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ON KLEIN-MASKIT COMBINATION THEOREM IN SPACE I

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Abstract

In this paper, we generalise the first Klein-Maskit combination theorem to discrete groups of Möbius transformations in higher dimensions. The application of the main theorem is discussed in the last section.

1. Introduction

In the theory of classical Kleinian groups, there are theorems called the combination theorems which give methods to generate new Kleinian groups as amalgamated free products or HNN extensions of Kleinian groups. The prototype of such theorems is Klein's combination theorem which can be rephrased as follows in the modern terms:

Theorem 1.1 (Klein [16]). *Let G_1 and $G_2 \subset PSL_2\mathbb{C}$ be two finitely generated Kleinian groups with non-empty regions of discontinuity, and let D_1 and D_2 be fundamental domains for G_1 and G_2 of their regions of discontinuity respectively. Suppose that the interior of D_2 contains the frontier and the exterior of D_1 and that the interior of D_1 contains the frontier and the exterior of D_2. Then the group $\langle G_1, G_2\rangle$ generated by G_1 and G_2 in $PSL_2\mathbb{C}$ is a Kleinian group isomorphic to $G_1 * G_2$ with non-empty region of discontinuity and $D = D_1 \cap D_2$ is a fundamental domain for the region of discontinuity of $\langle G_1, G_2\rangle$.*

Fenchel-Nielsen, in [12], gave a generalisation of Klein's theorem to amalgamated free products and HNN extensions for Fuchsian groups. In a series of papers, Maskit considered to generalise Klein's theorem to amalgamated free products and HNN extensions for Kleinian groups ([18]–[23]). Thurston gave an interpretation of the combination theorem using three-dimensional hyperbolic geometry and harmonic maps, cf. [27]. For applications of the combination theorems, we refer the reader to [1, 4, 7, 12, 17, 24, 34].

Among these, the first Maskit combination theorem says that under some conditions two Kleinian groups G_1, G_2 whose intersection J is geometrically finite generate

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a Kleinian group isomorphic to the free product of G_1 and G_2 amalgamated over J and also under the same conditions the resulting group is geometrically finite if and only if both G_1 and G_2 are geometrically finite.

The purpose of the present paper is to generalise this first Maskit combination theorem to discrete groups of Möbius transformations of dimension greater than 2. A first pioneering attempt to generalise Maskit's combination theorems to higher dimensions was made by Apanasov [5, 6]. Ivascu [15] also considered this generalisation. In particular, they showed that under the same assumptions as Maskit combined with some extra conditions, one can get a discrete group which is an amalgamated free product of two discrete groups of n-dimensional Möbius transformations. In fact, they proved the following.

Theorem 1.2. *Let G_1, G_2 be two discontinuous n-dimensional Möbius subgroups with a common subgroup H, and let the n-sphere S^n split along a hypersurface $S \subset S^n$ into two domains D_1 and D_2 whose closures $\overline{D}_1$ and $\overline{D}_2$ are precisely invariant with respect to H, in G_1 and G_2, respectively. Let also the following two conditions hold*:

(1) *For fundamental domains Δ, F_1 and F_2 of the groups H, G_1 and G_2, there exists a neighbourhood V of the surface S such that $\Delta \cap V \subset F_i$, $i = 1, 2$.*

(2) *For each $i = 1, 2$, the set $\Delta \cap \overline{D}_i = \overline{D}_i \cap F_i$ is a proper subdomain in F_i.*

Then the following hold.

(1) *The group $G = \langle G_1, G_2\rangle$ is discontinuous and isomorphic to the amalgamated free product $G_1 *_H G_2$.*

(2) *$F = F_1 \cap F_2$ is a fundamental domain for the group G.*

(3) *$m_n(\Lambda(G)) = 0$ if and only if $m_n(\Lambda(G_i)) = 0$, $i = 1, 2$.*

(4) *Each elliptic or parabolic element of G is conjugate in G to an element from $G_1 \cup G_2$.*

In this paper, we shall show that a generalisation of the first Maskit's theorem holds in higher dimensions without any such additional assumptions, imposing only natural ones. Our theorem also includes the equivalence of geometric finiteness of the given two groups and that of the group obtained by the combination. It should be noted that in this paper, we say that a Kleinian group is geometrically finite when the ε-neighbourhood of its convex core has finite volume for some $\varepsilon > 0$, and there is an upper bound for the orders of torsions in the group. We do not assume that it has a finite-sided fundamental polyhedron. For more details about these Kleinian groups of higher dimensions, we refer the reader to [11, 26, 28, 29, 30] and the references therein.

Our main result (Theorem 4.2) and its proof will appear in §4.

This is the first of a series in which we shall discuss generalisations and applications of Klein-Maskit combination theorem in higher dimensions. A generalisation of the second Klein-Maskit combination theorem, which corresponds to HNN extensions, to the case of discrete groups of Möbius transformations in higher dimensions and applications of these two combination theorems will be given in forthcoming papers.

2. Preliminaries

2.1. Basics on Möbius transformations. For $n \geq 2$, we denote by $\overline{\mathbb{R}}^n$ the one-point compactification of $\mathbb{R}^n$ obtained by adding ∞. The group of orientation-preserving Möbius transformations of $\overline{\mathbb{R}}^n$ is denoted by $M(\overline{\mathbb{R}}^n)$, with which we endow the compact-open topology. We regard $\overline{\mathbb{R}}^n$ as the boundary at infinity of the hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ which is identified with the open unit ball bounded by $\overline{\mathbb{R}}^n$. We denote the union of $\mathbb{H}^{n+1}$ and $\overline{\mathbb{R}}^n$ endowed with the natural topology by $\mathbb{B}^{n+1}$. Any Möbius transformation of $\overline{\mathbb{R}}^n$ is extended to a Möbius transformation of $\mathbb{B}^{n+1}$, which induces an isometry of $\mathbb{H}^{n+1}$. When it is more convenient, we regard $\mathbb{H}^{n+1}$ as the upper half-space of the $(n+1)$-dimensional Euclidean space and $\mathbb{R}^n$ as $\{(x_1, \dots, x_n, 0)\}$ in $\mathbb{R}^{n+1}$. A non-trivial element $g \in M(\overline{\mathbb{R}}^n)$ is called

(1) *loxodromic* if it has two fixed points in $\overline{\mathbb{R}}^n$ and none in $\mathbb{H}^{n+1}$;
(2) *parabolic* if it has only one fixed point in $\overline{\mathbb{R}}^n$ and none in $\mathbb{H}^{n+1}$;
(3) *elliptic* if it has a fixed point in $\mathbb{H}^{n+1}$.

For a discrete group G of $M(\overline{\mathbb{R}}^n)$ and a point $z \in \mathbb{H}^{n+1}$ or $x \in \overline{\mathbb{R}}^n$, the sets $G(z) = \{g(z)\colon g \in G\} \subset \mathbb{H}^{n+1}$ and $G(x) = \{g(x)\colon g \in G\} \subset \overline{\mathbb{R}}^n$ are called G-orbits of z and x respectively. If z' lies in the G-orbit of z, then we say that z' and z are G-equivalent.

2.2. Limit sets, regions of discontinuity and fundamental sets. The *limit set* $\Lambda(G)$ of a discrete group $G \subset M(\overline{\mathbb{R}}^n)$ is defined as follows:

$$\Lambda(G) = \overline{G(z)} \cap \overline{\mathbb{R}}^n$$

for some $z \in \mathbb{H}^{n+1}$, where the overline denotes the closure in $\mathbb{B}^{n+1} = \mathbb{H}^{n+1} \cup \overline{\mathbb{R}}^n$ and $G(z)$ the G-orbit of z. We call points of $\Lambda(G)$ *limit points*. The complement $\Omega(G) = \overline{\mathbb{R}}^n \setminus \Lambda(G)$ is called the *region of discontinuity* of G. The following is a well-known fact.

Lemma 2.1. *Let G be a discrete subgroup of $M(\overline{\mathbb{R}}^n)$. If $B \subset \overline{\mathbb{R}}^n$ is a closed and G-invariant subset containing at least two points, then $\Lambda(G)$ is contained in B.*

A discrete group $G \subset M(\overline{\mathbb{R}}^n)$ is said to act discontinuously at a point $x \in \overline{\mathbb{R}}^n$ if there is a neighbourhood U of x such that $\{g \in G\colon g(U) \cap U \neq \emptyset\}$ is a finite set. The group G acts discontinuously at every point of $\Omega(G)$, and at no point of $\Lambda(G)$.

The complement of the fixed points of elliptic elements in $\Omega(G)$ is called the *free regular set*, and is denoted by ${}^\circ\Omega(G)$. When ${}^\circ\Omega(G) \neq \emptyset$, a *fundamental set* of G is a set which contains one representative of each orbit $G(y)$ of $y \in {}^\circ\Omega(G)$. It is obvious that ${}^\circ\Omega(G) \neq \emptyset$ if and only if $\Omega(G) \neq \emptyset$.

We have the following lemmata for the limit points. These lemmata in the classical case when $n = 2$ can be found in Theorems II.D.2 and II.D.5 in Maskit [22]. Although the argument is quite parallel, we give their proofs for completeness.

Lemma 2.2. *Let x be a limit point of a discrete subgroup G in $M(\overline{\mathbb{R}}^n)$. Then there are a limit point y of G and a sequence $\{g_m\}$ of distinct elements of G such that g_m converges to the constant map x uniformly on any compact subset of $\overline{\mathbb{R}}^{n+1} \setminus \{y\}$.*

Proof. Since x is a limit point, there are a point $z \in \mathbb{H}^{n+1}$ and a sequence $\{g_m\}$ of distinct elements of G such that $g_m(z) \to x$. Regard $\mathbb{H}^{n+1}$ as the upper half-space. Let $(z_1, \dots, z_n, z_{n+1})$ be the coordinate of z, with $z_{n+1} > 0$. Consider the point $z' = (z_1, \dots, z_n, -z_{n+1})$ in the lower half-space. The actions of Möbius transformations can be extended to the lower half-space conformally. Then obviously, we have $g_m(z') \to x$.

By conjugation, we can assume that G acts on $\mathbb{B}^{n+1}$ with $\mathrm{Int}\,\mathbb{B}^{n+1} = \mathbb{H}^{n+1}$, that $z = \mathbf{0}$, and that $\mathrm{Stab}_G(\mathbf{0}) = \mathrm{Stab}_G(\infty) = \{id\}$. Then $z' = \infty$; hence we have $g_m(\infty) \to x$. By taking a subsequence we can make $g_m^{-1}(\infty)$ converge to some limit point y. Since g_m maps the outside of its isometric sphere onto the interior of that of g_m^{-1}, the radii of the isometric spheres of g_m and g_m^{-1}, which are equal, converge to 0 as $m \to \infty$, and the centre $g_m(\infty)$ of the isometric sphere of g_m^{-1} converges to x. On the other hand, the centre of the isometric sphere of g_m, which is $g_m^{-1}(\infty)$ converges to y. This completes the proof. □

Lemma 2.3. *Let $\{g_m\}$ be a sequence of distinct elements of a discrete group $G \subset M(\overline{\mathbb{R}}^n)$. Then there are a subsequence of $\{g_m\}$ and limit points x, y of G, which may coincide, such that g_m converges to the constant map x uniformly on any compact subset of $\overline{\mathbb{R}}^{n+1} \setminus \{y\}$.*

Proof. We may assume that G acts on $\mathbb{B}^{n+1}$ with $\mathrm{Int}\,\mathbb{B}^{n+1}$ identified with $\mathbb{H}^{n+1}$, and that $\mathrm{Stab}_G(\infty) = \{id\}$. By taking a subsequence if necessary, we have two limit points x and y such that $g_m(\infty) \to x$ and $g_m^{-1}(\infty) \to y$. The conclusion now follows from the proof of Lemma 2.2. □

We shall use the following term frequently.

DEFINITION 2.1. Let H be a subgroup of a discrete subgroup G of $M(\overline{\mathbb{R}}^n)$. An subset V of $\overline{\mathbb{R}}^n$ is said to be *precisely invariant* under H in G if $h(V) = V$ for all $h \in H$ and $g(V) \cap V = \emptyset$ for all $g \in G - H$.

For $\Omega(G)$, we have the following proposition: refer to Proposition II.E.4 in Maskit [22] or Theorem 5.3.12 in Beardon [7].

Proposition 2.4. *Suppose that $\Omega(G)$ is not empty. Then a point $x \in \overline{\mathbb{R}}^n$ is contained in $\Omega(G)$ if and only if*

(1) *the stabiliser* $\mathrm{Stab}_G(x) = \{g \in G : g(x) = x\}$ *of x in G is finite, and*

(2) *there is a neighbourhood U of x in $\overline{\mathbb{R}}^n$ which is precisely invariant under* $\mathrm{Stab}_G(x)$ *in G.*

DEFINITION 2.2. A fundamental domain for a discrete group G of $M(\overline{\mathbb{R}}^n)$ with non-empty region of discontinuity is an open subset D of $\Omega(G)$ satisfying the following.

(1) D is precisely invariant under the trivial subgroup in G.

(2) For every $z \in \Omega(G)$, there is an element $g \in G$ such that $g(z)$ is contained in $\overline{D}$, where $\overline{D}$ denotes the closure of D in $\overline{\mathbb{R}}^n$.

(3) $\operatorname{Fr} D$, the frontier of D in $\overline{\mathbb{R}}^n$, consists of limit points of G, and a finite or countable collection of codimension-1 compact smooth submanifolds with boundary, whose boundary is contained in $\Omega(G)$ except for a subset with $(n-1)$-dimensional Lebesgue measure 0. The intersection of each submanifold with $\Omega(G)$ is called a side of D.

(4) For any side σ of D, there are another side σ' of D, which may coincide with σ, and a nontrivial element $g \in G$ such that $g(S) = S'$. Such an element g is called the side-pairing transformation from σ to σ'.

(5) If $\{\sigma_m\}$ is a sequence of distinct sides of D, then the diameter of σ_m with respect to the ordinary spherical metric on $\overline{\mathbb{R}}^n$ goes to 0.

(6) For any compact subset K of $\Omega(G)$, there are only finitely many translates of D that intersect K.

A fundamental set F for a discrete subgroup G of $M(\overline{\mathbb{R}}^n)$ whose interior is a fundamental domain is called *a constrained fundamental set*.

2.3. Normal forms. Let G_1 and G_2 be two subgroups of $M(\overline{\mathbb{R}}^n)$, and J a subgroup of $G_1 \cap G_2$.

A *normal form* is a word consisting of alternate products of elements of $G_1 - J$ and those of $G_2 - J$. Two normal forms $g_n \cdots g_k g_{k-1} \cdots g_1$ and $g_n \cdots (g_k j)(j^{-1} g_{k-1}) \cdots g_1$ are said to be *equivalent* for any $j \in J$. The word length of the normal form is simply called the *length*. The length is invariant under the equivalence relation.

A normal form is called a 1-form if the last letter is contained in $G_1 - J$, and a 2-form otherwise. More specifically a normal form is called an (m, k)-form if the last letter is contained in $G_m - J$ and the first letter is contained in $G_k - J$.

The multiplication of two normal forms is defined to be the concatenation of two words which is contracted to the minimum length by the equivalence defined above. The product of two normal forms is equivalent to either a normal form or to an element of J.

It is obvious that any element of the free product of G_1 and G_2 amalgamated over J, which is denoted by $G_1 *_J G_2$, either is an element of J or can be expressed in a normal form, and that there is a one-to-one correspondence between $G_1 *_J G_2$ and the union of J and the set of the equivalence classes of normal forms. Also it is easy to see that this correspondence is an isomorphism with respect to the multiplication defined above.

Let $\langle G_1, G_2 \rangle$ denote the subgroup of $M(\overline{\mathbb{R}}^n)$ generated by G_1 and G_2. There is a natural homomorphism $\Phi\colon G_1 *_J G_2 \to \langle G_1, G_2 \rangle$ which is defined by $\Phi(g_n \cdots g_1) =$

$g_n \circ \cdots \circ g_1$ for a normal form $g_n \cdots g_1$ representing an element of $G_1 *_J G_2$, and $\Phi(j) = j$ for $j \in J$. It is easy to see that this is well defined and independent of a choice of a representative of the equivalence class. The map is obviously an epimorphism.

If Φ is an isomorphism, then we write $\langle G_1, G_2 \rangle = G_1 *_J G_2$ identifying elements of $G_1 *_J G_2$ and their images by Φ.

Since J is embedded in $\langle G_1, G_2 \rangle$, each nontrivial element in the kernel of Φ can be written in a normal form.

Lemma 2.5. *$\langle G_1, G_2 \rangle = G_1 *_J G_2$ if and only if Φ maps no non-trivial normal forms to the identity.*

2.4. Interactive pairs. Following Maskit, we shall define interactive pairs as follows.

Let G_1 and G_2 be two discrete subgroups of $M(\overline{\mathbb{R}}^n)$ and J a subgroup of $G_1 \cap G_2$ as in the previous subsection. Let X_1, X_2 be disjoint non-empty subsets of $\overline{\mathbb{R}}^n$. The pair (X_1, X_2) is said to be an *interactive pair* (for G_1, G_2, J) when

(1) each of X_1, X_2 is invariant under J,

(2) every element of $G_1 - J$ sends X_1 into X_2,

(3) and every element of $G_2 - J$ sends X_2 into X_1.

An interactive pair is said to be *proper* if there is a point in X_1 which is not contained in a G_2-orbit of any point of X_2, or there is a point in X_2 which is not contained in a G_1-orbit of any point of X_1.

Lemma 2.6 (Lemma VII.A.9 in [22]). *Suppose that (X_1, X_2) is an interactive pair for G_1, G_2, J. Let $g = g_n \cdots g_1$ be an (m, k)-form. Then we have $\Phi(g)(X_k) \subset X_{3-m}$. Furthermore if (X_1, X_2) is proper and g has length greater than* 1, *then the inclusion is proper.*

The existence of a proper interactive pair forces Φ to be isomorphic. (Theorem VII.A.10 in Maskit [22] in the case when $n = 2$.)

Theorem 2.7. *Let G_1, G_2, J be as above and suppose that there is a proper interactive pair for G_1, G_2, J. Then $\langle G_1, G_2 \rangle = G_1 *_J G_2$.*

This easily follows from Lemmata 2.5 and 2.6.

The following is a straightforward generalisation of Theorem VII.A.12 in Maskit [22].

Lemma 2.8. *Suppose that (X_1, X_2) is an interactive pair for G_1, G_2, J. Suppose moreover that there is a fundamental set D_m for G_m for $m = 1, 2$ such that $G_m(D_m \cap X_{3-m}) \subset X_{3-m}$. Then $D = (D_1 \cap X_2) \cup (D_2 \cap X_1)$ is precisely invariant under $\{id\}$ in $G = \langle G_1, G_2 \rangle$. Furthermore, if D is non-empty, then Φ is isomorphic.*

Proof. What we shall show is that for any $x \in D$ and any non-trivial element $g \in G_1 *_J G_2$, we have $\Phi(g)(x) \notin D$. Since this holds trivially for the case when D is empty, we assume that D is non-empty. We assume that x is contained in $D_1 \cap X_2$. The case when x lies in $D_2 \cap X_1$ can be dealt with in the same way.

If g is a non-trivial element in J, then $g(x)$ lies in X_2 since X_2 is J-invariant. On the other hand, since D_1 is a fundamental set, we have $g(x) \notin D_1$. These imply that $g(x) \notin D$.

Now we shall consider the case when g is represented in a normal form.

