SELF-MAPPING DEGREES
OF TORUS BUNDLES AND TORUS SEMI-BUNDLES

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Abstract

Each closed oriented 3-manifold $M$ is naturally associated with a set of integers $D(M)$, the degrees of all self-maps on $M$. $D(M)$ is determined for each torus bundle and semi-bundle $M$. The structure of torus semi-bundle is studied in detail. The paper is a part of a project to determine $D(M)$ for all 3-manifolds in Thurston’s picture.

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1. Introduction

1.1. Background. Each closed oriented $n$-manifold $M$ is naturally associated with a set of integers, the degrees of all self-maps on $M$, denoted as $D(M) = \{\deg(f) | f: M \to M\}$.

Indeed the calculation of $D(M)$ is a classical topic appeared in many literatures. The result is simple and well-known for dimension $n = 1, 2$, and for dimension $n > 3$, there are many interesting special results (see [2] and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$.  

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The case of dimension 3 becomes attractive in the topic and it is possible to calculate $D(M)$ for any closed oriented 3-manifold $M$. Since Thurston’s geometrization conjecture, which seems to be confirmed, implies that closed oriented 3-manifolds can be classified in reasonable sense.

Thurston’s geometrization conjecture claims that the each Jaco–Shalen–Johanson decomposition piece of a prime 3-manifold supports one of the eight geometries which are $H^3$, $\overline{PSL}(2, \mathbb{R})$, $H^2 \times E^1$, Sol, Nil, $E^3$, $S^3$, and $S^2 \times E^1$ (for details see [11] and [10]). Call a closed orientable 3-manifold $M$ is geometrizable if each prime factor of $M$ meets Thurston’s geometrization conjecture.

A known rather general fact about $D(M)$ for geometrizable 3-manifolds is the following:

**Theorem 1.1** ([12], Corollary 4.3). Suppose $M$ is a geometrizable 3-manifold. Then $M$ admits a self-map of degree larger than 1 if and only if $M$ is either
1. covered by a torus bundle over the circle, or
2. covered by an $F \times S^1$ for some compact surface $F$ with $\chi(F) < 0$, or
3. each prime factor of $M$ is covered by $S^3$ or $S^2 \times E^1$.

The proof of the “only if” part in Theorem 1.1 is based on the theory of simplicial volume, and various results on 3-manifold topology and group theory. The proof of “if” part in Theorem 1.1 is a sequence of elementary constructions, which were essentially known before.

Hence for any $M$ not listed in Theorem 1.1, $D(M)$ is either $\{0, 1, -1\}$ or $\{0, 1\}$, which depends on whether $M$ admits a self map of degree $-1$ or not. To determine $D(M)$ for geometrizable 3-manifolds listed in Theorem 1.1, let’s have a close look of those 3-manifolds from geometric and topological aspects.

Among Thurston’s eight geometries, six of them belong to the list in Theorem 1.1. 3-manifolds in (1) are exactly those supporting either $E^3$, or Sol or Nil geometries. $E^3$ 3-manifolds, Sol 3-manifolds, and some Nil 3-manifolds are torus bundles or semi-bundles; Nil 3-manifolds which are not torus bundles or semi-bundles are Seifert spaces having Euclidean orbifolds with three singular points. 3-manifolds in (2) are exactly those support $H^2 \times E^1$ geometry; 3-manifolds supporting $S^3$ or $S^2 \times E^1$ geometries form a proper subset of (3).

For 3-manifold $M$ with $S^3$-geometry, $D(M)$ has been presented recently in [1] in term of the orders of $\pi_1(M)$ and its elements (and determined earlier in [5] when the maps induce automorphisms on $\pi_1$). Note an algorithm is given to calculate the degree set of maps between $S^3$-manifolds in term of their Seifert invariants [8].

To determine $D(M)$ for the remaining geometrizable 3-manifolds $M$, the main task is to solve the question for the following three groups ($D(M)$ is rather easy to determine for Seifert manifold $M$ supporting $H^2 \times E^1$ or $S^2 \times E^1$ geometry):
(a) torus bundles and semi-bundles;
(b) Nil Seifert manifolds not in (a);
(c) connected sums of 3-manifolds in (3) do not supporting $S^3$ or $S^2 \times E^1$ geometries. Indeed $D(M)$ for $M$ in (a) will be determined in this paper (hopefully all the remaining cases will be solved in a forthcoming paper by the authors and Hao Zheng).

1.2. Main result. In this paper we calculate $D(M)$ for 3-manifold $M$ which is either a torus bundle or semi-bundle. To do this, we need first to coordinate torus bundles and semi-bundles by integer matrices in Propositions 1.3 and 1.5, then state the results of $D(M)$ in term of those matrices in Theorems 1.6 and 1.7.

