{\rho}^2 ; \rho < \tau \right]
\]

\[
= 2E\left[ \left( \sup_{t<\infty} |M_t| \right)^2 + M_{\rho}^2 ; \rho < \tau = \infty \right]
\]

\[
+ 2E\left[ \left( \sup_{t<\infty} |M_{t \wedge \tau}| \right)^2 + M_{\rho}^2 ; \rho < \tau < \infty \right]
\]

(18)

\[
\leq 4E\left[ \left( \sup_{t<\infty} |M_t| \right)^2 ; \rho < \tau = \infty \right]
\]

\[
+ 2E\left[ \left( \sup_{t<\tau} |M_t| \right)^2 + \left( \sup_{t<\tau} |M_t| \right)^2 ; \rho < \tau < \infty \right]
\]

\[
= 4E\left[ \left( \sup_{t<\tau} |M_t| \right)^2 ; \rho < \tau = \infty \right]
\]

\[
+ 4E\left[ \left( \sup_{t<\tau} |M_t| \right)^2 ; \rho < \tau < \infty \right]
\]

\[
= 4E\left[ \left( \sup_{t<\tau} |M_t| \right)^2 ; \rho < \tau \right],
\]

where the eighth line of the last inequality holds by the quasi left continuity of $t \to$
sup_{t \leq \tau} |M_t|$. Hence, (17) and (18) imply

$$E[(\sqrt{\langle M \rangle_{\tau^-}} - \sqrt{\langle M \rangle_{\rho^-}})^2] = 4E\left[ \left( \sup_{t \leq \tau} |M_t| \right)^2 ; \rho < \tau \right].$$

Then, according to Corollary 6 of Azéma, Gundy, and Yor [1], the above implies the desired conclusion. \hfill \Box

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**References**


Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyamachou 1–1
Toyonaka
Osaka 560–0043
Japan
e-mail: kaji@math.sci.osaka-u.ac.jp