Claim 1. *If $g = g_n g_{n-1} \cdots g_1$ is an m-form ($m = 1$ or 2), then $\Phi(g)(x) \in X_{3-m} \setminus D_m$.*

Proof. We shall prove this claim by induction.

We first consider the case when $n = 1$. Suppose first that g is an element in $G_1 - J$. Then $\Phi(g)(x) \in X_2$ by assumption, whereas $\Phi(g)(x) \notin D_1$ since D_1 is a fundamental set of G_1. Therefore $\Phi(g)(x)$ is not contained in D in this case. Suppose next that g is in $G_2 - J$. Then $\Phi(g)(x)$ lies in X_1 since the assumption that (X_1, X_2) is an interactive pair implies $\Phi(g)(X_2) \subset X_1$. We shall show that $\Phi(g)(x)$ does not lie in D_2. Suppose, seeking a contradiction, that $\Phi(g)(x)$ lies in D_2. Then since $\Phi(g^{-1})$ is contained in $G_2 - J$ and $\Phi(g)(x) \in X_1 \cap D_2$, by assumption, we have $x = \Phi(g^{-1})\Phi(g)(x)$ lies in X_1. This contradicts the assumption that x lies in X_2.

Now, we assume that our claim holds in the case when g has length $n - 1$, and suppose that g has length n. We consider the case when g is a $(3-m)$-form. The case when g is an m-form can also be dealt with in the same way. Since $\Phi(g_{n-1} \cdots g_1)(x) \in X_{3-m} \setminus D_m$ by the assumption of induction, we have $\Phi(g)(x) \in g_n(X_{3-m} \setminus D_m) \subset X_m$.

Suppose that $\Phi(g)(x)$ lies in D_{3-m}. Then we have $\Phi(g)(x) \in X_m \cap D_{3-m}$. This implies that $\Phi(g_{n-1} \cdots g_1)(x) \in g_n^{-1}(X_m \cap D_{3-m}) \subset X_m$. This is a contradiction. Thus we have shown that $\Phi(g)(x)$ is contained in $X_m \setminus D_{3-m}$. □

By what we have proved above, if $D \neq \emptyset$, then for any $g \in G_1 *_J G_2 - \{id\}$, we have $\Phi(g)(D) \cap D = \emptyset$. This in particular shows that $\Phi(g) \neq id$. Then Lemma 2.5 shows that $G = G_1 *_J G_2$. □

REMARK 2.1. Maskit called a fundamental set D_m for G_m maximal with respective to X_m (which is precisely invariant under J in G_m) if $D_m \cap X_m$ is a fundamental set for the action of J on X_m, and in Theorem VII.A.12 in [22], the fundamental sets D_1, D_2 were assumed to be maximal. The proof of the theorem above shows that the assumption of maximality is in fact redundant.

In Maskit [22], the following sufficient condition for two open balls to be an interactive pair is given.

Proposition 2.9 (Proposition VII.A.6 in [22]). *Let $G_m \subset M(\overline{\mathbb{R}}^n)$ $(m = 1, 2)$ be two discrete groups with a common subgroup J and $S \subset \overline{\mathbb{R}}^n$ be an $(n-1)$-sphere bounding two open balls X_1 and X_2. If each X_m is precisely invariant under J in G_m, then (X_1, X_2) is an interactive pair.*

2.5. Convex cores and geometric finiteness.

DEFINITION 2.3. Let G be a discrete subgroup of $M(\overline{\mathbb{R}}^n)$ and $\Lambda(G)$ its limit set. We denote by $\mathrm{Hull}(\Lambda(G))$, the minimal convex set of $\mathbb{H}^{n+1}$ containing all geodesics whose endpoints lie on $\Lambda(G)$. This set is evidently G-invariant, and its quotient $\mathrm{Hull}(G)/G$ is called the *convex core* of G, and is denoted by $\mathrm{Core}(G)$. The group G is said to be *geometrically finite* if the following two conditions are satisfied:
(1) there exists $\varepsilon > 0$ such that the ε-neighbourhood of $\mathrm{Core}(G)$ in $\mathbb{H}^{n+1}/G$ has finite volume, and
(2) there is an upper bound for the orders of torsions in G.

We do not assume that G is finitely generated above. The latter condition, the existence of the bound on the orders is automatically satisfied if G is finitely generated. For infinitely generated groups, Hamilton showed in [13] that the second condition is not redundant.

As we shall see below, Bowditch proved in [9] that this condition is equivalent to other reasonable definitions of geometric finiteness, except for the one that $\mathbb{H}^{n+1}/G$ has a finite-sided fundamental polyhedron, whose equivalence to the above condition has not been known until now.

2.6. Euclidean isometries. The classification of discrete groups of Euclidean isometries is known as Bieberbach's theorem (see [33] or [25], for example).

Theorem 2.10 (Bieberbach). *Let G be a discrete group of Euclidean isometries of $\mathbb{R}^n$. Then the following hold.*
(1) *If $\mathbb{R}^n/G$ is compact, then there is a normal subgroup $G^* \subset G$ of finite index consisting only of Euclidean translations, which is isomorphic to a free abelian group of rank n.*
(2) *If $\mathbb{R}^n/G$ is not compact, then there exists a normal subgroup $G^* \subset G$ of finite index in G which is a free abelian group of rank k with $0 \le k \le n-1$.*

By taking conjugates of G and G^* with respect to an isometry of $\mathbb{R}^n$, the groups can be made to have the following properties.

Decompose $\mathbb{R}^n$ into $\mathbb{R}^k \times \mathbb{R}^{n-k}$, where $\mathbb{R}^k$ is identified with $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ and $\mathbb{R}^{n-k}$ with $\{0\} \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$. Let $g(x) = U(x) + a$ be an arbitrary element of G, where U is a rotation and a is an element of $\mathbb{R}^n$. Then the rotation U leaves $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ invariant and the vector a lies in the subspace $\mathbb{R}^k$. Furthermore, if g lies in G^*, then U acts on $\mathbb{R}^k$ trivially.

In the following we always identify the factors of the decomposition $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ with $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^{n-k}$.

DEFINITION 2.4. For a discrete subgroup G of Euclidean isometries, we define G^* to be a free abelian normal subgroup of G which is maximal among those having the property in Theorem 2.10.

2.7. Extended horoballs, peak domains and standard parabolic regions. A point x of $\Lambda(G)$ of a discrete group G of Möbius transformations is called *a parabolic fixed point* if $\mathrm{Stab}_G(x)$ contains parabolic elements. An easy argument shows that $\mathrm{Stab}_G(x)$ cannot contain a loxodromic element then. For a parabolic fixed point z, a horoball in $\mathbb{B}^{n+1}$ touching $\overline{\mathbb{R}}^n$ at z is invariant under $\mathrm{Stab}_G(z)$. In the case when $\mathrm{Stab}_G(z)$ has rank less than n, it is useful to consider a domain larger than a horoball as follows.

DEFINITION 2.5. Let G be a discrete subgroup of $M(\overline{\mathbb{R}}^n)$. Let z be a point of $\overline{\mathbb{R}}^n$ which is not a loxodromic fixed point. Let $\mathrm{Stab}^*_G(z)$ be the maximal free abelian subgroup as in Definition 2.4 of the stabiliser $\mathrm{Stab}_G(z)$ of z in G. Suppose that the rank of $\mathrm{Stab}^*_G(z)$ is k with $k \leq n-1$. Then there is a closed subset $B_z \subset \mathbb{B}^{n+1}$ invariant under $\mathrm{Stab}_G(z)$ which is in the form

$$B_z = h^{-1}\left\{x \in \mathbb{B}^{n+1} : \sum_{i=k+1}^{n+1} {x_i}^2 \geq t\right\},$$

where t (> 0) is a constant and $h \in M(\overline{\mathbb{R}}^n)$ is a Möbius transformation such that $h(z) = \infty$. We call B_z an *extended horoball* of G around z.

DEFINITION 2.6. Let $T_1, \dots, T_m$ be subsets of $\overline{\mathbb{R}}^n$ and $J_1, \dots, J_m$ subgroups of the group $G \subset M(\overline{\mathbb{R}}^n)$. We say that $(T_1, \dots, T_m)$ is precisely invariant under $(J_1, \dots, J_m)$ in G, if each T_k is precisely invariant under J_k in G, and if for $i \neq j$ and all $g \in G$, we have $g(T_i) \cap T_j = \emptyset$.

DEFINITION 2.7. A *peak domain* of a discrete group G of $M(\overline{\mathbb{R}}^n)$ with non-empty region of discontinuity at the parabolic fixed point $z \in \overline{\mathbb{R}}^n$ is an open subset $U_z \subset \overline{\mathbb{R}}^n$ such that

(1) U_z is precisely invariant under $\mathrm{Stab}_G(z)$ in G,
(2) there exist a $t > 0$, and a transformation $h \in M(\overline{\mathbb{R}}^n)$ with $h(z) = \infty$ such that

$$\left\{x \in \mathbb{R}^n : \sum_{i=k+1}^{n} {x_i}^2 > t\right\} = h(U_z),$$

where $k = \mathrm{rank}\,\mathrm{Stab}^*_G(z)$, $1 \leq k \leq n-1$.

DEFINITION 2.8. If G has a precisely invariant extended horoball B around z, then the interior of its intersection with $\overline{\mathbb{R}}^n$ is a peak domain. Following Bowditch [9], we use the term *standard parabolic region* at z to mean an extended horoball when the rank of $\mathrm{Stab}_G(z)$ is less than n, and a horoball when the rank of $\mathrm{Stab}_G(z)$ is n.

DEFINITION 2.9. A point $z \in \overline{\mathbb{R}}^n$ fixed by a parabolic element of a discrete group $G \subset M(\overline{\mathbb{R}}^n)$ is said to be a *parabolic vertex* of G if one of the following conditions is satisfied.
(1) The subgroup $\mathrm{Stab}^*_G(z)$ has rank n.
(2) There exists a peak domain U_z at the point z.

REMARK 2.2. It is easy to see that the two conditions in Definition 2.9 are mutually exclusive: a peak domain exists only if $\mathrm{rank}\,\mathrm{Stab}^*_G(z) < n$. Also we can easily see that, in the case when $n = 2$, the definition coincides with that of cusped parabolic fixed points as in Beardon-Maskit [8].

DEFINITION 2.10. A parabolic fixed point z for the group G is called bounded if $(\Lambda(G) \setminus \{z\})/\mathrm{Stab}_G(z)$ is compact (see Bowditch [9, 10]).

There is a relationship between a bounded parabolic fixed point and a parabolic vertex, which was proved by Bowditch [9].

Lemma 2.11. *z is a bounded parabolic fixed point for a discrete group G if and only if z is a parabolic vertex.*

DEFINITION 2.11. Let G be a discrete subgroup of $M(\overline{\mathbb{R}}^n)$. A point $x \in \overline{\mathbb{R}}^n$ is said to be a conical limit point (or a point of approximation in some literature) if there are $z \in \mathbb{H}^{n+1}$ and a geodesic ray l in $\mathbb{H}^{n+1}$ tending to x in $\mathbb{B}^{n+1}$ whose r-neighbourhood with some $r \in \mathbb{R}$ contains infinitely many translates of z.

Conical limit points can be characterised as follows. See Theorem 12.2.5 in Ratcliffe [25].

Proposition 2.12. *Let G be a discrete group of $M(\overline{\mathbb{R}}^n)$ regarded as acting on $\mathbb{B}^{n+1}$ by hyperbolic isometries. Then a point $z \in \partial\mathbb{B}^{n+1}$ is a conical limit point of G if and only if there exist $\delta > 0$, distinct elements g_m of G, and $x \in \partial\mathbb{B}^{n+1} \setminus \{z\}$ such that $g_m^{-1}(\mathbf{0})$ converges to z while $|g_m(x) - g_m(z)| > \delta$ for all m. Furthermore, if this condition holds, then for every $x \in \partial\mathbb{B}^{n+1} \setminus \{z\}$, there is $\delta > 0$ such that $|g_m(x) - g_m(z)| > \delta$ for all m.*

The following result due to Bowditch [9] or [10] will be essentially used in the proof of our main theorem.

Proposition 2.13. *Let $G \subset M(\overline{\mathbb{R}}^n)$ $(n \geq 2)$ be a discrete group. Then G is geometrically finite if and only if every point of $\Lambda(G)$ is either a parabolic vertex or a conical limit point.*

2.8. Dirichlet domains and standard parabolic regions. Dirichlet domains are fundamental polyhedra of hyperbolic manifolds, which will turn out to be very useful for us.

DEFINITION 2.12. Let G be a discrete subgroup of $M(\overline{\mathbb{R}}^n)$, and x a point in $\mathbb{H}^{n+1}$, which is not fixed by any nontrivial element of G. The set $\{y \in \mathbb{H}^{n+1} \colon d_h(y, x) \leq d_h(y, g(x)),\ \forall g \in G\}$ is called the Dirichlet domain for G centred at x, where d_h denotes the hyperbolic distance.

It is easy to see that any Dirichlet domain is convex and the interior of the intersection of the closure of a Dirichlet domain with $\overline{\mathbb{R}}^n$ is a fundamental domain as defined before.

The following follows immediately from the definition of conical limit points.

Lemma 2.14. *Let D be a Dirichlet domain of a discrete group $G \subset M(\overline{\mathbb{R}}^n)$. Then $\overline{D} \cap \overline{\mathbb{R}}^n$ contains no conical limit points, where $\overline{D}$ denotes the closure of D in $\mathbb{B}^{n+1} = \mathbb{H}^{n+1} \cup \overline{\mathbb{R}}^n$.*

Now, we consider how a Dirichlet domain of a geometrically finite group intersects standard parabolic regions. We shall make use of the following result of Bowditch [9]. For a G-invariant set S on $\overline{\mathbb{R}}^n$, we say a collection of subsets $\{A_s\}_{s \in S}$ is *strongly invariant* if $gA_s = A_{gs}$ for any $s \in S$ and $g \in G$, and $A_s \cap A_t = \emptyset$ for any $s \neq t \in S$. We should note that each A_s is in particular precisely invariant under $\mathrm{Stab}_G(s)$ in G in the sense as defined before.

Lemma 2.15. *Let Π be the set of all bounded parabolic fixed points contained in the limit set $\Lambda(G)$ of a discrete group $G \subset M(\overline{\mathbb{R}}^n)$. Then we can choose a standard parabolic region B_p at p for each $p \in \Pi$ in such a way that $\{B_p \colon p \in \Pi\}$ is strongly invariant.*

Using this lemma, we can show the following, which is essentially contained in the argument of §4 in Bowditch [9].

Proposition 2.16. *Let D be a Dirichlet domain of a geometrically finite group $G \subset M(\overline{\mathbb{R}}^n)$. Let $\{B_p\}$ be the collection of standard parabolic regions obtained as in the preceding lemma. Then there is a finite number of points $p_1, \ldots, p_k \in \overline{D} \cap \Pi$ such that $\overline{D} \setminus \bigcup_{i=1}^k (\mathrm{Int}\, B_{p_i} \cup \{p_i\})$ is compact and contains no limit point of G.*

Proof. Choose a family of standard parabolic regions $\{B_p\}$ as in Lemma 2.15. Since G is geometrically finite, every limit point of G is either a conical limit point or a parabolic vertex. By Lemma 2.14, no limit point on $\overline{D}$ is a conical limit point. Therefore $\{B_p\}$ covers all limit points contained in $\overline{D}$.

Suppose that there are infinitely many distinct B_{p_i} among $\{B_p\}$ with $p_i \in \overline{D}$. By taking a subsequence, we can assume that $\{p_i\}$ converges to a point $q \in \overline{D}$, which is also contained in $\Lambda(G)$, hence in Π. By taking a subsequence again, we can further assume that all the p_i belong to either the same $\mathrm{Stab}_G(q)$-orbit or distinct $\mathrm{Stab}_G(q)$-orbits. We first consider the former case. Let α_i be the geodesic line connecting p_i to q, which must be contained in D. Since all p_i belong to the same orbit, there are $h_i \in \mathrm{Stab}_G(q)$ such that $h_i(p_i) = p_1$. By taking a subsequence again, we can assume that all h_i are distinct. Then, the geodesic α_1 is shared by infinitely many translates of $h_i D$. This contradicts the local finiteness of the translates of the Dirichlet domain D.

Since q is a parabolic vertex, by Lemma 2.11, we see that $(\Lambda(G) \setminus \{q\})/\mathrm{Stab}_G(q)$ is compact. Therefore, by taking a subsequence again, we can assume that there are $g_i \in \mathrm{Stab}_G(q)$ such that $\{g_i p_i\}$ converges to a point $r \in \overline{\mathbb{R}}^n \setminus \{q\}$. We can assume that all the g_i are distinct by taking a subsequence. Let α_i be the geodesic line connecting p_i and q as before. Then $g_i\alpha_i$ converges to the geodesic line connecting r to q. Since $g_i\alpha_i$ is contained in $g_i D$, this again contradicts the local finiteness of the translates of D. □

Another easy consequence of Lemma 2.15 is the following.

Corollary 2.17. *Let G be a discrete subgroup of $M(\overline{\mathbb{R}}^n)$. In the upper half-space model of $\mathbb{H}^{n+1}$, suppose that ∞ is a parabolic vertex of G. Then the Euclidean radii of the isometric spheres $I(g)$ of $g \in G - \mathrm{Stab}_G(\infty)$ are bounded from above.*

Proof. Consider the set of standard parabolic regions $\{B_p\}_{p\in\Pi}$ obtained by Lemma 2.15. Since ∞ is a bounded parabolic fixed point, a standard parabolic region B_∞ and its translates gB_∞ by elements $g \in G - \mathrm{Stab}_G(\infty)$ are among $\{B_p\}$. Let B'_∞ be the maximal horoball contained in B_∞. Then there is a number h such that $B'_\infty = \{(z_1, \dots, z_{n+1}) \colon z_{n+1} \geq h\} \cup \{\infty\}$, which is equal to the height of $\mathrm{Fr}\, B'_\infty$.

Fix an element $g \in G - \mathrm{Stab}_G(\infty)$. By enlarging B'_∞, we get a horoball B'' which touches $g^{-1}B''$ at one point. Let $h' < h$ be the height of $\mathrm{Fr}\, B''$. Then the point $B'' \cap g^{-1}B''$ has height h'. The isometric sphere $I(g)$ of g must contain the point $B'' \cap g^{-1}B''$ since the reflection in $I(g)$ sends $g^{-1}B''$ to B''. Therefore the Euclidean radius of $I(g)$ is equal to h', which is bounded above by the constant h independent of g. □

This implies the following fact in the conformal ball model, which is Corollary G.8 in Maskit [18].

Corollary 2.18. *We regard G as above as acting on the ball $\mathbb{B}^{n+1}$ or $L = \overline{\mathbb{R}}^{n+1} \setminus \mathbb{B}^{n+1}$, and let $p \in \partial\mathbb{B}^{n+1} = \partial L$ be a parabolic vertex of G. Suppose that $g_n \in G$ are distinct elements. Then the radius with respect to the ordinary Euclidean metric on $\mathbb{B}^{n+1}$ or L of the isometric sphere $I(g_k)$ goes to 0 as $k \to \infty$.*

3. Blocks

Throughout this section, we assume that G is a discrete subgroup of $M(\overline{\mathbb{R}}^n)$ and J is a subgroup of G.

DEFINITION 3.1. A closed J-invariant set B, containing at lease two points, is called a block, or more specifically (J, G)-block if it satisfies the following conditions.
(1) $B \cap \Omega(G) = B \cap \Omega(J)$, and $B \cap \Omega(G)$ is precisely invariant under J in G.
(2) If U is a peak domain for a parabolic fixed point z of J with the rank of $\mathrm{Stab}_J(z)$ being $k < n$, then there is a smaller peak domain $U' \subset U$ such that $U' \cap \mathrm{Fr}\, B = \emptyset$.