CONVENTION. (1) To simplify notions, for a diffeomorphism $\phi$ on torus $T$, we also use $\phi$ to present its isotopy class and its induced 2 by 2 matrix on $\pi_1(T)$ for a given basis.

(2) Each 3-manifold $M$ is oriented, and each 3-submanifold of $M$ and its boundary have induced orientations.

(3) Suppose $S$ (resp. $P$) is a properly embedded surface (resp. an embedded 3-manifold) in a 3-manifold $M$. We use $M \setminus S$ (resp. $M \setminus P$) to denote the resulting manifold obtained by splitting $M$ along $S$ (resp. removing int $P$, the interior of $P$).

DEFINITION 1.2. A torus bundle is

$M_\phi = T \times I / (x, 1) \sim (\phi(x), 0)$

where $\phi$ is a self-diffeomorphism of the torus $T$ and $I$ is the interval $[0, 1]$.

For a torus bundle $M_\phi$, we can isotopic $\phi$ to be a linear diffeomorphism, which means $\phi \in GL_2(\mathbb{Z})$ while not changing $M_\phi$. Since we consider the orientable case only, $\phi$ must be in the special linear group $SL_2(\mathbb{Z})$.

Proposition 1.3. (1) $M_\phi$ admits $E^3$ geometry if and only if $\phi$ is periodical, or equivalently $\phi$ is conjugate to one of the following matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ of finite order 1, 2, 3, 4 and 6 respectively;

(2) $M_\phi$ admits Nil geometry if and only if $\phi$ is reducible, or equivalently $\phi$ is conjugate to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ where $n \neq 0$;

(3) $M_\phi$ admits Sol geometry if and only if $\phi$ is Anosov or equivalently $\phi$ is conjugate to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $|a + d| > 2$, $ad - bc = 1$.

Proof. See [4].

DEFINITION 1.4. Let $K$ be the Klein bottle and $N = K \times I$ be the twisted $I$-bundle over $K$. A torus semi-bundle $N_\phi = N \cup_\phi \mathcal{N}$ is obtained by gluing two copies along their torus boundary $\partial N$ via a diffeomorphism $\phi$. Note $N_\phi$ is foliated by tori parallel to $\partial N$ with a Klein bottle at the core of each copy of $N$. 


Let \((x, y, z)\) be the coordinate of \(S^1 \times S^1 \times I\). Then \(N = S^1 \times S^1 \times I / \tau\), where \(\tau\) is an orientation preserving involution such that \(\tau(x, y, z) = (x + \pi, -y, 1 - z)\), and we have the double covering \(p: S^1 \times S^1 \times I \to N\). Let \(C_x\) and \(C_y\) be the two circles on \(S^1 \times S^1 \times \{1\}\) defined by \(y\) to be constant and \(x\) to be constant, see Fig. 1. Denote by \(l_0 = p(C_x)\) (0 slope) and \(l_\infty = p(C_y)\) (\(\infty\) slope) on \(\partial N\). A canonical coordinate is an orientation of \(l_0 \cup l_\infty\), hence there are four choices of canonical coordinate on \(\partial N\). Once canonical coordinates on each \(\partial N\) are chosen, \(\Phi\) is identified with an element \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) of \(\text{GL}_2(\mathbb{Z})\) given by \(\phi(l_0, l_\infty) = (l_0, l_\infty) \begin{pmatrix} a & b \\ c & d \end{pmatrix}\).

**Proposition 1.5.** With suitable choice of canonical coordinates of \(\partial N\), we have:

1. \(N_{\Phi}\) admits \(E^3\) geometry if and only if \(\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) or \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\);
2. \(N_{\Phi}\) admits \(\text{Nil}\) geometry if and only if \(\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}\) or \(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\) where \(z \neq 0\);
3. \(N_{\Phi}\) admits \(\text{Sol}\) geometry if and only if \(\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) where \(abcd \neq 0, ad - bc = 1\).

Moreover a torus semi-bundle \(N_{\Phi}\) is also a torus bundle if and only if \(\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\) under suitable choice of canonical coordinates.

We will prove Proposition 1.5 in Section 2.

**Theorem 1.6.** Using matrix coordinates given by Proposition 1.3, \(D(M_{\Phi})\) is listed in Table 1 for torus bundle \(M_{\Phi}\), where \(\delta(3) = \delta(6) = 1, \delta(4) = 0\).