Let S be a (J, G)-block, and let S be a topological $(n-1)$-dimensional sphere in $\overline{\mathbb{R}}^n$. Then S separates $\overline{\mathbb{R}}^n$ into two open sets. We say that S is *precisely embedded* in G if $g(S)$ is disjoint from one of the two open sets for any $g \in G$.

A (J, G)-block is said to be *strong* if every parabolic fixed point of J is a parabolic vertex of G.

Then we have the following.

Theorem 3.1. *Suppose that G is a discrete subgroup of $M(\overline{\mathbb{R}}^n)$. Let J be a geometrically finite subgroup of G and $B \subset \overline{\mathbb{R}}^n$ a (J, G)-block such that for every parabolic fixed point z of J with the rank of $\mathrm{Stab}_J(z)$ being less than n, there is a peak domain U_z for J with $U_z \cap B = \emptyset$. Let $G = \bigcup g_k J$ be a coset decomposition. Then we have $\mathrm{diam}(g_k(B)) \to 0$, where $\mathrm{diam}(M)$ denotes the diameter of the set M with respect to the ordinary spherical metric on $\overline{\mathbb{R}}^n$.*

Proof. By conjugating G by an element of $M(\overline{\mathbb{R}}^n)$, we can assume that $\mathrm{Stab}_G(\mathbf{0}) = \mathrm{Stab}_G(\infty) = \{id\}$ when we regard G as acting on $\overline{\mathbb{R}}^{n+1}$ by considering the Poincaré extension. Let L denote the exterior of $\mathbb{B}^{n+1}$ with the point ∞, which we regard also as a model of hyperbolic $(n+1)$-space. Then J is also geometrically finite as a discrete group acting on L. Let P be a Dirichlet domain for J in L.

Let g be some element of $G - J$. For a fixed g, the set $\{(g \circ j)^{-1}(\infty) = j^{-1} \circ g^{-1}(\infty) \colon j \in J\}$ is J-invariant. Then for each coset $g_k J$, we can choose a representative g_k in such a way that $a_k = g_k^{-1}(\infty)$, which is the centre of the isometric sphere of g_k, lies in P.

Now, by Proposition 2.16, there are finitely many standard parabolic regions $B_{p_1}, \dots, B_{p_s}$ in L around parabolic vertices $p_1, \dots, p_s$ on $\bar{P}$ such that $\bar{P} \setminus \bigcup_i (\mathrm{Int}\, B_{p_i} \cup \{p_i\})$ is compact and contains no limit point of J. We number them in such a way that $\mathrm{Stab}^*_J(p_1), \dots, \mathrm{Stab}^*_J(p_r)$ have rank n whereas $\mathrm{Stab}^*_J(p_{r+1}), \dots, \mathrm{Stab}^*_J(p_s)$ have rank less than n. We can assume that for $j \geq r+1$, we have $B_{p_j} \cap \overline{\mathbb{R}}^n \cap B = \{p_j\}$ because of the following: By our assumption in the theorem, we can make B_{p_j} smaller so that it satisfies this condition. Also it is clear that for the old B_{p_j}, there is no limit point of J in $\overline{\mathbb{R}}^n \cap B_{p_j}$ other than p_j, which is also contained in the new B_{p_j}. On the other hand no point in $\overline{P}$ can converge to p_j from outside this smaller B_{p_j} since p_j is not a conical limit point, which implies that the compactness is preserved.

For horoballs $B_{p_1}, \ldots, B_{p_r}$, we have the following.

Claim 2. *We can choose the horoballs $B_{p_1}, \ldots, B_{p_r}$ sufficiently small so that $B_{p_i} \cap G(\infty) = \emptyset$ for each i $(1 \leq i \leq r)$.*

Proof. We identify L with the standard upper half-space model of hyperbolic $(n+1)$-space, which we denote by $\mathbb{H}^{n+1}$. By conjugation, we can assume that $e = (0, \ldots, 0, 1)$ corresponds to $\infty \in L$ under the identification of $\mathbb{H}^{n+1}$ with L. Regarding G as acting on this $\mathbb{H}^{n+1}$ and $B_{p_1}, \ldots, B_{p_r}$ lying in $\mathbb{B}^{n+1}$, what we have to show is that $B_{p_i} \cap G(e) = \emptyset$ for each i.

We shall show that how we can make B_{p_1} satisfy this condition. Conjugating G by an isometry of $\mathbb{H}^{n+1}$, we may assume that $p_1 = \infty$. Then Corollary 2.17 implies that the radii of the isometric spheres $I(g)$ of $g \in G - \mathrm{Stab}_G(\infty)$ are bounded from above by some constant r_0. We set $B_{p_1} = \{x \in \mathbb{H}^{n+1} : x_{n+1} \geq 2\max\{1, r_0^2\}\} \cup \{\infty\}$.

Any $h \in \mathrm{Stab}_G(\infty)$ can be represented as a transformation of $\mathbb{R}^n$ in the form $h(x) = Ax + b$ for $A \in O(n)$ and $b \in \mathbb{R}^n$. Let $\tilde{h}$ denote h regarded as an isometry of $\mathbb{H}^{n+1}$. Then we have $\tilde{h}(e) = (b, 1)$, hence $\tilde{h}(e) \notin B_{p_1}$.

For any $g \in G - \mathrm{Stab}_G(\infty)$, let r_g denote the radius of the isometric sphere $I(g)$. Then $g(x)$ is represented as a transformation of $\overline{\mathbb{R}}^n$ in the form $a + r_g^2 A(x-b)/|x-b|^2$ for some $A \in O(n)$ and $a, b \in \mathbb{R}^n$ (see [2] or [7]). As before we denote by $\tilde{g}$ the transformation g regarded as an isometry of $\mathbb{H}^{n+1}$. Then we have

$$\tilde{g}(e) = \left(a - \frac{r_g^2 Ab}{|b|^2+1}, \frac{r_g^2}{|b|^2+1} \right)$$

and

$$\frac{r_g^2}{|b|^2+1} \leq r_0^2,$$

which implies that $\tilde{g}(e) \notin B_{p_1}$. We make each B_{p_i} smaller in the same way. It is clear that even after changing the horoballs, $\overline{P} \setminus \bigcup_i (\mathrm{Int}\, B_{p_i} \cup \{p_i\})$ is compact and contains no limit point of J since B_{p_j} intersects $\overline{P} \cap \mathbb{R}^n$ only at p_j $(1 \leq j \leq r)$ and p_i is not a conical limit point. □

Recall that $a_k = g_k^{-1}(\infty)$ is in P. By taking a subsequence, we have only to consider the cases when every a_k lies outside all the standard parabolic regions B_{p_j} and when all the a_k lie in some B_{p_j}.

First consider the case when every a_k lies outside the B_{p_j}. Since $a_k \in \overline{P}$ and $\overline{P} \setminus \bigcup(\mathrm{Int}(B_{p_j}) \cup \{p_j\})$ is compact, the sequence $\{a_k\}$ converges to a point $x \in \overline{P} \setminus \bigcup(\mathrm{Int}(B_{p_j}) \cup \{p_j\})$. Suppose that x is contained in B. Then x must lie in $B \cap \Lambda(G) = B \cap \Lambda(J)$, which contradicts the fact that $\overline{P} \setminus \bigcup(\mathrm{Int}(B_{p_j}) \cup \{p_j\})$ contains no limit point of J. Therefore, it follows that the a_k are uniformly bounded away from B. Since the g_k are distinct elements, the radius with respect to the Euclidean metric of the conformal ball model of the isometric sphere $I(g_k)$ converges to 0 by Corollary 2.18. Therefore, we see that B lies outside the isometric sphere $I(g_k)$ for sufficiently large k. This means $g_k(B)$ lies inside the isometric sphere $I(g_k^{-1})$. This implies that $\mathrm{diam}(g_k(B)) \to 0$.

Next we consider the case when the a_k lie in some standard parabolic region B_{p_j}. By Claim 2, we see that B_{p_j} is not a horoball; hence B_{p_j} is an extended horoball, i.e., $j \geq r+1$. Furthermore, if $\{a_k\}$ does not converge to p_j, then we can take B_{p_j} smaller. Therefore, we can assume that $\{a_k\}$ converges to p_j.

By composing a rotation of the sphere $\overline{\mathbb{R}}^n$, we may assume that p_j is at the north pole $(0, \ldots, 0, 1)$. Let S be the n-sphere of radius 1 centred at p_j, and let ϕ be the reflection in S. Let $B' \subset B_{p_j}$ be the largest horoball contained in B_{p_j} touching $\overline{\mathbb{R}}^n$ at p_j.

We denote points in $\mathbb{R}^{n+1}$ as $(\mathbf{z}, t)$ with $\mathbf{z} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then we have $p_j = (\mathbf{0}, 1)$. Take B_{p_j} to be small enough so that $B' = \{(\mathbf{z}, t) \colon |\mathbf{z}|^2 + (t - s' - 1)^2 \leq s'^2\}$ for some s' satisfying $0 < s' < 1/2$, and

$$\phi(z, t) = \left(\frac{\mathbf{z}}{|\mathbf{z}|^2 + (t-1)^2}, \frac{|\mathbf{z}|^2 + t^2 - t}{|\mathbf{z}|^2 + (t-1)^2} \right).$$

We deduce that

$$\phi(\mathbb{B}^{n+1}) = \left\{ (\mathbf{z}, t) \colon t \leq \frac{1}{2} \right\} \cup \{\infty\}$$

and

$$\phi(B') = \left\{ (\mathbf{z}, t) \colon t \geq 1 + \frac{1}{2s'} \right\} \cup \{\infty\}.$$

For any $j \in \mathrm{Stab}_J(p_j)$, we have $\phi j \phi(\infty) = \infty$ since $\phi(\infty) = p_j$. Consider the decomposition $\mathbb{R}^{n+1} = \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}$, where m $(< n)$ is the rank of $\mathrm{Stab}_J(p_j)$. Let $\phi j \phi(z) = U(z) + \mathbf{a}$ be an arbitrary element of $\phi\, \mathrm{Stab}_J(p_j)\phi$, where U denotes a rotation. By Theorem 2.10, we may assume that the rotation U leaves $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ invariant and the vector $\mathbf{a}$ lies in the subspace $\mathbb{R}^m$. Also, if $\phi j \phi \in \phi\, \mathrm{Stab}^*_J(p_j)\phi$, then its restriction to the subspace $\mathbb{R}^m$ is a translation. Hence, we have

$$\phi(B_{p_j}) = \left\{ (\mathbf{z}, t) \colon \sum_{i=m+1}^{n} z_i^2 + t^2 \geq \left(1 + \frac{1}{2s'}\right)^2, t \geq \frac{1}{2} \right\} \cup \{\infty\},$$

where z_i denotes the i-th component of $\mathbf{z}$.

Since $B_{p_j} \cap B = \{p_j\}$, we have

$$\phi(B) \subset \left\{(\mathbf{z}, t) : \sum_{i=m+1}^{n} z_i^2 + \frac{1}{4} < \left(1 + \frac{1}{2s'}\right)^2,\ t = \frac{1}{2}\right\} \cup \{\infty\}. \tag{3.1}$$

We should recall that $\phi \operatorname{Stab}_J^*(p_j)\phi$ acts on $\mathbb{R}^m$ cocompactly. Therefore, we can take representatives g_k so that the projections of $\phi(a_k) = \phi(g_k^{-1}(\infty))$ to $\mathbb{R}^m$ stay within a compact subset of $\mathbb{R}^m$ by multiplying elements of $\operatorname{Stab}_J^*(p_j)$ to the original g_k. Note that by changing representatives, we do not have the condition that $a_k \in P$ any more, but still the a_k are contained in B_{p_j}. This means that there is a constant L such that $\phi(a_k) \in \left\{(z, t) : \sum_{i=1}^m z_i^2 < L, t > 1/2\right\} \cap \phi(B_{p_j})$.

Claim 3. *There is a constant $K > 0$ such that for every $a_k \in B_{p_j}$ and every $y \in B$, we have $|a_k - y| \geq K|a_k - p_j|$.*

Proof. Suppose, seeking a contradiction, that such a K does not exist. Then there exist a sequence $\{y_s\} \subset B$ and a subsequence $\{a_{k_s}\}$ of $\{a_k\}$ such that

$$\frac{|a_{k_s} - y_s|}{|a_{k_s} - p_j|} \to 0 \quad \text{as} \quad s \to \infty. \tag{3.2}$$

We shall denote a_{k_s} by a_s for simplicity.

We can assume that $y_s \neq p_j$ for all s. Then, since

$$|\phi(a_s) - \phi(y_s)| = \frac{|a_s - y_s|}{|y_s - p_j|\,|a_s - p_j|}$$

and

$$|\phi(y_s) - p_j|\,|y_s - p_j| = 1,$$

we have

$$\begin{aligned} \frac{|a_s - y_s|^2}{|a_s - p_j|^2} &= \frac{|\phi(a_s) - \phi(y_s)|^2}{|\phi(y_s) - p_j|^2} \\ &= \frac{\sum_{i=1}^m (\phi(a_s) - \phi(y_s))_i^2 + \sum_{i=m+1}^{n+1} (\phi(a_s) - \phi(y_s))_i^2}{\sum_{i=1}^m (\phi(y_s))_i^2 + \sum_{i=m+1}^{n+1} (\phi(y_s) - p_j)_i^2}. \end{aligned} \tag{3.3}$$

We shall show that there exists $M > 0$ such that

(1) $\sum_{i=1}^m (\phi(a_s))_i^2 \leq M$ for all s;

(2) $\sum_{i=m+1}^{n+1} (\phi(y_s) - p_j)_i^2 \leq M$ for all s; and

(3) $\sum_{i=m+1}^{n+1}(\phi(a_s) - \phi(y_s))_i^2 \to \infty$ as $s \to \infty$.
The inequality (1) follows from the fact that we choose a_k so that the projections of $\phi(a_k)$ to $\mathbb{R}^m$ stay in a compact subset. The second one is a consequence of (3.1). We now turn to the third inequality. Since $\{a_s\}$ was assumed to converge to p_j, we see that $\phi(a_s)$ tends to ∞, which means that $\sum_{i=1}^{n+1}(\phi(a_s))_i^2 \to \infty$. On the other hand, we know that $\sum_{i=1}^{m}(\phi(a_s))_i^2 \leq M$ by (1), and that $\sum_{i=m+1}^{n+1}(\phi(y_s))_i^2$ is bounded above independently of s by (2). These imply (3).

Then (3.2), (3.3), (2) and (3) imply that

$$\sum_{i=1}^{m}(\phi(y_s))_i^2 \to \infty \quad \text{as} \quad s \to \infty.$$

It follows from (1) that for all sufficiently large s,

$$\frac{|a_s - y_s|}{|a_s - p_j|} \geq \frac{1}{2}.$$

This is a contradiction and we have completed the proof of Claim 3. □

Let ρ_k be the Euclidean radius of the isometric sphere of g_k in L. Then we have the following.

Claim 4. *If all a_k lie inside the extended horoball B_{p_j}, then we have $\rho_k^2/|a_k - p_j| \to 0$.*

Proof. Suppose that there is $\delta > 0$ such that $\rho_k^2/|a_k - p_j| \geq \delta$. Then $|g_k(p_j) - g_k(\infty)| = \rho_k^2/|a_k - p_j| \geq \delta$.

We can apply Proposition 2.12 by identifying L with $\mathbb{B}^{n+1}$ by the reflection in $\partial\mathbb{B}^{n+1}$ and taking into account the fact that the Euclidean metric does not distort much by the reflection near $\partial\mathbb{B}^{n+1}$ and see that p_j is a conical limit point of G. This contradicts Lemma 2.14 since p_j lies in $\overline{P}$. □

We shall conclude the proof of Theorem 3.1. Let δ_k be the distance from a_k to B. Since δ_k is the infimum of $|a_k - y|$ for $y \in B$, by Claim 3, we have $\delta_k \geq K|a_k - p_j|$. Since Proposition I.C.7 in [22] holds for $g \in M(\mathbb{R}^n)$, we have

$$\operatorname{diam}(g_k(B)) \leq \frac{2\rho_k^2}{\delta_k} \leq \frac{2K^{-1}\rho_k^2}{|a_k - p_j|}.$$

This implies that $\operatorname{diam}(g_k(B)) \to 0$ by Claim 4. □

4. The combination theorem

In this section, we shall state and prove our main theorem, which is a combination theorem for discrete groups in $M(\overline{\mathbb{R}}^n)$. Before that we shall prove the following lemma which constitutes the key step for the proof of our main theorem.

Lemma 4.1. *Let G_1 and G_2 be discrete subgroups of $M(\overline{\mathbb{R}}^n)$. Suppose that J is a subgroup of $G_1 \cap G_2$, which coincides with neither G_1 nor G_2. Suppose that there is a topological $(n-1)$-sphere S dividing $\overline{\mathbb{R}}^n$ into two closed balls B_1 and B_2 such that each B_m is a (J, G_m)-block. Suppose that there are fundamental sets D_1, D_2 for G_1, G_2 respectively such that $J(D_m \cap B_m) = B_m \cap {}^\circ\Omega(J)$ for $m = 1, 2$, and $D_1 \cap S = D_2 \cap S$. Set $D = (D_1 \cap B_2) \cup (D_2 \cap B_1)$ and $G = \langle G_1, G_2 \rangle$. Then the following hold.*
(1) *S is also a (J, G_m)-block for $m = 1, 2$.*
(2) *$S \cap \Lambda(G_1) = S \cap \Lambda(G_2) = S \cap \Lambda(J) = \Lambda(J)$.*
(3) *Both G_1 and G_2 have non-empty regions of discontinuity, and B_m° is contained in $\Omega(G_m)$ for $m = 1, 2$, where B_m° is the interior of B_m in $\overline{\mathbb{R}}^n$.*
(4) *B_m° is precisely invariant under J in G_m.*
(5) *For any $g \in G_m - J$ $(m = 1, 2)$, $g(B_m) \cap B_m = g(S) \cap S \subset \Lambda(G_m)$.*
(6) *For any $g \in G_m$, we have $g(D_m \cap B_{3-m}) \subset B_{3-m}$ and $g(D_m \cap B_{3-m}^\circ) \subset B_{3-m}^\circ$.*
(7) *Let $G_m = \bigcup g_{km} J$ be a coset decomposition for $m = 1, 2$. If J is geometrically finite, then $\mathrm{diam}(g_{km}(B_m)) \to 0$ as $k \to \infty$ where diam denotes the diameter with respect to the ordinary spherical metric on $\overline{\mathbb{R}}^n$.*
(8) *(B_1°, B_2°) is an interactive pair.*
(9) *If $\Lambda(J) \neq \Lambda(G_1)$ or $\Lambda(J) \neq \Lambda(G_2)$, then (B_1°, B_2°) is a proper interactive pair.*
(10) *If $D \neq \emptyset$ and J is geometrically finite, then (B_1°, B_2°) is a proper interactive pair.*

Proof. (1). This is obvious since S is contained in B_m.

(2). By Lemma 2.1, we see that $\Lambda(J)$ is contained in S; hence $S \cap \Lambda(J) = \Lambda(J)$. Since S is a (J, G_m)-block for $m = 1, 2$ by (1), we have $S \cap \Lambda(G_m) = S \cap \Lambda(J)$. This implies (2).

(3). Since $\Lambda(J)$ is contained in S, we see that $B_m^\circ \cap \Omega(J) = B_m^\circ$. On the other hand, since B_m is a (J, G_m)-block, we have $B_m^\circ \cap \Omega(G_m) = B_m^\circ \cap \Omega(J) = B_m^\circ \neq \emptyset$. Thus both G_1 and G_2 have non-empty regions of discontinuity and $\Omega(G_m)$ contains B_m°.

(4). Since $B_m^\circ \subset \Omega(G_m)$, by the definition of blocks, $B_m \cap \Omega(G_m)$ is precisely invariant under J in G_m, and $J(S) = S$, we see that B_m° is precisely invariant under J in G_m.

(5). Since $B_m \cap \Omega(G_m)$ is precisely invariant under J in G_m, for every $g \in G_m - J$, $g(B_m \cap \Omega(G_m)) \cap (B_m \cap \Omega(G_m)) = \emptyset$. It follows $(g(B_m) \cap B_m) \cap \Omega(G_m) = \emptyset$. Then we see that (4) implies (5).

(6). For any $j \in J \subset G_m$, $j(D_m \cap B_{3-m}) \subset j(B_{3-m}) = B_{3-m}$ and $j(D_m \cap B_{3-m}^\circ) \subset j(B_{3-m}^\circ) = B_{3-m}^\circ$. Hence we have only to consider the case when g lies in $G_m - J$. Suppose that there exists an element $g \in G_m - J$ such that $g(D_m \cap B_{3-m}) \cap B_m \neq \emptyset$.

Take points $x \in g(D_m \cap B_{3-m}) \cap B_m$ and $y \in D_m \cap B_{3-m}$ such that $x = g(y)$. Since x lies in $B_m \cap g(D_m \cap B_{3-m}) \subset B_m \cap {}^\circ\Omega(G_m) \subset B_m \cap {}^\circ\Omega(J) = J(D_m \cap B_m)$, there are an element $j \in J$ and a point $z \in D_m \cap B_m$ such that $j(z) = x$. Then $j(z) = g(y)$. Since z and y are G_m-equivalent points of D_m, we have $z = y$ and $j = g$, which is a contradiction. Therefore, for any $g \in G_m - J$, we have $g(D_m \cap B_{3-m}) \cap B_m = \emptyset$ and $g(D_m \cap B_{3-m}) \subset B^\circ_{3-m}$. Thus we have proved (6).

(7). By (1), we know that S is a (J, G_m)-block. Also we should note that since $\mathrm{Fr}\, S = S$, by the definition of blocks, for any parabolic vertex z of J on S with the rank of $\mathrm{Stab}_J(z)$ being less than n, there is a peak domain centred at z which is disjoint from S, and that every parabolic fixed point is a parabolic vertex if J is geometrically finite. Therefore by Theorem 3.1, $\mathrm{diam}(g_{km}(S)) \to 0$ as $k \to \infty$. On the other hand since B_m is a (J, G_m)-block, $\mathrm{diam}(g_{km}(S)) \to 0$ implies $\mathrm{diam}(g_{km}(B_m)) \to 0$, and we have completed the proof of (7).

(8). This follows from (4) and Proposition 2.9.

(9). If (B°_1, B°_2) is not proper, then $B^\circ_1 \cup B^\circ_2 = G_1(B^\circ_1) \subset \Omega(G_1)$ and $B^\circ_1 \cup B^\circ_2 = G_2(B^\circ_2) \subset \Omega(G_2)$. It follows that for each m, we have $\Lambda(G_m) \subset S$. On the other hand, by (2), we have $\Lambda(G_m) = S \cap \Lambda(G_m) = S \cap \Lambda(J) = \Lambda(J)$. Therefore if one of $\Lambda(G_1), \Lambda(G_2)$ is not equal to $\Lambda(J)$, then (B°_1, B°_2) is a proper interactive pair.

(10). Suppose that D is non-empty and J is geometrically finite. Then we can assume that $D_1 \cap B_2 \neq \emptyset$, for the case $D_2 \cap B_1$ can be proved just by interchanging the indices. We divide the argument into two cases: the case when $D_1 \cap S \neq \emptyset$ and the one when $D_1 \cap B^\circ_2 \neq \emptyset$.

Suppose first that there is a point $x \in D_1 \cap S = D_2 \cap S$. Recall that D_1 is contained in $\Omega(G_1)$, and that for $g \in G_1 - J$, we have $g(B_1) \cap B_1 \subset \Lambda(G_1)$ by (5). These imply that no $(G_1 - J)$-translates of B_1 pass through $x \in D_1 \cap S \subset D_1 \cap B_1$. By the same argument, we see that no $(G_2 - J)$-translates of B_2 pass through x.

Next we shall show that $(G_m - J)(B_m)$ cannot accumulate at x. First we should note that the translate of B_m by an element of G_m depends only on the cosets of G_m over J since J stabilises B_m. Suppose that $(G_m - J)(B_m)$ accumulates at x. Then there are elements g_k in $G_m - J$, which we can assume to belong to distinct cosets, and points $y_k \in B_m$ such that $\{g_k(y_k)\}$ converges to x. Since we assumed that J is geometrically finite, by (7) we see that $\mathrm{diam}(g_k(B_m)) \to 0$. Therefore if we choose one point y in B_m, then $\{g_k(y)\}$ also converges to x. This means that x is a limit point of G_m, which contradicts the assumption that x lies in D_m.

By these two facts which we have just proved, we see that there is a neighborhood of x which is disjoint from $(G_m - J)(B_m)$ for each m. This implies in particular that there is a point in B°_{3-m} which is not contained in the G_m-translates of B_m. Hence, in this case, (B°_1, B°_2) is proper.

Now we assume that there is a point $x \in D_1 \cap B^\circ_2$. If $x \in (G_1 - J)(B^\circ_1)$, then there are an element $g \in G_1 - J$ and a point $y \in B^\circ_1$ with $x = g(y)$. Since y lies in $B^\circ_1 \cap {}^\circ\Omega(G_1) \subset B^\circ_1 \cap {}^\circ\Omega(J) = J(D_1 \cap B^\circ_1)$, there are an element $j \in J$ and a point $z \in D_1 \cap B^\circ_1$ with $y = j(z)$, which implies $x = gj(z)$. Since D_1 is a fundamental set of G_1, it follows that $x = z$ and $g = j^{-1}$, which is a contradiction. Therefore x is not contained in $(G_1 - J)(B_1^\circ)$ and (B_1°, B_2°) is proper. Thus we have proved (10). $\square$

DEFINITION 4.1. Let $\{S_j\}$ be a collection of topological $(n-1)$-spheres. We say that the sequence $\{S_j\}$ *nests about* the point x if the following are satisfied.
(1) The spheres S_j are pairwise disjoint.
(2) For each j, the sphere S_j separates x from the precedent S_{j-1};
(3) For any point $z_j \in S_j$, the sequence $\{z_j\}$ converges to x.

Now we can state and prove our main theorem.

Theorem 4.2. *Let J be a geometrically finite proper subgroup of two discrete groups G_1 and G_2 in $M(\overline{\mathbb{R}}^n)$. Assume that there is a topological $(n-1)$-sphere S dividing $\overline{\mathbb{R}}^n$ into two closed topological balls B_1 and B_2 such that each B_m is a (J, G_m)-block and (B_1°, B_2°) is a proper interactive pair. Assume that for $m = 1, 2$, there is a fundamental set D_m for G_m such that $J(D_m \cap B_m) = B_m \cap {}^\circ\Omega(J)$, $D_m \cap B_{3-m}$ is either empty or has nonempty interior, and $D_1 \cap S = D_2 \cap S$. Set $D = (D_1 \cap B_2) \cup (D_2 \cap B_1)$ and $G = \langle G_1, G_2\rangle$. Then the following hold.*
(1) *$G = G_1 *_J G_2$.*
(2) *G is discrete.*
(3) *If an element g of G is not loxodromic, then one of the following must hold.*
   (a) *g is conjugate to an element of either G_1 or G_2.*
   (b) *g is parabolic and is conjugate to an element fixing a parabolic fixed point of J.*
(4) *S is a precisely embedded (J, G)-block.*
(5) *If $\{S_k\}$ is a sequence of distinct G-translates of S, then* $\operatorname{diam}(S_k) \to 0$, *where* diam *denotes the diameter with respect to the ordinary spherical metric on $\overline{\mathbb{R}}^n$.*
(6) *There is a sequence of distinct G-translates of S nesting about the point x if and only if x is a limit point of G which is not G-equivalent to a limit point of either G_1 or G_2.*
(7) *D is a fundamental set for G. If both D_1 and D_2 are constrained, and $S \cap \operatorname{Fr} D$ consists of finitely many connected components the sum of whose $(n-1)$-dimensional measures on S vanishes, then D is also constrained.*
(8) *Let Q_m be the union of the G_m-translates of B_m°, and let R_m be the complement of Q_m in $\overline{\mathbb{R}}^n$. Then $\Omega(G)/G = (R_1 \cap \Omega(G_1))/G_1 \cup (R_2 \cap \Omega(G_2))/G_2$, where the latter two possibly disconnected orbifolds are identified along their common possibly disconnected or empty boundary $(S \cap \Omega(J))/J$.*

Furthermore, under the assumption that S is a strong (J, G)-block if and only if for $m = 1, 2$, each B_m is a strong (J, G_m)-block, two more statements hold.
(9) *If both B_1 and B_2 are strong, then, except for G-translates of limit points of G_1 or G_2, every limit point of G is a conical limit point.*
(10) *G is geometrically finite if and only if both G_1 and G_2 are geometrically finite.*

Proof of (1). Since (B_1°, B_2°) is proper, (1) follows from Theorem 2.7. $\square$

Proof of (2). Suppose that G is not discrete. Then there is a sequence $\{g_k\}$ of distinct elements of G which converges to the identity uniformly on compact subsets. Express g_k in a normal form $g_k = \gamma_{n_k} \circ \gamma_{n_{k-1}} \circ \cdots \circ \gamma_{n_1}$. We may assume that each g_k has even length, for if g_k has odd length, then by Lemma 2.6, either $g_k(B_1^\circ) \subset B_2^\circ$, or $g_k(B_2^\circ) \subset B_1^\circ$, and such elements cannot converge to the identity. By interchanging B_1 and B_2 if necessary, we may assume that $(G_1 - J)(B_1^\circ)$ is a proper subset of B_2° since (B_1°, B_2°) is proper. By choosing a subsequence, we may assume that all the g_k are $(1, 2)$-forms or all of them are $(2, 1)$-forms. It suffices to prove the case that every g_k is a $(1, 2)$-form since if g_k is a $(2, 1)$-form, then g_k^{-1} is a $(1, 2)$-form.

Since we assumed that each g_k is a $(1, 2)$-form, we have $g_k(B_2^\circ) \subset \gamma_{n_k} \circ \gamma_{n_{k-1}}(B_2^\circ)$. If $\gamma_{n_{k-1}}(B_2^\circ) = B_1^\circ$, then $g_k(B_2^\circ) \subset \gamma_{n_k}(B_1^\circ) \subset B_2^\circ$, with the last inclusion being proper, and if $\gamma_{n_{k-1}}(B_2^\circ)$ is a proper subset of B_1°, then $g_k(B_2^\circ) \subset \gamma_{n_k} \circ \gamma_{n_{k-1}}(B_2^\circ) \subset \gamma_{n_k}(B_1^\circ) \subset B_2^\circ$, with the last two inclusions being proper. Therefore, in either case, we have $g_k(B_2^\circ) \subset \gamma_{n_k}(B_1^\circ) \subset B_2^\circ$, with the last inclusion being proper. Thus $B_2^\circ - g_k(B_2^\circ) \supset B_2^\circ - \gamma_{n_k}(B_1^\circ) \supset B_2^\circ - (G_1 - J)(B_1^\circ)$. Since $g_k \to id$ on B_2 and $B_2^\circ \setminus (G_1 - J)(B_1^\circ) \neq \emptyset$, this is a contradiction. $\square$

Now for a normal form $g = g_n \cdots g_1 \in G$, we call g *positive* if $g_1 \in G_1 - J$ and we express it as $g > 0$; we call g *negative* if $g_1 \in G_2 - J$ and we express it as $g < 0$.

Using this distinction, we consider a coset decomposition of G:

$$G = J \cup \left(\bigcup_{n,k} a_{nk} J \right) \cup \left(\bigcup_{n,k} b_{nk} J \right),$$

where $|a_{nk}| = |b_{nk}| = n$, $a_{nk} > 0$, and $b_{nk} < 0$. Following Apanasov [6], we set $T_n = \left(\bigcup_k a_{nk}(B_1)\right) \cup \left(\bigcup_k b_{nk}(B_2)\right)$, $C_n = \overline{\mathbb{R}}^n \setminus T_n$, $C = \bigcup C_n$, and $T = \overline{\mathbb{R}}^n \setminus C = \bigcap T_n$.

Then we have the following.

Lemma 4.3. *$\{T_n\}$ is a decreasing sequence with respect to the inclusion, that is,* $T_1 \supset T_2 \supset \cdots$.

Proof. Take a point $x \in T_n$ $(n > 1)$. Then either there are an element $a_{nk} > 0$ with length n and a point $y \in B_1$ satisfying that $x = a_{nk}(y)$, or there are an element $b_{nk} < 0$ with length n and a point $y \in B_2$ satisfying that $x = b_{nk}(y)$. In the former case, if we express a_{nk} in a normal form as $g_n \circ \cdots \circ g_1$, then $g_1 \in G_1 - J$. Since $g_1(y)$ lies in $g_1(B_1) \subset B_2$, there is a point $z \in B_2$ with $g_1(y) = z$. Therefore, $x = a_{nk}(y) = g_n \circ \cdots \circ g_2(z) \in b_{(n-1)s}(B_2) \subset T_{n-1}$. In the latter case, by the same argument we have $x \in T_{n-1}$. $\square$

Lemma 4.4. *The sphere S is precisely embedded in G. If S is precisely invariant under J in G_1 and G_2, respectively, then S is precisely invariant under J in G.*

Proof. We shall first show that S is precisely embedded. For any $g \in G$ with $|g| = 0$, we have $g(S) = S$ and is disjoint from both B_1° and B_2°. If $|g| = 1$, then $g \in G_m - J$ $(m = 1, 2)$, and $g(S) = g(\mathrm{Fr}\, B_m) \subset g(B_m) \subset B_{3-m}$. This means that $g(S)$ is disjoint from B_m°.

Now let $g = g_n \circ \cdots \circ g_1$ be an (m, k)-form with $|g| > 1$. Then $g(S) = g(\mathrm{Fr}\, B_k) \subset g(B_k) \subset B_{3-m}$ since $g(B_k^\circ) \subset B_{3-m}^\circ$ by Lemma 2.6. This means that $g(S)$ is disjoint from B_m° again, and we have thus shown that S is precisely embedded in G.

Now suppose that S is precisely invariant under J both in G_1 and G_2. Since, as was shown above, for $g \in J$, we have $g(S) = S$, we have only to show that $g(S) \cap S = \emptyset$ for $g \in G - J$. Note that $g(S) = g(\mathrm{Fr}\, B_m) \subset g(B_m) \subset B_{3-m}^\circ$ for any $g \in G_m - J$. Therefore, it remains to consider the case when $|g| > 1$. If $g = g_n \circ \cdots \circ g_1$ is an (m, k)-form with $|g| > 1$, then $h = g_n^{-1} \circ g$ is a $(3 - m, k)$-form. It follows from Lemma 2.6 that $g(S) = g_n \circ h(S) = g_n \circ h(\mathrm{Fr}\, B_k) \subset g_n \circ h(B_k) \subset g_n(B_m) \subset B_{3-m}^\circ$. Thus, we have shown that for any $g \in G - J$, $g(S) \cap S = \emptyset$. □

Lemma 4.5. $D \subset C_1$.

Proof. We assume that $D \neq \emptyset$. By interchanging B_1 and B_2 if necessary, we can assume that $D_1 \cap B_2 \neq \emptyset$. If there is a point $x \in D_1 \cap S = D_2 \cap S$, then no $(G_m - J)$-translates of B_m pass through x as was shown in the proof of Lemma 4.1-(10). This implies that $x \in C_1$.

It remains to consider the case when $x \in D_1 \cap B_2^\circ$. If $x \in (G_1 - J)(B_1)$, then there are an element $g \in G_1 - J$ and a point $y \in B_1$ with $x = g(y)$. Since $y \in {}^\circ\Omega(G_1) \cap B_1 \subset {}^\circ\Omega(J) \cap B_1$, there are an element $j \in J$ and a point $z \in D_1 \cap B_1$ with $y = j(z)$ by the assumption that $J(D_1 \cap B_1) = {}^\circ\Omega(J) \cap B_1$ in Theorem 4.2. Therefore we have $x = gj(z)$, which implies that $x = z$ and $gj = id$. This contradicts the assumption that g lies in $G_1 - J$. Thus we have shown that $x \in C_1$. □

Lemma 4.6. *D is contained in ${}^\circ\Omega(G)$, and precisely invariant under $\{id\}$ in G.*

Proof. We shall first prove that D is contained in $\Omega(G)$. Suppose, on the contrary, that there is a point z in $D \cap \Lambda(G)$. Since $D = (D_1 \cap B_2) \cup (D_2 \cap B_1)$, we can assume that $z \in D_1 \cap B_2$ by interchanging the indices if necessary.

Claim 5. *In this situation, we have $z \in D_1 \cap S$.*

Proof of Claim 5. Suppose not. Then z must be contained in $D_1 \cap B_2^\circ$. Since $z \in \Lambda(G)$, it follows from Lemma 2.2 that there is a sequence $\{g_k\}$ of distinct elements in G such that $g_k(y) \to z$ for all y with at most one exception. Since $z \in B_2^\circ \subset \Omega(G_2)$ (by

Lemma 4.1-(3)) and $z \in D_1 \subset \Omega(G_1)$, we have $|g_k| > 1$, and we can assume that each g_k is a 1-form. Since $g_k(B) \subset T_1$ for B which is equal to B_1 or B_2, Lemma 4.5 implies that $z \in \mathrm{Fr}\, T_1$. Since $z \in D_1 \subset \Omega(G_1)$ and every point of $B_2^\circ \cap \mathrm{Fr}\, T_1$ is either a $(G_1 - J)$-translate of a point of S or a limit point of G_1, we deduce that z is a $(G_1 - J)$-translate of a point of S. On the other hand, since z is contained in $C_1 = \overline{\mathbb{R}}^n \setminus T_1$, we see that z is not a $(G_1 - J)$-translate of a point of S. This is a contradiction. $\square$

Since $z \in D_1 \cap S = D_2 \cap S$, as was shown in the proof of Lemma 4.1-(10), no $(G_m - J)$-translates of B_m pass through z nor accumulate at z. Therefore, we have $z \in C_1^\circ$. Since $\{T_n\}$ is decreasing, the $(G - J)$-translates of S do not accumulate at z, for $(G - J)$-translates of S accumulate at points in $\overline{T}_1$, which is disjoint from C_1°. This means that z cannot be a limit point of G; hence $z \in \Omega(G)$. Thus we have shown that D is contained in $\Omega(G)$.

By Lemma 4.1-(6) and Lemma 2.8, we see that $(D_1 \cap B_2^\circ) \cup (D_2 \cap B_1^\circ)$ is precisely invariant under $\{id\}$ in G. Setting $A = (D_1 \cap B_2^\circ) \cup (D_2 \cap B_1^\circ)$, we have $D = A \cup (D_1 \cap S)$ and $A \subset C_1^\circ$. Then for any $g \in G - \{id\}$, we have $g(D) \cap D = (g(A) \cap (D_1 \cap S)) \cup (g(D_1 \cap S) \cap A) \cup (g(D_1 \cap S) \cap (D_1 \cap S))$.

If $g \in J - \{id\}$, then $g(D_1 \cap S) \subset S \setminus D_1$ and $g(A) \cup A \subset B_1^\circ \cup B_2^\circ$. Therefore, $g(D_1 \cap S) \cap (D_1 \cap S) = \emptyset$, $g(D_1 \cap S) \cap A = \emptyset$ and $g(A) \cap (D_1 \cap S) = \emptyset$. It follows that $g(D) \cap D = \emptyset$ in this case.

If $g \in G_m - J$, then $g(D_1 \cap S) = g(D_m \cap S) \subset T_1$ and Lemma 4.1-(4) and (6) imply that $g(A) \subset B_{3-m}^\circ$. Since $A \cup (D_1 \cap S) = D$ is contained in C_1 by Lemma 4.5, and $g(D_1 \cap S)$ is contained in T_1, we have $g(D_1 \cap S) \cap A = \emptyset$. We also have $g(D_1 \cap S) \cap (D_1 \cap S) = \emptyset$ since $D_1 \cap S = D_2 \cap S$ and D_1, D_2 are fundamental sets of G_1, G_2 respectively, and $g(A) \cap (D_1 \cap S) = \emptyset$ since $g(A)$ is contained in B_{3-m}° as was seen above. Therefore also in this case, we have $g(D) \cap D = \emptyset$.

Now, we consider $g = g_n \circ \cdots \circ g_1 \in G - (G_1 \cup G_2)$, where $g_1 \in G_m - J$. Then $g(D_1 \cap S) = g(D_m \cap S) \subset g(B_m) \subset T_n \subset T_1$ and $g(A) = g(D_m \cap B_{3-m}^\circ) \cup g(D_{3-m} \cap B_m^\circ) \subset g_n \circ \cdots \circ g_2(B_{3-m}^\circ) \cup g(B_m^\circ)$ (Lemma 4.1-(6)) $\subset T_{n-1}^\circ \cup T_n^\circ \subset T_1^\circ \subset B_1^\circ \cup B_2^\circ$. These facts imply that $g(D_1 \cap S) \cap (D_1 \cap S) = \emptyset$ by Lemma 4.5, $g(D_1 \cap S) \cap A = \emptyset$ by the fact that $A \subset C_1^\circ$, and $g(A) \cap (D_1 \cap S) = \emptyset$. Thus we have shown that D is precisely invariant under $\{id\}$ in G. Since we have already shown that $D \subset \Omega(G)$, this means that $D \subset {}^\circ\Omega(G)$. $\square$

Lemma 4.7. *$S \cap \Omega(J) = S \cap \Omega(G)$, and $S \cap \Omega(J)$ is precisely invariant under J in G.*

Proof. Let z be a point in $S \cap \Omega(J)$. Since $S \cap \Omega(G_m) = S \cap \Omega(J)$ for each m by Lemma 4.1-(2), we have $z \in \Omega(G_m)$. As was shown in the proof of Lemma 4.1-(10), no $(G_m - J)$-translates of B_m pass through z nor accumulate at z. Therefore z is contained in C_1°.

Suppose, seeking a contradiction, that z lies in $\Lambda(G)$. Then there is a sequence $\{g_k\}$ of distinct elements of G such that $g_k(y) \to z$ for all y with at most one exception. Since z is contained in $\Omega(G_1) \cap \Omega(G_2)$, we can assume $|g_k| > 1$ for all k by taking a subsequence. We deduce from the fact that $g_k(B) \subset T_1$ for $B = B_1$ or B_2 that z must be contained in $\overline{T}_1$, which is a contradiction. Thus we have shown that $S \cap \Omega(J)$ is contained in $S \cap \Omega(G)$. The opposite inclusion is trivial.

Now we turn to prove the latter half of our lemma. It is clear that J keeps $S \cap \Omega(J)$ invariant. Suppose that there are points y and z in $S \cap \Omega(G) = S \cap \Omega(J)$ and that there is an element $g \in G - J$ such that $g(y) = z$. Express g in a normal form $g = g_n \circ \cdots \circ g_1$. Then $n > 1$ since S is a (J, G_m)-block $(m = 1, 2)$. Clearly z lies on $g(S) \cap S$. Moreover since $g(S) = g_n(g_{n-1} \circ \cdots \circ g_1(S))$ and S is contained in both B_1 and B_2, by Lemma 2.6, $g(S)$ is contained in either $g_n(B_m)$, where g_n is assumed to lie in G_m. If $z \in g(S)$ is contained in $g_n(B_m^\circ)$, then it must lie in B_{3-m}°, which contradicts our assumption. Therefore z must lie in $g_n(S)$. We may assume that $g_n \in G_1 - J$ by interchanging the indices if necessary. Since B_1 is a (J, G_1)-block, $B_1 \cap \Omega(G_1)$ is precisely invariant under J in G_1, which means that $g_n(\Omega(G_1) \cap B_1)$ is contained in B_2°. Because we have shown that z lies in $S \cap g_n(S)$, this implies that $z \in \Lambda(G_1) \subset \Lambda(G)$. Since $z = g(y) \in \Omega(G)$, this is a contradiction. Thus we have shown that $g(S \cap \Omega(G)) \cap (S \cap \Omega(G)) = \emptyset$ for any $g \in G - J$. □

Proof of (3). Let g be an element of G which is not conjugate to any element of either G_1 or G_2, such that $|g|$ is minimal among all conjugates of g in G. Clearly, we have $|g| > 1$. Express g in a normal form $g = g_n \circ \cdots \circ g_1$. If the length of g is odd, say, $g_n, g_1 \in G_m - J$, then $g_n^{-1} \circ g \circ g_n = g_{n-1} \circ \cdots \circ (g_1 \circ g_n)$. The corresponding normal form of $g_n^{-1} \circ g \circ g_n$ has length less than n, which contradicts the minimality of $|g|$. Therefore the length of g must be even and g must be a $(3-m, m)$-form. This implies that $g(B_m) \subset g_n \circ g_{n-1}(B_m) \subset B_m$. Since (B_1°, B_2°) is a proper interactive pair by assumption, the last inclusion is proper by Lemma 2.6. Hence g has the infinite order and has a fixed point in $g(B_m) \subset B_m$. Similarly, $g^{-1}(B_{3-m}) \subset g_1^{-1} \circ g_2^{-1}(B_{3-m}) \subset B_{3-m}$, where the last inclusion is proper. Therefore g also has a fixed point in $g^{-1}(B_{3-m}) \subset B_{3-m}$, which may coincide with the above-mentioned fixed point.

Since G is discrete and g has infinite order, g is not elliptic. If g is parabolic, then its fixed point is unique, which we denote by x. Hence the two fixed points mentioned above are equal and x lies on $S \cap g(S)$. By Lemma 4.7, x is a limit point of J. Since J is geometrically finite, x is either a parabolic fixed point of J or a conical limit point for J by Proposition 2.13. Since a conical limit point for J is also that for G and a conical limit point cannot be a parabolic fixed point, we see that x is a parabolic fixed point of J. □

Proof of (4). Since B_1 and B_2 are both blocks, for every parabolic fixed point z of J with the rank of $\mathrm{Stab}_J(z)$ being less than n, the peak domain centered at z for J has trivial intersection with $S = \mathrm{Fr}\, B_1 = \mathrm{Fr}\, B_2$. This shows the second condition in the definition of blocks holds for S. Lemma 4.7 implies that the first condition in the definition holds for S, hence that S is a (J, G)-block. By Lemma 4.4, S is precisely embedded in G. □

Proof of (5). By (4) shown above, we know that S is a (J, G)-block. Then (5) follows from Theorem 3.1. □

Lemma 4.8. *$C_1 \cap B_m^\circ$ is precisely invariant under G_{3-m} in G.*

Proof. It is obvious that $C_1 \cap B_m^\circ = \overline{\mathbb{R}}^n - G_{3-m}(B_{3-m})$. Since $G_{3-m}(B_{3-m})$ is invariant under G_{3-m}, its complement $C_1 \cap B_m^\circ$ is also invariant under G_{3-m}.

If $g \in G_m - J$, then $g(C_1 \cap B_m^\circ) \subset g(B_m^\circ) \subset B_{3-m}^\circ$, and we are done. Now we consider a general g which is expressed in a normal form $g = g_n \circ \cdots \circ g_1$ with $|g| > 1$. If g is an (m, m)-form, then $g(C_1 \cap B_m^\circ) \subset g(B_m^\circ) \subset B_{3-m}^\circ$ by Lemma 2.6. If g is an $(m, 3-m)$-form, then $g(C_1 \cap B_m^\circ) = g_n \circ \cdots \circ g_1(C_1 \cap B_m^\circ) = g_n \circ \cdots \circ g_2(C_1 \cap B_m^\circ)$ as was shown in the last paragraph, and this last term is contained in B_{3-m}° since $g_n \circ \cdots \circ g_2$ is an (m, m)-form. If $g = g_n \circ \cdots \circ g_1$ is a $(3-m, k)$-form, where either $k = 1$ or $k = 2$, then, by the discussion above, we see $g_{n-1} \circ \cdots \circ g_1(C_1 \cap B_m^\circ) \subset B_{3-m}^\circ$; hence $g(C_1 \cap B_m^\circ) \subset g_n(B_{3-m}^\circ) \subset T_1^\circ$. Thus in every case, if $g \notin G_{3-m}$, then $g(C_1 \cap B_m^\circ) \cap (C_1 \cap B_m^\circ) = \emptyset$. □

Lemma 4.9. *The set C is contained in the union of $\Omega(G) \setminus {}^\circ\Omega(G)$ and the G-translates of $D \cup \Lambda(G_1) \cup \Lambda(G_2)$.*

Proof. Every point $x \in C$ is contained either in C_1 or in $C_n \setminus C_{n-1}$ for some index n $(n > 1)$ since $\{C_n\}$ is increasing. If $x \in C_n \setminus C_{n-1}$, then $x \in T_{n-1} \setminus T_n$. Hence there are a point $y \in B_k$ and an element expressed in an (m, k)-form $g = g_{n-1} \circ \cdots \circ g_1 \in G$ such that $x = g(y)$. If y lies in T_1, then either $y \in (G_k - J)(B_k) \cap B_k$ or $y \in (G_{3-k} - J)(B_{3-k})$. In the former case, y is contained in $\Lambda(G_k) \cap S = \Lambda(J) \cap S$ by Lemma 4.1-(5). In the latter case, we have $x \in T_n$, which is a contradiction. Therefore, every point $x \in C$ is either contained in $G(\Lambda(J))$ or $G(C_1)$. In the former case, we are done. Therefore, we have only to consider the latter case. Moreover, since the sets in our statement are G-invariant, we can assume that x lies in C_1.

It suffices to prove our lemma under the assumption that $x \in C_1 \cap B_2$; the proof for the case $x \in C_1 \cap B_1$ is the same. If x lies in $C_1 \cap B_2$, then either $x \in \Lambda(G_1)$ or $x \in {}^\circ\Omega(G_1)$ or $x \in \Omega(G_1) \setminus {}^\circ\Omega(G_1)$. We only need to discuss the latter two cases.

CASE 1: $x \in {}^\circ\Omega(G_1)$.

In this case, there are an element $g \in G_1$ and a point $z \in D_1$ with $g(z) = x$. We claim that $z \notin B_1^\circ$. Suppose, on the contrary, that z is contained in B_1°. If g lies in $G_1 - J$, then $g(z)$ is contained in T_1 by the definition of T_1. Since we assumed that x lies in C_1, this is not possible. Therefore, we have $g \in J$. On the other hand, $J(B_1^\circ) =$

B_1°, which contradicts the assumption that x lies in B_2. This shows that $z \in D_1 \cap B_2 \subset D$, and we are done in this case.

CASE 2: $x \in \Omega(G_1) \setminus {}^\circ\Omega(G_1)$.

Since $S \cap \Omega(J) = S \cap \Omega(G_1) = S \cap \Omega(G_2) = S \cap \Omega(G)$ by Lemma 4.7, if $x \in S$, then x lies in $\Omega(G)$. Furthermore, since ${}^\circ\Omega(G)$ is contained in ${}^\circ\Omega(G_1)$, this implies that $x \in \Omega(G) \setminus {}^\circ\Omega(G)$, and we are done in this case. If $x \notin S$, then $x \in C_1 \cap B_2^\circ$. Since $x \in \Omega(G_1)$, no $(G_1 - J)$-translates of B_1 accumulate at x as was shown in the proof of Lemma 4.1-(10). Therefore, we have $x \in C_1^\circ$. Then, by Proposition 2.4, there is a neighbourhood U of x contained in $C_1 \cap B_2^\circ$ such that U is precisely invariant under $\mathrm{Stab}_{G_1}(x)$ in G_1 and $\mathrm{Stab}_{G_1}(x)$ is a non-trivial finite subgroup. Now Lemma 4.8 implies that $\mathrm{Stab}_{G_1}(x) = \mathrm{Stab}_G(x)$. Hence U is precisely invariant under $\mathrm{Stab}_G(x)$ in G. This shows that x is contained in $\Omega(G) \setminus {}^\circ\Omega(G)$, and we have completed the proof. □

Lemma 4.10. *$T \subset \Lambda(G)$. Furthermore, every point of T is either a G-translate of a point in $\Lambda(J)$ or the limit of nested translates of S.*

Proof. Consider a point $z \in T$. We assume that $z \in (G_1 - J)(B_1)$, for the case when $z \in (G_2 - J)(B_2)$ can be dealt with in the same way. Then there is an element $h_1 = g_1 \in G_1 - J$ such that $z \in g_1(B_1)$. Since $T_1 \supset T_2$, we have $z \in T_2$, and there is an element $g_2 \in G_2 - J$ such that $z \in g_1 \circ g_2(B_2) = h_2(B_2) \subset h_1(B_1)$. Similarly, since $z \in T_3$, there is an element $g_3 \in G_1 - J$ such that $z \in g_1 \circ g_2 \circ g_3(B_1) = h_3(B_1) \subset h_2(B_2) \subset h_1(B_1)$; etc. Since the element h_k has length increasing as $k \to \infty$ and (B_1°, B_2°) is a proper interactive pair, the sets $h_k(S)$ can be assumed to be all distinct by taking a subsequence if necessary. Thus we have shown that if $z \in T$, then there is a sequence $\{h_k\}$ of elements of G, with $|h_k| \to \infty$, and $z \in \cdots \subset h_k(\check{B}_k) \subset \cdots \subset h_2(\check{B}_2) \subset h_1(\check{B}_1)$, where $\check{B}_j$ is either B_1 or B_2. Passing to a subsequence if necessary, we may assume that $\check{B}_j = B_1$.

There are two possibilities for this sequence: either z lies in the interiors of infinitely many $h_k(B_1)$, or from some k on, z lies on the boundary of every $h_k(B_1)$. In either case, since the $h_k(S)$ are distinct, we have $\mathrm{diam}(h_k(S)) \to 0$. Since the ball $h_k(B_1)$ bounded by $h_k(S)$ decreases as $k \to \infty$, this is possible only when $\mathrm{diam}(h_k(B_1)) \to 0$. Since z is a limit of $\{h_k(x_k)\}$ with $x_k \in B_1$ in either case above, it follows that for every $x \in B_1$, we have $h_k(x) \to z$. This means that z lies in $\Lambda(G)$. Moreover, in the former case, we have shown that $\{h_k(S)\}$ nests around z. In the latter case, since $z \in h_{k_0}(S) \cap h_{k_0+1}(S) \cap \cdots$, we have $w = h_{k_0}^{-1}(z) \in S \cap h_{k_0}^{-1} h_{k_0+1}(S) \cap \cdots$. Since such w is contained in $\Lambda(G)$, by Lemma 4.7, it also lies in $\Lambda(J)$. This means that z is contained in the G-translate of $\Lambda(J)$. □

Lemma 4.11. *If $z \in C \cap \Lambda(G)$, then there is no sequence of distinct translates of S nesting about z.*

Proof. Lemma 4.9 implies that z is a G-translate of a point in either D or $\Lambda(G_1) \cup \Lambda(G_2)$. Since D is contained in $\Omega(G)$ by Lemma 4.6, the only possibility is $z \in G(\Lambda(G_1) \cup \Lambda(G_2))$.

We first consider the special case when z lies in $G(\Lambda(J))$. Under this assumption, suppose, seeking a contradiction, that there is a sequence $\{h_k(S)\}$ of distinct G-translates of S nesting about $z = g(y)$ for an element $g \in G$ and a point $y \in \Lambda(J) \subset S$. Then we have $z \in h_k(B^\circ)$ by taking a subsequence for B which is either B_1 or B_2. We can assume that B is B_1 after taking a subsequence, for we can deal with the other case in the same way. It follows that $y \in g^{-1} \circ h_k(B_1^\circ)$. Now since $\{h_k(S)\}$ nests around z, we have $\operatorname{diam}(h_k(B_1)) \to 0$. This is possible only when after taking a subsequence all h_k are $(m_k, 1)$-forms with $m_k = 1, 2$. (If h_k were $(m_k, 2)$-form, then $h_k(B_1)$ would contain S; hence its diameter would not go to 0.) Therefore $g^{-1}h_k$ is also expressed as an $(m', 1)$-form for large k and $g^{-1}h_k(B_1^\circ)$ is contained in $B_{3-m'}^\circ$. In particular, we have $y \notin S$. This contradiction shows that if $z \in G(\Lambda(J))$, then there is no sequence of distinct translates of S nesting about z.

Now we turn to the general case when $z \in G(\Lambda(G_1) \cup \Lambda(G_2))$. It suffices to consider the case $z \in G(\Lambda(G_1))$ since the proof for the case $z \in G(\Lambda(G_2))$ is entirely the same. Then there are an element $g \in G$ and a point $y \in \Lambda(G_1)$ with $g(y) = z$. Since $B_1^\circ \subset \Omega(G_1)$, we have $\Lambda(G_1) \subset \overline{\mathbb{R}}^n \setminus G_1(B_1^\circ)$. Therefore, y is not contained in $G_1(B_1^\circ)$; hence unless y lies in $G_1(S)$, it must lie in $C_1 \cap B_2^\circ$. If $y \in G_1(S)$, then $y \in G_1(S \cap \Lambda(G_1)) = G_1(S \cap \Lambda(J))$. The discussion in the previous paragraph implies that this case cannot occur.

Now we assume that $y \in C_1 \cap B_2^\circ$. If there is a sequence $\{h_k(S)\}$ of distinct G-translates of S nesting about $z = g(y)$, then $z \in h_k(B^\circ)$ for every k where B is B_1 or B_2, and hence $y \in g^{-1} \circ h_k(B^\circ)$. We may assume that $B = B_1$ by changing the index and taking a subsequence and h_k is an $(m, 1)$-form. Then $g^{-1} \circ h_k$ is also an $(m', 1)$-form for sufficiently large k. Since $\{T_n\}$ is a decreasing sequence, $y \in T_1^\circ$, which is a contradiction. Thus we have completed the proof. □

Proof of (6). If x lies in $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$, then $x \in T$ by Lemma 4.9. Since every point of T is either a translate of a point of $\Lambda(J)$ or is the limit of a nested sequence of translates of S by Lemma 4.10, we have proved the "if" part.

Now we turn to the "only if" part. Suppose that x lies in $\Lambda(G_m)$ for $m = 1$ or 2. Since $B_m^\circ \subset \Omega(G_m)$ by Lemma 4.1-(3), we have $x \in \overline{\mathbb{R}}^n \setminus G_m(B_m^\circ)$. If $x \in G_m(S)$, then as was shown in the proof of Lemma 4.11, there is no distinct G-translates of S nesting about x. Therefore x is contained in $\overline{\mathbb{R}}^n \setminus G_m(B_m) = C_1 \cap B_{3-m}^\circ$, which implies that $x \in C \cap \Lambda(G)$. By Lemma 4.11, there is no distinct translates of S nesting about x. □

Proof of (7). By Lemma 4.9, every point of $C \cap {}^\circ\Omega(G)$ is a translate of a point of D. Also by Lemma 4.10, T is contained in $\Lambda(G)$. This shows that every point of ${}^\circ\Omega(G)$ is contained in a G-translate of D. Furthermore, since $D \subset {}^\circ\Omega(G)$ and D

is precisely invariant under the identity in G by Lemma 4.6, it follows that D is a fundamental set for G.

Now assume that both D_1 and D_2 are constrained.

Claim 6. $\Omega(G) \subset G(\bar{D})$.

Proof. Since we have already shown that D is a fundamental set for G, we have only to prove that if $x \in \Omega(G) \setminus {}^\circ\Omega(G)$, then there is an element $g \in G$ with $g(x) \in \bar{D}$. Now let x be a point in $\Omega(G) \setminus {}^\circ\Omega(G)$. By Lemma 4.10, x is not contained in T. As was shown in the proof of Lemma 4.9, we have $x \in G(C_1) \cap (\Omega(G) \setminus {}^\circ\Omega(G))$. This means that there are an element $g \in G$ and a point $y \in C_1 \cap (\Omega(G) \setminus {}^\circ\Omega(G))$ such that $x = g(y)$. We may assume that $y \in B_2$, for the proof in the case $y \in B_1$ is entirely the same.

Suppose first that $y \in S \cap C_1 \cap (\Omega(G) \setminus {}^\circ\Omega(G))$. Then since $S \cap \Omega(J) = S \cap \Omega(G_1) = S \cap \Omega(G)$ by Lemma 4.7 and D_1 is a constrained fundamental set for G_1, there are an element $h \in G_1$ and a point $z \in \bar{D}_1$ such that $y = h(z)$. Since $G_1(B_1^\circ) \subset B_1^\circ \cup T_1^\circ$, we see that z must be contained in B_2, hence $z \in \bar{D}_1 \cap B_2 \subset \bar{D}$. Thus we have completed the proof in this case.

Next we assume that $y \notin S$, which means that $y \in C_1 \cap B_2^\circ \cap (\Omega(G) \setminus {}^\circ\Omega(G))$. Since $y \in \Omega(G) \subset \Omega(G_1)$ and D_1 is a fundamental set for G_1, we see that y is G_1-equivalent to a point $w \in \bar{D}_1$. By Lemma 4.8, we have $w \in \bar{D}_1 \cap C_1 \cap B_2^\circ$. Since $\bar{D}_1 \cap B_2^\circ \subset \bar{D}$, this implies $w \in \bar{D}$, and our claim has been proved. □

We now return to the proof of (7). We have

$$G_m(\bar{D}_m) = G_m((\bar{D}_m \cap B_m^\circ) \cup (\bar{D}_m \cap B_{3-m})), \tag{4.1}$$

$$G_m(\bar{D}_m \cap B_m^\circ) \subset B_m^\circ \cup (T_1^\circ \cap B_{3-m}^\circ) \tag{4.2}$$

by the definition of T_1, and

$$\bar{D}_m \cap B_{3-m}^\circ \subset \overline{D_m \cap B_{3-m}} \subset \bar{C}_1 \cap B_{3-m} \tag{4.3}$$

by Lemma 4.5.

Since $\bar{C}_1 \cap B_{3-m} = \bar{\mathbb{R}}^n \setminus G_m(B_m^\circ)$, we see that $\bar{C}_1 \cap B_{3-m}$ is G_m-invariant. Therefore from (4.3), we obtain

$$G_m(\bar{D}_m \cap B_{3-m}^\circ) \subset \bar{C}_1 \cap B_{3-m}. \tag{4.4}$$

Since $\operatorname{Fr} D \cap S$ consists of only finitely many connected components the sum of whose $(n-1)$-dimensional measures on S vanishes by assumption, it follows from (4.1), (4.2), and (4.4) that the sides of D_m in B_{3-m} are paired with those in B_{3-m} by elements of G_m for each m. Since the sides of D in B_1 are equal to those of D_2 in B_1 and the sides of D in B_2 those of D_1 in B_2, we see the sides are paired to each other. These sides can accumulate only at limit points because of the same property for $\bar{D}_1$ and D_2. The only thing left to show is that the tessellation of $\Omega(G)$ by translates of $\bar{D}$ is locally finite.

Take any $z \in \bar{D} \cap \Omega(G)$. We see from Lemma 4.5 that either $z \in C_1^\circ$ or $z \in \mathrm{Fr}\, C_1 = \mathrm{Fr}\, T_1$. We may assume that $z \in \overline{D_1 \cap B_2} \subset \bar{D}_1 \cap B_2$, for the proof in the case $z \in \bar{D}_2 \cap B_1$ is entirely the same.

CASE 1: $z \in C_1^\circ$.

Since z is contained in $\Omega(G_m)$ for each m and D_m is a constrained fundamental set for G_m, there is a neighborhood U of z with $U \subset C_1^\circ$ such that for each m there is a finite set $\{g_{m1}(D_m), \ldots, g_{mk_m}(D_m)\}$ with $U \subset \bigcup_i g_{mi}(\bar{D}_m)$ for $g_{mi} \in G_m$. We consider $U \cap B_{3-m}$. Since $G_m(\bar{D}_m \cap B_m^\circ) \subset B_m^\circ \cup T_1^\circ$ and $U \subset C_1$, we have $U \cap B_{3-m} \subset \bigcup_i g_{mi}(\bar{D}_m \cap B_{3-m})$. Therefore $U \subset \bigcup_{m=1}^2 \big(\bigcup g_{mi}(\bar{D}_m \cap B_{3-m})\big) \subset \bigcup_{m=1}^2 \big(\bigcup_i g_{mi}(\bar{D})\big)$, and we have obtained the local finiteness of D at such a point.

CASE 2: $z \in \mathrm{Fr}\, C_1 = \mathrm{Fr}\, T_1$.

We claim that $z \notin S$ in this case. Suppose, on the contrary, that z is contained in S. Since $z \in \Omega(G) \subset \Omega(G_m)$, as was shown in the proof of Lemma 4.1-(10), no $(G_m - J)$-translates of B_m pass through z and no G_m-translates of B_m accumulate at z. Therefore, we have $z \in C_1^\circ$, which contradicts our assumption for Case 2.

Hence, we can assume that z lies in B_2°. Since a point of $\mathrm{Fr}\, T_1$ in B_2° is either a point of $(G_1 - J)(S)$, or a point of $\Lambda(G_1)$ and $z \in \Omega(G) \subset \Omega(G_1)$, we see that z must lie in $B_2^\circ \cap (G_1 - J)(S)$. Then there are a point $s \in S$ and an element $g \in G_1 - J$ with $g(s) = z$. By Lemma 4.7, s lies in $S \cap \Omega(G) = S \cap \Omega(J) = S \cap \Omega(G_1) = S \cap \Omega(G_2)$. Therefore no $(G_m - J)$-translates of B_m pass through s and no G_m-translates of B_m accumulate at s as was shown in the proof of Lemma 4.1-(10). This implies that s is contained in $C_1^\circ \cap S$. By applying the proof of Case 1 to s, we see that there is a neighbourhood U of s covered by finitely many G-translates of $\bar{D}$. It follows that $g(U)$ is a neighbourhood of z covered by finitely many G-translates of $\bar{D}$. This shows that D is locally finite at a point as in Case 2.

Thus we have shown the proof of the local finiteness of D, hence completed the proof. □

Proof of (8). We shall prove this by showing the following three claims.

Claim 7. *For each m, we have $R_m \cap \Omega(G_m) \subset \Omega(G)$.*

Proof. Take a point $z \in R_m \cap \Omega(G_m)$. Since $R_m = \bar{\mathbb{R}}^n \setminus G_m(B_m^\circ)$, we have either $z \in G_m(S)$ or $z \in C_1 \cap B_{3-m}^\circ$. If $z \in G_m(S)$, then $z \in \Omega(G)$ since $S \cap \Omega(G) = S \cap \Omega(J) = S \cap \Omega(G_m)$ by Lemma 4.7. If $z \in C_1 \cap B_{3-m}^\circ$, since $z \in \Omega(G_m)$, no G_m-translates of B_m passe through or accumulate at z as was shown in the proof of Lemma 4.1-(10). It follows that $z \in C_1^\circ$. By Proposition 2.4, there is a neighbourhood U of z lying in $C_1^\circ \cap B_{3-m}^\circ$ which is precisely invariant under $\mathrm{Stab}_{G_m}(z)$ in G_m such that $\mathrm{Stab}_{G_m}(z)$

is finite. By Lemma 4.8, we see that $\mathrm{Stab}_{G_m}(z) = \mathrm{Stab}_G(z)$ and that U is precisely invariant under $\mathrm{Stab}_G(z)$ in G. By Proposition 2.4, this implies that $z \in \Omega(G)$. □

Claim 8. *Every point of $\Omega(G)$ is G-equivalent to a point of either $R_1 \cap \Omega(G_1)$ or $R_2 \cap \Omega(G_2)$.*

Proof. Let z be a point in $\Omega(G)$. By Lemma 4.10, we see that $z \notin T$. As was shown in the first half of the proof of Lemma 4.9, we have $z \in G(C_1)$. We have only to consider the case when $z \in C_1$ by translating z by elements of G. Since $C_1 \cap B_m \subset R_{3-m}$ by the definitions of R_{3-m} and C_1 and $\Omega(G) \subset \Omega(G_1) \cap \Omega(G_2)$, we see that $z \in (R_1 \cap \Omega(G_1)) \cup (R_2 \cap \Omega(G_2))$. □

Claim 9. *For each $m = 1, 2$, the set $R_m \cap \Omega(G_m)$ is precisely invariant under G_m in G.*

Proof. It is obvious that R_m is G_m-invariant, hence so is $R_m \cap \Omega(G_m)$. We shall show that $R_m \cap \Omega(G_m)$ is moved to a set disjoint from it by other elements of G.

For any $g \in G_{3-m} - J$, we have $g(R_m \cap \Omega(G_m)) \subset g(B_{3-m} \cap \Omega(G_m)) \subset B_m$. By Lemma 4.1-(5), $g(B_{3-m}) \cap S \subset \Lambda(G_{3-m}) \cap S$, which is equal to $S \cap \Lambda(G_m)$ by Lemma 4.1-(2). This implies that no point of $\Omega(G_m) \cap B_{3-m}$ is mapped into S by g, hence $g(B_{3-m} \cap \Omega(G_m)) \subset B_m^\circ$. Since R_m is contained in B_{3-m}, it follows that $g(R_m \cap \Omega(G_m)) \cap R_m \cap \Omega(G_m) = \emptyset$.

Now let $g = g_n \circ \cdots \circ g_1$ be a normal form with $|g| > 1$. If g is a $(3-m, 3-m)$-form, then since $g_1(R_m \cap \Omega(G_m)) \subset B_m^\circ$, we have $g(R_m \cap \Omega(G_m)) \subset g_n \circ \cdots \circ g_2(B_m^\circ) \subset B_m^\circ$. If g is a $(3-m, m)$-form, then since g_1 preserves $R_m \cap \Omega(G_m)$, we have $g(R_m \cap \Omega(G_m)) = g_n \circ \cdots \circ g_2(R_m \cap \Omega(G_m))$, which is contained in B_m° by the argument above for $(3-m, 3-m)$-forms. Finally if g is an (m, k)-form, then $g_{n-1} \circ \cdots \circ g_1$ is a $(3-m, k)$-form with $k = 3-m$ or $k = m$. Then, as was discussed above, we have $g_{n-1} \circ \cdots \circ g_1(R_m \cap \Omega(G_m)) \subset B_m^\circ$, and $g(R_m \cap \Omega(G_m)) \subset g_n(B_m^\circ)$, which is contained in the complement of R_m by definition. Thus we have shown that $g(R_m \cap \Omega(G_m)) \cap R_m \cap \Omega(G_m) = \emptyset$ for any $g \in G - G_m$. □

By these three claims, we have shown that $\Omega(G)/G = (R_1 \cap \Omega(G_1))/G_1 \cup (R_2 \cap \Omega(G_2))/G_2$. Now we consider the intersection of the two terms in the right hand side. We should first note that $(R_1 \cap \Omega(G_1)) \cap (R_2 \cap \Omega(G_2))$ is contained in $B_2 \cap B_1 = S$ since R_1 is contained in B_2, and R_2 is in B_1. Since $\Omega(G_m) \cap S = \Omega(J) \cap S \subset R_m \cap \Omega(G_m)$, the intersection is equal to $\Omega(J) \cap S$. Furthermore since S is a (J, G_m)-block, $\Omega(J) \cap S$ projects to $(\Omega(J) \cap S)/J$ in $(R_m \cap \Omega(G_m))/G_m$. Therefore $(R_1 \cap \Omega(G_1))/G_1$ and $(R_2 \cap \Omega(G_2))/G_2$ are pasted along $(S \cap \Omega(J))/J$. □

In the following, we assume further that S is a strong (J, G)-block if and only if each B_m is a strong (J, G_m)-block.

Proof of (9). Since we are assuming both B_1 and B_2 are strong blocks, by assumption, S is a strong (J, G)-block. Let x be a limit point of G which is not a translate of a limit point of either G_1 or G_2. By Lemma 4.9, we see that x is contained in T. Furthermore, by Lemma 4.10, there is a sequence $\{h_k\}$ of distinct elements of G such that $x \in \cdots \subset h_k(B) \subset \cdots \subset h_1(B)$ for B which is either B_1 or B_2. We can assume that $B = B_1$ and $h_1 = id$ by interchanging the indices and replacing $g(B_2)$ with B_1 for $g \in G_2$ if necessary. Then S separates $h_k^{-1}(S)$ from $h_k^{-1}(x)$.

Since J is geometrically finite, by Proposition 2.16, there are a Dirichlet domain P and standard parabolic regions $B_{p_1}, \dots B_{p_k}$ such that $\overline{P} \setminus \bigcup_j (\mathrm{Int}\, B_{p_j} \cup \{p_j\})$ is compact. Since P is a Dirichlet domain, the interior of $Q = \overline{P} \cap \overline{\mathbb{R}}^n$ is a fundamental domain for J. Since $h_k^{-1}(x) \in \Omega(J)$ for each k, there is an element $j_k \in J$ such that $j_k \circ h_k^{-1}(x) \in Q$. We denote $j_k \circ h_k^{-1}$ by l_k.

We claim that $\{l_k(x)\}$ stays away from S. Suppose, on the contrary, that $l_k(x) \to w \in S$. Then, by Lemma 4.7, $w \in \Lambda(J)$. It follows that $w \in P \cap \Lambda(J)$. So w is a parabolic fixed point of J, where the rank of $\mathrm{Stab}_J(w)$ is less than n since Q intersects $\Lambda(J)$ only at the p_j.

This means that all the $l_k(x)$ lie in some B_{p_j} if we take a subsequence, where $p_j = w$. Let the rank of $\mathrm{Stab}_J(w)$ be s and the rank of $\mathrm{Stab}_G(w)$ be m.

If $s = m$, then we can assume that the interior of $B_w \cap \overline{\mathbb{R}}^n$, which is denoted by U_w, is also a peak domain for G. Hence we may assume that $\overline{U}_w \setminus \{w\}$ is contained in $\Omega(G)$. On the other hand, since x lies in $\Lambda(G)$, we have $l_k(x) \in \Lambda(G)$, which is a contradiction.

Therefore, there is $\delta > 0$ such that $d(l_k(x), z) > \delta$ for all $z \in S$, where d denotes the ordinary spherical metric on $\overline{\mathbb{R}}^n$. Since S separates $h_k^{-1}(x)$ from $h_k^{-1}(S)$, we see that for all z on S we have $\delta < d(l_k(x), z) \le d(l_k(x), l_k(z))$. On the other hand, since $h_k(S)$ nest around x, we see that for any point y on S, the points $l_k^{-1}(y)$ converge to x. We can now apply Proposition 2.12 to conclude that x is a conical limit point.

If $s < m$, by conjugation and Theorem 2.10, we may assume that $w = \infty$,

$$\mathrm{Stab}^*_G(w) = \langle j_1, \dots, j_m \rangle$$

and

$$\mathrm{Stab}^*_J(w) = \langle h_1, \dots, h_s \rangle,$$

where $j_i(y) = A_i(y) + e_{i-1}$ $(i = 1, \dots, m)$, $h_j(y) = U_j(y) + e_{j-1}$ $(j = 1, \dots, s)$, $y \in \mathbb{R}^n$, A_i and U_j are rotations, and A_i and U_j act on $\mathbb{R}^m$ trivially. It follows from $\{l_k(x)\} \subset Q$ that $\sum_{i=1}^{s} |l_k(x)_i|^2$ are bounded away from ∞ for all k. Since S is strong, there is $t > 0$ such that

$$U = \left\{ z \in \mathbb{R}^n : \sum_{i=m+1}^{n} |z_i|^2 > t \right\}$$

is a peak domain for G and $\bar{U} \setminus \{\infty\} \subset \Omega(G)$. We know that $\{l_k(x)\} \subset \Lambda(G)$. Hence $\sum_{i=m+1}^{n} |l_k(x)_i|^2 < t$. It follows from $l_k(x) \to \infty$ as $k \to \infty$ that

$$\sum_{i=s+1}^{m} |l_k(x)_i|^2 \to \infty.$$

For each $i = s+1, \ldots, m$, if $|l_k(x)_i|^2 \to \infty$ $(k \to \infty)$, then we choose a sequence $\{i_k\}$ of integers such that for all k, $|j_i^{i_k} l_k(x)_i|^2 < M_1$, where $M_1 > 0$; if $|l_k(x)_i|^2 < M_2$ for some $M_2 > 0$, we let $i_k = 0$. Let $f_k = j_m^{m_k} \cdots j_{s+1}^{(s+1)_k}$. It follows that $|f_k(l_k(x))|^2 < M_3$ $(M_3 > 0)$, and for any $y \in S$

$$|f_k(y)|^2 = \left|j_{s+1}^{(s+1)_k} l_k(y)_{s+1}\right|^2 + \cdots + |j_m^{m_k} l_k(y)_m|^2 \to \infty.$$

Therefore, there is $\delta > 0$ such that $d(f_k l_k(x), f_k(z)) > \delta$ for all $z \in S$, where d denotes the ordinary spherical metric on $\bar{\mathbb{R}}^n$. Since S separates $h_k^{-1}(x)$ from $h_k^{-1}(S)$ and hence S separates $l_k^{-1}(x)$ from $l_k^{-1}(S)$, we see that for all z on S we have $\delta < d(f_k l_k(x), f_k(z)) \leq d(f_k l_k(x), f_k l_k(z))$. By Lemma 2.3 and choosing a subsequence, we know that $f_k l_k(z) \to z'$ for all $z \in \mathbb{R}^{n+1} \setminus \{x\}$ and $f_k l_k(x) \to x'$, where $z' \neq x'$. We now conclude that x is a conical limit point. □

Proof of (10). We first assume that G_1 and G_2 are geometrically finite. Then every parabolic fixed point of G_m is a parabolic vertex by Proposition 2.13. Therefore B_1 and B_2 are both strong blocks. By assumption, this implies that S is a strong (J, G)-block.

Let x be a point on $\Lambda(G)$. What we have to show is that x is either a parabolic vertex or a conical limit point, for this proves that G is geometrically finite by Proposition 2.13. Suppose first that x is a parabolic fixed point, where the rank k of $H = \mathrm{Stab}_G(x)$ is less than n. We shall show that x is a parabolic vertex then. Since x is a parabolic fixed point, it cannot be a conical limit point. Hence by (9), x is a translate of a limit point of either G_1 or G_2.

By interchanging the indices and translating x by elements of G, we may assume that x lies in $\Lambda(G_1)$. Since G_1 is assumed to be geometrically finite, x is a parabolic vertex or a conical limit point for G_1 by Proposition 2.13. If x is a conical limit point for G_1, then so is it for G, which contradicts the assumption that x is a parabolic fixed point. Therefore, x is a parabolic vertex for G_1. Suppose first that x lies on $G_1(S)$. Then there is an element $\gamma \in G_1$ such that $\gamma^{-1}(x)$ lies on S. Since x is not a conical limit point for G_1, neither is $\gamma^{-1}(x)$. This also implies that $\gamma^{-1}(x)$ is not a conical limit point for J either. Since J is geometrically finite, again by Proposition 2.13, we see that $\gamma^{-1}(x)$ is a parabolic vertex for J. Since S is a strong (J, G)-block, it follows that $\gamma^{-1}(x)$ is a parabolic vertex also for G, hence so is x. Thus we are done for this case.

Suppose next that x does not lie on any G_1-translate of S. We shall show that x is a parabolic vertex for G even in this case. Since $G_1(B_1^\circ) \subset \Omega(G_1)$ by Lemma 4.1-(3) and x is a parabolic vertex of G_1, we have $x \in B_2^\circ \cap C_1$. Since $B_2^\circ \cap C_1$ is precisely invariant under G_1 in G by Lemma 4.8, $H = \mathrm{Stab}_G(x)$ must be contained in G_1. This implies that $H = \mathrm{Stab}_{G_1}(x)$. Since x is a parabolic vertex for G_1, there is a peak domain U at x for G_1. Since $U \cap \Lambda(G_1) = \emptyset$ and $x \in B_2^\circ \cap C_1$, by choosing U to be sufficiently small, we can assume that $\bar{U} \setminus \{x\} \subset \Omega(G_1)$ and $\bar{U} \subset B_2^\circ$. By conjugating G by an element of $M(\bar{\mathbb{R}}^n)$, we may assume that $x = \infty$ and U is in the form $U = \left\{x \in \mathbb{R}^n\colon \sum_{i=k+1}^n x_i^2 > t\right\}$, for some $t > 0$. By Theorem 2.10, for any $g \in \mathrm{Stab}_G(\infty)$, we have an expression $g(x) = Ax + \mathbf{a}$, for $\mathbf{a} \in \mathbb{R}^k$ and an orthogonal matrix A preserving the subspaces $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$. Now we shall show the following.

Claim 10. *The projections of G_1-translates of B_1 to the last $n-k$ coordinates $\mathbb{R}^{n-k}$ are bounded away from ∞.*

Proof. Since U is contained in B_2°, the last $n-k$ coordinates of its complement B_1 are bounded away from ∞. Moreover since $\sum_{i=k+1}^n |g(x)_i|^2 = \sum_{i=k+1}^n |x|_i^2$ for any $g \in H$, by taking t sufficiently large, we know that $g(B_1) \cap U = \emptyset$. This means that the projections of H-translates of B_1 to the last $n-k$ coordinates of $\mathbb{R}^{n-k}$ are bounded away from ∞.

Now we consider general translates by elements of G_1. Suppose, seeking a contradiction, that there is a sequence $\{g_k(B_1)\}$ of distinct G_1-translates of B_1 whose projections to $\mathbb{R}^{n-k}$ go to ∞. Since J stabilises B_1, we see that $g_k \in G_1 - (H \cup J)$.

On the other hand, since U is a peak domain for G_1, it is precisely invariant under H in G_1. Take a point y_0 in U. Since $g_k(y_0)$ is disjoint from U, the last $n-k$ coordinates of $g_k(y_0)$ are bounded as $k \to \infty$. Since H acts on the first k-coordinates cocompactly, we can choose $j_k \in H$ such that $j_k g_k(y_0)$ stays in a bounded set.

Since j_k lies in H, we have $\sum_{i=k+1}^n (j_k(x))_i^2 = \sum_{i=k+1}^n (x)_i^2$. Therefore the projections of $j_k g_k(B_1)$ to $\mathbb{R}^{n-k}$ also go to ∞. Now Lemma 4.1-(7) implies that $j_k g_k(y) \to \infty$ for all $y \in B_1$. By Lemma 2.3, we see that, by choosing a subsequence if necessary, we may assume that $j_k g_k(y) \to \infty$ for all y except for at most one point which is contained in the limit set of G_1. Since y_0 is contained in $U \subset \Omega(G_1)$, we have in particular that $j_k g_k(y_0) \to \infty$. This is a contradiction. □

Our claim shows that U can be taken to be disjoint from T_1. Therefore, we have $U \subset C_1 \cap B_2^\circ$. Since $C_1 \cap B_2^\circ$ is precisely invariant under G_1 in G, for any $g \in G - G_1$, $g(U) \cap U = \emptyset$. Therefore, U is a peak domain at x of G, which means that x is a parabolic vertex for G. Thus we have proved that all parabolic fixed points of G are parabolic vertices.

Next assume that x is a limit point of G which is not a parabolic fixed point. Suppose that x is a translate of a limit point y of G_m. Since y is not a parabolic fixed point and G_m is geometrically finite, by Proposition 2.13, y is a conical limit point of

G_m, hence also for G. If x is not a translate of a limit point of either G_1 or G_2, then by (9), it is a conical limit point for G. Thus we have shown that any non-parabolic limit point of G is a conical limit point, and completed the proof of the "if" part.

We shall now turn to show the "only if" part. Assume that G is geometrically finite. Then S is a strong (J, G)-block. This implies that B_m is a strong (J, G_m)-block for $m = 1, 2$ by assumption.

Let x be a parabolic fixed point of G_1. We assume that the rank of $\mathrm{Stab}_{G_1}(x)$ is $k < n$, and shall prove that there is a peak domain at x for G_1. Since B_1° is contained in $\Omega(G_1)$ by Lemma 4.1-(3), x cannot lie in $G_1(B_1^\circ)$. Therefore, x lies in either $G_1(S)$ or $B_2^\circ \cap C_1$. If $x \in G_1(S)$, then, since B_1 is a strong (J, G_1)-block and J is geometrically finite, there is a peak domain at x for G_1, and we are done. If $x \in B_2^\circ \cap C_1$, then $\mathrm{Stab}_G(x) = \mathrm{Stab}_{G_1}(x)$ since $B_2^\circ \cap C_1$ is precisely invariant under G_1 in G by Lemma 4.8. Therefore $\mathrm{Stab}_G(x)$ has rank $k < n$ in particular. Since G is geometrically finite, there is a peak domain U at x for G, which is also a peak domain for G_1.

Now let x be a limit point of G_1 which is not a parabolic fixed point of G_1. We shall show that x is a conical limit point of G_1. Again we have only to consider the cases when $x \in G_1(S)$ and when $x \in B_2^\circ \cap C_1$. If $x \in G_1(S)$, then there are a point y lying on S and $g \in G_1$ such that $x = g_1(y)$. Since y lies on $\Lambda(J)$ by Lemma 4.1-(2), and J is geometrically finite, it is a conical limit point for J by Proposition 2.13. This implies that x is a conical limit point for G_1, and we are done in this case.

Suppose now that $x \in B_2^\circ \cap C_1$. Since $B_2^\circ \cap C_1$ is precisely invariant under G_1, we have $\mathrm{Stab}_G(x) = \mathrm{Stab}_{G_1}(x)$. Therefore x is not a parabolic fixed point of G either. Since G was assumed to be geometrically finite, x is a conical limit point for G by Proposition 2.13. It follows from Proposition 2.12 that there is a sequence $\{h_k\}$ of distinct elements of G such that $d(h_k(z), h_k(x))$ is bounded away from zero for all $z \in \overline{\mathbb{R}}^n \backslash \{x\}$ and $h_k^{-1}(z_0) \to x$ for some $z_0 \in \mathbb{H}^{n+1}$. We may assume that h_k belong to distinct cosets of J in G. By Theorem 3.1, we have that $\mathrm{diam}(h_k(S)) \to 0$. So all the $h_k(S)$ must be distinct.

Claim 11. *By taking a subsequence we can assume $h_k > 0$ for all k.*

Proof. Suppose, on the contrary, that $h_k < 0$ for all k after passing to a subsequence. We recall that $\mathrm{diam}(h_k(S)) \to 0$. It follows that the set $h_k(B_2)$ cannot contain S inside. Therefore, we have $\mathrm{diam}(h_k(B_2)) \to 0$. Recall that we are considering the case when $x \in B_2^\circ \cap C_1$. This shows that $d(h_k(z), h_k(x)) \to 0$ for all $z \in B_2$. This contradicts the fact that $d(h_k(z), h_k(x))$ is bounded away from 0 for $z \in \overline{\mathbb{R}}^n \setminus \{x\}$. Thus we have completed the proof of Claim 11. □

Now we return to the proof of (10). Note that we have only to consider the case when h_k is not contained in G_1, for otherwise x is a conical limit point of G_1 by

Proposition 2.12. Therefore, we can assume that $|h_k| > 1$. Express h_k in a normal form $h_k = \gamma_{k_l} \circ \cdots \circ \gamma_{k_1}$. Set $g_k = h_k \circ \gamma_{k_1}^{-1}$. Then g_k is negative.

First consider the case when $g_k = g \circ j_k$ for some $g \in G$ with some $j_k \in J$. Then $d(h_k(z), h_k(x)) = d(g \circ j_k \circ \gamma_{k_1}(z), g \circ j_k \circ \gamma_{k_1}(x))$. By Lemma 2.3, we may assume that there are two distinct points x', z' such that $g \circ j_k \circ \gamma_{k_1}(z) \to z'$ for all $z \in \overline{\mathbb{R}}^n \setminus \{x\}$ and $g \circ j_k \circ \gamma_{k_1}(x) \to x'$. It follows that $j_k \circ \gamma_{k_1}(z) \to g^{-1}(z')$ for all $z \in \overline{\mathbb{R}}^n \setminus \{x\}$, $j_k \circ \gamma_{k_1}(x) \to g^{-1}(x')$ and $(j_k \circ \gamma_k)^{-1}(g^{-1}(z_0)) \to x$, where $g^{-1}(z_0) \in \mathbb{H}^{n+1}$. It follows from Proposition 2.12 that x is a conical limit point of G_1.

Suppose next that g_k is not expressed as $g \circ j_k$, that is, g_k belong to distinct cosets of J in G. Then by Theorem 3.1, $g_k(S)$ are all distinct. Applying the proof of Claim 11 to g_k, we see that $\operatorname{diam}(g_k(B_2)) \to 0$. For any $z \in B_1$, we have that $\gamma_{k_1}(z) \in \gamma_{k_1}(B_1) \subset B_2$. On the other hand, $\gamma_{k_1}(x) \in B_2$ for $\gamma_{k_1}(C_1 \cap B_2^\circ) = C_1 \cap B_2^\circ$. These imply that $d(h_k(z), h_k(x)) = d(g_k\gamma_{k_1}(z), g_k\gamma_{k_1}(x)) \to 0$ for all $z \in B_1$. This contradicts the fact that $d(h_k(z), h_k(x))$ is bounded away from 0 for $z \in \overline{\mathbb{R}}^n \setminus \{x\}$. Thus we have completed the proof of (10). □

Corollary 4.12. *Under the hypotheses of* Theorem 4.2, *if each B_m is precisely invariant under J in G_m, especially J is the trivial subgroup $I = \{id\}$, and if we set $D = (D_1 \cap B_2) \cup (D_2 \cap B_1)$ and $G = \langle G_1, G_2 \rangle$, then the following hold.*

(1) *$G = G_1 *_J G_2$.*

(2) *G is discrete.*

(3) *Except perhaps for conjugates of elements of G_1 and G_2, every element of G is loxodromic.*

(4) *S is a (J, G)-block and S is precisely invariant under J in G.*

(5) *If $\{S_k\}$ is a sequence of distinct G-translates of S, then* $\operatorname{diam}(S_k) \to 0$, *where* diam *denotes the diameter with respect to the ordinary spherical metric on $\overline{\mathbb{R}}^n$.*

(6) *There is a sequence of distinct G-translates of S nesting about the point x if and only if x is a limit point of G which is not G-equivalent to a limit point of either G_1 or G_2.*

(7) *D is a fundamental set for G. If both D_1 and D_2 are constrained, and $S \cap \operatorname{Fr} D$ consists of finitely many connected components the sum of whose $(n-1)$-dimensional measures on S vanishes, then D is also constrained.*

(8) *Let Q_m be the union of the G_m-translates of B_m°, and let R_m be the complement of Q_m in $\overline{\mathbb{R}}^n$. Then $\Omega(G)/G = (R_1 \cap \Omega(G_1))/G_1 \cup (R_2 \cap \Omega(G_2))/G_2$, where the latter two possibly disconnected orbifolds are identified along their common possibly disconnected or empty boundary $(S \cap \Omega(J))/J$.*

(9) *S is a strong (J, G)-block if and only if each B_m is a strong (J, G_m)-block.*

(10) *If both B_1 and B_2 are strong, then, except for G-translates of limit points of G_1 or G_2, every limit point of G is a conical limit point.*

(11) *G is geometrically finite if and only if both G_1 and G_2 are geometrically finite.*

Proof. By Theorem 4.2, we only need to prove (9).

Let x be a parabolic fixed point of J. Such a point x is contained in S by Lemma 4.1-(2). Since each B_m is precisely invariant under J in G_m by our assumption, we have $\mathrm{Stab}_J(x) = \mathrm{Stab}_{G_m}(x)$, which is also equal to $\mathrm{Stab}_G(x)$ by Lemma 4.4. Let H denote $\mathrm{Stab}_J(x)$.

The proof of the "if" part. Suppose that B_m is a strong (J, G_m)-block for each $m = 1, 2$. There is nothing to prove if the rank of H is n since the rank of $\mathrm{Stab}_G(x)$ is also n. Now assume that the rank of H is $k < n$. By conjugation, we may assume that $x = \infty$. By Theorem 2.10, we can assume that each $g \in H$ is expressed as $g(y) = Ay+\mathbf{a}$ for $\mathbf{a} \in \mathbb{R}^k$ and an orthogonal matrix A preserving the subspaces $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$.

Since both B_1 and B_2 are assumed to be strong and $\mathrm{Stab}_{G_1}(\infty) = \mathrm{Stab}_{G_2}(\infty)$, there is a common peak domain U at ∞ for G_1 and G_2. Since $U \cap (\Lambda(G_1) \cup \Lambda(G_2)) = \emptyset$, by choosing U small enough, we may assume that $\overline{U} \setminus \{\infty\} \subset \Omega(G_1) \cap \Omega(G_2)$, where $\overline{}$ means the closure on $\overline{\mathbb{R}}^n$. We can assume that U has a form $U = \left\{y \in \mathbb{R}^n \colon \sum_{i=k+1}^n y_i^2 > t^2\right\}$, where t is a sufficiently large positive number.

Claim 12. *We can choose U small enough to satisfy $U \subset C_1$.*

Proof of Claim. We divide our discussions into two cases.

CASE 1: The case when $k = n - 1$.

In this case, U is the union of two components U_1 and U_2, and we may assume that $U_m \subset B_m^\circ$ by our assumption that B_m is a strong block. We have only to prove that we can choose U_1 small enough in such a way that every G_2-translate of B_2 is disjoint from U_1. We may assume that $U_1 = \{y \in \mathbb{R}^n \colon y_n > t\}$. Suppose, seeking a contradiction, that such a U_1 does not exist. Then, there is a sequence $\{g_k(B_2)\}$ of distinct G_2-translates of B_2 intersecting $\{y \in \mathbb{R}^n \colon y_n > s\}$ for any large s. This means that the projections of $g_k(B_2)$ to the n-th coordinate $\mathbb{R}$ accumulate at ∞. We may assume that $g_k \in G_2 - J$ since J fixes B_2.

Now Lemma 4.1-(7) implies that $\mathrm{diam}(g_k(B_2)) \to 0$ with respect to the ordinary spherical metric. It follows that $g_k(y) \to \infty$ for all $y \in B_2$ since $\{g_k(B_2)\}$ accumulates at ∞. By Lemma 2.3, by taking a subsequence of $\{g_k\}$, we may assume that $g_k(y) \to \infty$ for all y with at most one exception, which must be a limit point.

Since $\overline{U}_2 \setminus \{\infty\}$ is contained in $\Omega(G_2)$, for all $y \in \overline{U}_2 \setminus \{\infty\}$, we have $g_k(y) \to \infty$. Since $g_k(U_2) \cap U = \emptyset$, it follows that the projections of $g_k(\overline{U}_2)$ to the n-th coordinate are bounded. Hence the projections of $g_k(\overline{U}_2 \setminus \infty)$ to the first $n-1$ coordinates $\mathbb{R}^{n-1}$ accumulate at ∞. By Theorem 2.10, for each g_k, we can choose an element $j_k \in H$ such that $\{j_k g_k(y_0)\}$ lies in a bounded set for a fixed $y_0 \in U_2$. For each k, we have $\infty \notin g_k(B_2)$ since B_2 was assumed to be precisely invariant under J in G_2 and ∞ lies on S. Therefore, we have $\infty \notin j_k g_k(B_2)$. Since $|(j_k g_k(y))_n| = |(g_k(y))_n|$ and the projections of the $g_k(B_2)$ to the n-th coordinate $\mathbb{R}$ accumulate at ∞, we see that $\{j_k g_k(B_2)\}$ also accumulates at ∞. By Lemma 4.1-(7), this implies that $j_k g_k(y) \to \infty$ for all $y \in B_2$.

This is a contradiction since $\{j_k g_k(y_0)\}$ stays in a compact set. This proves our claim for the case when $k = n - 1$.

CASE 2: The case when $k < n - 1$.

Since U is connected and is disjoint from S, we see that U lies in either B_1° or B_2°. We may assume that $U \subset B_1^\circ$. Then, by the same argument as in the proof of Case 1, we see that the projections of G_2-translates of B_2 in the last $n-k$ coordinates cannot accumulate at ∞. Therefore, we have $U \subset C_1 \cap B_1^\circ$.

The claim has thus been proved. $\square$

Now we return to the proof of the "if" part. Take a small common peak domain U for both G_1 and G_2 as in Claim 12. By assumption, U is precisely invariant under H in both G_1 and G_2. We need to show it is precisely invariant under $\mathrm{Stab}_G(x)$ in G.

For any $g \in G-(G_1 \cup G_2)$, we have $g(U) = g(U_1) \cup g(U_2) \subset g(C_1 \cap B_1^\circ) \cup g(C_1 \cap B_2^\circ)$, where U_1, U_2 are the components of U if $k = n-1$, and we regard one of them as the emptyset when $k < n-1$. Suppose that g is expressed as a $(1, 1)$-form $g_n \circ \cdots \circ g_1$. As was shown in Lemma 2.6, $g_n \circ \cdots \circ g_1(C_1 \cap B_1^\circ) \subset B_2^\circ$. Furthermore, we have $g_n \circ \cdots \circ g_1(C_1 \cap B_1^\circ) \subset g_n \circ \cdots \circ g_1(B_1^\circ) \subset T_n^\circ \subset T_1^\circ$. On the other hand, $g_n \circ \cdots \circ g_1(C_1 \cap B_2^\circ) \subset g_n \circ \cdots \circ g_2(C_1 \cap B_2^\circ)$ by Lemma 4.8. Then applying the same argument for $C_1 \cap B_1^\circ$, we see that $g_n \circ \cdots \circ g_2(C_1 \cap B_2^\circ) \subset T_1^\circ$. Thus we have shown that $g(C_1 \cap B_1^\circ) \cup g(C_1 \cap B_2^\circ) \subset B_2^\circ \cap T_1^\circ$ for g expressed as a $(1, 1)$-form. A similar argument works also for $(1, 2)$-form. Also, we can see by the same argument that if g is expressed as a 2-form, then $g(U) = g(U_1) \cup g(U_2) \subset g(C_1 \cap B_1^\circ) \cup g(C_1 \cap B_2^\circ) \subset B_1^\circ \cap T_1^\circ$.

Since U, which is disjoint from S from the beginning, is taken to be lie inside C_1, it follows that U is precisely invariant under H in G in the case when $k \leq n-1$.

This completes the proof of the "if" part.

The proof of the "only if" part. Let x be a parabolic fixed point of J such that $\mathrm{Stab}_J(x)$ has rank less than n. This point x must lie on S since $\Lambda(J) \subset S$. Since we are assuming that S is a strong (J, G)-block, there is a peak domain U for G, which is also a peak domain for both G_1 and G_2. Since we already know that B_m is a (J, G_m)-block, this shows that B_m is a strong (J, G_m)-block.

By Theorem 4.2, we know that the conclusions hold. $\square$

REMARK 4.1. The condition that (B_1°, B_2°) is a proper interactive pair in Theorem 4.2 is necessary, as the following example shows.

EXAMPLE 4.1. Set

$$J = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and

$$G_1 = \langle J, g_1 \rangle, \quad G_2 = \langle J, g_2 \rangle.$$

We use the following symbols:

$$S = \{x \in \overline{\mathbb{R}}^2 \colon x_2 = 0\},\ B_1 = \{x \in \overline{\mathbb{R}}^2 \colon x_2 \leq 0\} \quad \text{and} \quad B_2 = \{x \in \overline{\mathbb{R}}^2 \colon x_2 \geq 0\}.$$

Then the following hold.

(1) J is geometrically finite.
(2) $S = \Lambda(J) = \Lambda(G_1) = \Lambda(G_2)$.
(3) $G_1 = J \cup g_1 J$ and $G_2 = J \cup g_2 J$.
(4) Each B_m is a (J, G_m)-block for $m = 1, 2$.
(5) (B_1°, B_2°) is an interactive pair, but (B_1°, B_2°) is not proper.
(6) $G \neq G_1 *_J G_2$.

The assertion (1) is obvious since J is a finitely generated Fuchsian group. To prove (2), set $w = p/r$, where p and r are integers and $r \neq 0$, and $j = \begin{pmatrix} 1 - pr & p^2 \\ -r^2 & 1 + pr \end{pmatrix}$. Then $j \in J$ is a parabolic element having w as its fixed point. Therefore, every rational number is a parabolic fixed point of J. Now (2) follows from Lemma 5.3.3 in [7]. The proofs of (3), (4) and (5) are trivial. We can verify (6) by checking that for a $(1, 2)$-form $g_1 g_2 g_1 g_2$, we have $\Phi(g_1 g_2 g_1 g_2) = id$.

5. An application

5.1. The statement of Theorem 5.1. Following [31] or [32], we denote by $PSL(2, \Gamma_n)$ the n-dimensional Clifford matrix group. Then $PSL(2, \Gamma_n)$ is isomorphic to $M(\overline{\mathbb{R}}^n)$ (cf. [3]).

Let

$$j_1 = \begin{pmatrix} e_1 & 0 \\ 0 & -e_1 \end{pmatrix}, \quad j_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad j_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j_4 = \begin{pmatrix} e_1 & 1 \\ 0 & -e_1 \end{pmatrix},$$

$$g_1 = \begin{pmatrix} e_2 & 0 \\ 0 & -e_2 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 - 8e_1 - 64e_2 & -130 \\ -32 & 1 + 8e_1 + 64e_2 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} -7 - 64e_1e_2 & -126e_1 + 32e_2 \\ 32e_1 & 9 - 64e_1e_2 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 65 - 8e_1e_2 & -32e_1 - 126e_2 \\ -32e_2 & -63 - 8e_1e_2 \end{pmatrix},$$

$J = \langle j_1, j_2, j_3, j_4 \rangle$, $G_1 = \langle J, g_1 \rangle$, $G_2 = \langle J, g_2, g_3, g_4 \rangle$ and $G = \langle G_1, G_2 \rangle$.

Then

Theorem 5.1. *G is geometrically finite.*

5.2. Several propositions.

Proposition 5.2. *As a* 2*-dimensional Möbius subgroup,*

$$\Lambda(J) = J(\infty) \cup \{\textit{the approximation points of } J\}.$$

Moreover, every parabolic fixed point of J is J-equivalent to ∞.

Proof. In the proof of this proposition, we regard J as a 2-dimensional Möbius subgroup. J has a fundamental polyhedron

$$P = \left\{x \in \mathbb{H}^3 \colon -\frac{1}{2} < x_1 < \frac{1}{2},\ 0 < x_2 < \frac{1}{2},\ |x| > 1\right\},$$

which has finitely many sides. This yields that J is geometrically finite as a 2-dimensional Möbius group. Hence every limit point of J is either an approximation point or a parabolic fixed point of J, cf. [8]. We see that $\overline{P} \cap \Lambda(J) = \{\infty\}$. It follows from Proposition VI.C.2 in [22] that every limit point of J which is not J-equivalent to ∞ is an approximation point of J. On the other hand, parabolic fixed points of J cannot be approximation points of J. These facts imply that every parabolic fixed point of J is J-equivalent to ∞. The proof is completed. □

Proposition 5.3. *As a* 3*-dimensional Möbius subgroup, J is geometrically finite.*

Proof. We see that every approximation point of $J \subset PSL(2, \mathbb{C})$ is a conical limit point of $J \subset PSL(2, \Gamma_3)$. By Proposition 5.2, it suffices to prove that ∞ is a parabolic vertex of $J \subset PSL(2, \Gamma_3)$.

We see that $J_\infty = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e_1 & b \\ 0 & -e_1 \end{pmatrix} \colon a,\ b \textit{ are Gaussian integers}\right\}$, and for any $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in J \setminus J_\infty$, $|\gamma| \geq 1$. It follows that the rank of ∞ is 2 and

$$U = \{x \in \mathbb{R}^3 \colon x_3^2 > 16\}$$

is a peak domain of J at ∞. Hence ∞ is a parabolic vertex of $J \subset PSL(2, \Gamma_3)$. □

In the following, all subgroups involved are regarded as 3-dimensional Möbius subgroups.

Proposition 5.4. *G_1 is geometrically finite.*

Proof. By computation, we know that

$$g_1 j_1 = j_1 g_1, \quad g_1 j_2 = j_2^{-1} g_1, \quad g_1 j_3 = -j_3 g_1, \quad g_1 j_4 = -j_4 g_1.$$

It follows that $G_1 = J \cup g_1 J$. We choose a point $y \in \mathbb{H}^4$. Then

$$\Lambda(G_1) = \overline{J(y) \cup g_1 J(y)} \cap \overline{\mathbb{R}}^3 = \Lambda(J).$$

For any conical limit point of J, it is also a conical limit point of G_1. It suffices to show that ∞ is a parabolic vertex of G_1. We see that $G_{1\infty} = J_\infty \cup g_1 J_\infty$ and for any $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_1 \setminus G_{1\infty}$, Hersonsky [14] implies that $|\gamma| \geq 1$. It follows that the rank of ∞ is 2 and U is also a peak domain of G_1 at ∞. The proof is completed. □

Proposition 5.5. *Let* $I = \{id\}$, $H = \langle g_2, g_3, g_4 \rangle$ *and* $R_3 = \{x \in \mathbb{R}^3 \colon |x - (e_1/4 + 2e_2)| = 1/8\}$ *which divides* $\overline{\mathbb{R}}^3$ *into two closed balls*

$$R_1 = \left\{ x \in \mathbb{R}^3 \colon \left| x - \left(\frac{e_1}{4} + 2e_2 \right) \right| \leq \frac{1}{8} \right\}$$

and

$$R_2 = \left\{ x \in \mathbb{R}^3 \colon \left| x - \left(\frac{e_1}{4} + 2e_2 \right) \right| \geq \frac{1}{8} \right\} \cup \{\infty\}.$$

Further, let

$$\begin{aligned} R = \Bigg\{ x \in \mathbb{R}^3 \colon & \left| x - \frac{7}{32} e_1 - 2e_2 \right| > \frac{1}{32}, \ \left| x - \frac{9}{32} e_1 - 2e_2 \right| \geq \frac{1}{32}, \\ & \left| x - \frac{1}{32} - \frac{e_1}{4} - 2e_2 \right| > \frac{1}{32}, \ \left| x + \frac{1}{32} - \frac{e_1}{4} - 2e_2 \right| \geq \frac{1}{32}, \\ & \left| x - \frac{e_1}{4} - \frac{65}{32} e_2 \right| > \frac{1}{32}, \ \left| x - \frac{e_1}{4} - \frac{63}{32} e_2 \right| \geq \frac{1}{32} \Bigg\} \end{aligned}$$

and

$$\triangle = \left\{ x \in \mathbb{R}^3 \colon -\frac{1}{2} < x_1 \leq \frac{1}{2}, \ 0 \leq x_2 \leq \frac{1}{2}, \ |x| \geq 1 \right\} \setminus (A_1 \cup A_2 \cup A_3),$$

where $A_1 = \{x \in \mathbb{R}^3 \colon x_2 = 0, \ -1/2 \leq x_1 \leq 0\}$, $A_2 = \{x \in \mathbb{R}^3 \colon x_2 = 1/2, \ -1/2 \leq x_1 \leq 0\}$, *and* $A_3 = \{x \in \mathbb{R}^3 \colon |x| = 1, \ -1/2 \leq x_1 \leq 0\}$. *Then the following hold.*

(1) $G_2 = \langle J, H \rangle = J *_I H$.
(2) G_2 *is discrete.*
(3) $D_2 = R \cap \triangle$ *is a fundamental set of* G_2.
(4) *Every point of* $\Lambda(G_2) \setminus G_2(\Lambda(J) \cup \Lambda(H))$ *is a conical limit point of* G_2.
(5) G_2 *is geometrically finite.*
(6) $\Lambda(G_2) = G_2(\infty) \cup G_2(e_1/4 + 2e_2) \cup \{$*conical limit points of* $G_2\}$.
(7) U *is also a peak domain for* G_2 *at* ∞ (*recall that* U *is defined in the proof of* Proposition 5.3).

Proof. It is obvious that $\triangle$ is a fundamental set of J. By G.3 in [22], we see that R is a fundamental set of H.

We see that $R_1 \subset \Omega(J)$ and $R_2 \subset \Omega(H)$. Since R_2 is outside the isometric spheres of $g \in H \setminus I$, R_2 is precisely invariant under I in H. It follows that R_2 is an (I, H)-block. Let $f = \begin{pmatrix} 1 & -2e_2 \\ 4e_1 & 1-8e_1e_2 \end{pmatrix}$. By a simple computation, we have that

$$fg_2f^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad fg_3f^{-1} = \begin{pmatrix} 1 & 2e_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad fg_4f^{-1} = \begin{pmatrix} 1 & 2e_2 \\ 0 & 1 \end{pmatrix}.$$

This yields that $\Lambda(H) = \{e_1/4 + 2e_2\}$ and $e_1/4 + 2e_2$ is a parabolic fixed point of rank 3. So H is geometrically finite and R_2 is strong.

Since $R_1 \subset \triangle$, for any $j \in J \setminus I$, $j(R_1) \cap R_1 = \emptyset$. It follows that R_1 is a strong (I, J)-block.

We can see that $\triangle$ and R satisfy that $\triangle \cap R_1 = R_1$, $R \cap R_2 = R_2$ and $\triangle \cap R_3 = R \cap R_3$. Since $\Lambda(H) \neq \emptyset$, we know that (R_1°, R_2°) is a proper interactive pair by Lemma 4.1-(9).

Therefore, groups J, H, I, sets R_1, R_2 and R_3, and fundamental sets $\triangle$ and R satisfy the conditions in Corollary 4.12, we have that

(1) $G_2 = \langle J, H \rangle = J *_I H$,
(2) G_2 is discrete,
(3) $D_2 = R \cap \triangle$ is a fundamental set of G_2,
(4) every point of $\Lambda(G_2) \setminus G_2(\Lambda(J) \cup \Lambda(H))$ is a conical limit point of G_2,
(5) G_2 is geometrically finite.

Since $\Lambda(H) = \{e_1/4 + 2e_2\}$, $\Lambda(J) = J(\infty) \cup \{$the conical limit points of $J\}$ and the conical limit points of J are also conical limit points of G_2, by the discussions above, we have that

$$\Lambda(G_2) = G_2(\infty) \cup G_2\Big(\frac{e_1}{4} + 2e_2\Big) \cup \{\text{conical limit points of } G_2\}.$$

Let $U_1 = \{x \in \mathbb{R}^3 : x_3 > 4\}$ and $U_2 = \{x \in \mathbb{R}^3 : x_3 < -4\}$. Then $U = U_1 \cup U_2$. Let $T_1 = (J \setminus I)(R_1) \cup (H \setminus I)(R_2)$ and $C_1 = \overline{\mathbb{R}}^3 \setminus T_1$. We can see that $U \subset R_2^\circ$ and $U \cap J(R_1) = \emptyset$, that is, $U \subset R_2^\circ \cap C_1$. Since $R_2^\circ \cap C_1$ is precisely invariant under J in G_2 by the proof of Lemma 4.8, we have $G_{2\infty} = J_\infty$ and $(G_2 \setminus J)(U) \cap U = \emptyset$. Therefore, U is also a peak domain for G_2 at ∞. □

Now we are ready to prove Theorem 5.1.

5.3. The proof of Theorem 5.1. Let

$$B_1 = \{x \in \mathbb{R}^3 : x_3 \geq 0\} \cup \{\infty\}, \quad B_2 = \{x \in \mathbb{R}^3 : x_3 \leq 0\} \cup \{\infty\}$$

and

$$S = \{x \in \mathbb{R}^3 : x_3 = 0\} \cup \{\infty\}.$$

It follows from $B_1^\circ = B_1 \cap \Omega(J) = B_1 \cap \Omega(G_1)$ and $g_1 J(B_1^\circ) = B_2^\circ$ that B_1 is a (J, G_1)-block. Since G_1 is geometrically finite, B_1 is strong.

Let

$$D_1 = \triangle \cap \{x \in \mathbb{R}^3 \colon x_3 > 0\}.$$

Then D_1 is a fundamental set of G_1 which satisfies that $D_1 \cap B_1 = \triangle \cap B_1$ and $D_1 \cap S = D_2 \cap S = \emptyset$.

It is obvious that $\triangle \cap B_2 = D_2 \cap B_2$. This yields that

$$B_2 \cap \Omega^\circ(J) = J(\triangle \cap B_2) = J(D_2 \cap B_2) \subset B_2 \cap \Omega^\circ(G_2)$$

and hence $B_2 \cap \Omega^\circ(J) = B_2 \cap \Omega^\circ(G_2) \subset B_2^\circ$. For any $g \in G_2 \setminus J$, we have that

$$g(B_2 \cap \Omega^\circ(G_2)) \cap (B_2 \cap \Omega^\circ(G_2)) = gJ(D_2 \cap B_2) \cap J(D_2 \cap B_2) = \emptyset.$$

Claim 13. *$B_2 \cap \Omega(G_2) = B_2 \cap \Omega(J)$ and $B_2 \cap \Omega(G_2)$ is precisely invariant under J in G_2.*

Proof. For any $x \in B_2 \cap (\Omega(J) \setminus \Omega^\circ(J))$, there exists a neighborhood U_x which is covered by finitely many images of $\overline{\triangle} \cap B_2$, see [22]. It follows from $\overline{\triangle} \cap B_2 = \overline{D_2} \cap B_2$ that $x \in B_2 \cap \Omega(G_2)$. Thus, $B_2 \cap \Omega(G_2) = B_2 \cap \Omega(J)$.

We now come to prove that $B_2 \cap \Omega(G_2)$ is precisely invariant under J in G_2. Suppose, on the contrary, that there exist points $x, y \in B_2 \cap (\Omega(G_2) \setminus \Omega^\circ(G_2))$ and an element $g \in G_2 \setminus J$ with $g(x) = y$. We choose a neighborhood U_x of x. Then $g(U_x)$ is a neighborhood of y. In U_x, we can choose a point $x_0 \in \Omega^\circ(G_2)$. Then $g(x_0) = y_0 \in \Omega^\circ(G_2)$, which contradicts the fact that $B_2 \cap \Omega^\circ(G_2)$ is precisely invariant under J in G_2. □

We have shown that B_2 is a (J, G_2)-block. Since G_2 is geometrically finite, B_2 is strong.

Since $\Lambda(G_2) \neq \Lambda(J)$, by Lemma 4.1-(9), (B_1°, B_2°) is a proper interactive pair. By Theorem 4.2, we know that $G = G_1 *_J G_2$, G is discrete and $D = (D_1 \cap B_2) \cup (D_2 \cap B_1) = D_2 \cap B_1$ is a fundamental set of G.

Claim 14. *S is a strong (J, G)-block.*

Proof. By Theorem 4.2-(4), we know that S is a (J, G)-block. It suffices to prove that ∞ is a parabolic vertex of G. We consider U again. It follows from

$$U_1 \cap \Omega^\circ(J) = J_\infty(U_1 \cap \triangle) = J_\infty(U_1 \cap D) \subset U_1 \cap \Omega^\circ(G)$$

that $U_1 \cap \Omega^\circ(J) = U_1 \cap \Omega^\circ(G)$ and that $g(U_1 \cap \Omega^\circ(G)) \cap (U_1 \cap \Omega^\circ(G)) = \emptyset$ for any $g \in G \setminus J_\infty$. By the similar reasoning as that in the proof of Claim 13, we know that $U_1 \cap \Omega(J) = U_1 \cap \Omega(G)$ and $U_1 \cap \Omega(J)$ is precisely invariant under J_∞ in G.

Since $g_1(U_1) = U_2$, for any $g \in G \setminus G_1$, we have that

$$g(U_1) \cap U_2 = g(U_1) \cap g_1(U_1) = \emptyset, \quad g(U_2) \cap U_1 = gg_1(U_1) \cap U_1 = \emptyset$$

and

$$g(U_2) \cap U_2 = gg_1(U_1) \cap g_1(U_1) = \emptyset.$$

This implies that U is a peak domain for G at ∞. □

By Theorem 4.2, we know that G is geometrically finite. The proof is completed.
From the proof of Theorem 5.1, we can easily get the following corollaries.

Corollary 5.6. *B_1 is not precisely invariant under J in G_1.*

Corollary 5.7. $D_1 \cap B_1 = D_1$.

REMARK 5.1. In Theorem 5.1 the following conditions are not satisfied:
(1) B_m ($m = 1, 2$) is precisely invariant under J in G_m;
(2) $D_m \cap B_m \neq D_m$.
But these conditions are required in Theorem 1.2.

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