**Theorem 1.7.** Using matrix coordinates given by Proposition 1.5, \(D(N_{\Phi})\) is listed in Table 2 for torus semi-bundle \(N_{\Phi}\), where \(\delta(a, d) = ad / \gcd(a, d)^2\).
Table 1. Degrees of self maps of orientable torus bundles.

<table>
<thead>
<tr>
<th>$M_{\phi}$</th>
<th>$\phi$</th>
<th>$D(M_{\phi})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^3$</td>
<td>finite order $k = 1, 2$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$E^3$</td>
<td>finite order $k = 3, 4, 6$</td>
<td>${(kt + 1)(p^2 - \delta(k)pq + q^2) \mid t, p, q \in \mathbb{Z}}$</td>
</tr>
<tr>
<td>Nil</td>
<td>$\pm \left(\begin{array}{cc} 1 &amp; 0 \ n &amp; 1 \end{array}\right), n \neq 0$</td>
<td>${l^2 \mid l \in \mathbb{Z}}$</td>
</tr>
<tr>
<td>Sol</td>
<td>$\left(\begin{array}{cc} a &amp; b \ c &amp; d \end{array}\right),</td>
<td>a + d</td>
</tr>
</tbody>
</table>

Table 2. Degrees of self maps of torus semi-bundles.

<table>
<thead>
<tr>
<th>$N_{\phi}$</th>
<th>$\phi$</th>
<th>$D(N_{\phi})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^3$</td>
<td>$\left(\begin{array}{cc} 1 &amp; 0 \ 0 &amp; 1 \end{array}\right)$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$E^3$</td>
<td>$\left(\begin{array}{cc} 0 &amp; 1 \ 1 &amp; 0 \end{array}\right)$</td>
<td>${2l + 1 \mid l \in \mathbb{Z}}$</td>
</tr>
<tr>
<td>Nil</td>
<td>$\left(\begin{array}{cc} 1 &amp; 0 \ z &amp; 1 \end{array}\right), z \neq 0$</td>
<td>${l^2 \mid l \in \mathbb{Z}}$</td>
</tr>
<tr>
<td>Nil</td>
<td>$\left(\begin{array}{cc} 0 &amp; 1 \ 1 &amp; 0 \end{array}\right)$ or $\left(\begin{array}{cc} 1 &amp; 0 \ z &amp; 1 \end{array}\right), z \neq 0$</td>
<td>${(2l + 1)^2 \mid l \in \mathbb{Z}}$</td>
</tr>
<tr>
<td>Sol</td>
<td>$\left(\begin{array}{cc} a &amp; b \ c &amp; d \end{array}\right), abcd \neq 0, ad - bc = 1$</td>
<td>${(2l + 1)^2 \mid l \in \mathbb{Z}}$, if $\delta(a, d)$ is even or ${(2l + 1)^2 \mid l \in \mathbb{Z}} \cup {(2l + 1)^2 \cdot \delta(a, d) \mid l \in \mathbb{Z}}$, if $\delta(a, d)$ is odd</td>
</tr>
</tbody>
</table>

1.3. Remark on orientation reversing homeomorphisms. Suppose $M$ is a torus bundle or semi-bundle. Then any non-zero degree map is homotopic to a covering ([12] Corollary 0.4). Hence if $-1 \in D(M)$ (which is computable by Theorems 1.6 and 1.7), then $M$ admits an orientation reversing self homeomorphism.

If $M$ is a torus semi-bundle, or $M$ supports the geometry of either $E^3$ or Nil, then when $M$ admits an orientation reversing self homeomorphism is explicitly presented in the following:

**Corollary 1.8.** (1) A torus semi-bundle $N_{\phi}$ admits an orientation reversing homeomorphism if and only if $\phi$ is either $\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$, or $\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$, or $\left(\begin{array}{cc} a & b \\ c & -a \end{array}\right)$ where $abc \neq 0$.

(2) A torus bundle $M_{\phi}$ supporting $E^3$ geometry admits an orientation reversing homeomorphism if and only if $\phi$ is either $\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$, or $\left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right)$.

(3) If $M$ supports Nil geometry, then $M$ admits no orientation reversing homeomorphism.

For torus bundle with given Anosov monodromy, even we can calculate whether $-1 \in D(M_{\phi})$, but there seems no simple description as in Corollary 1.8. (The referee
informed us that there is a convenient description of when \(-1 \in D(M_\phi)\), see Lemma 1.7, [9].)

**Example 1.9.** For the torus bundle \(M_\phi, \phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\), \(-1 \in D(M_\phi)\). Indeed for \(\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\), if \(|a + d| = 3\), then \(-1 \in D(M_\phi)\). Since \(p^2 + (d - a)pr/c - br^2/c = -1\) has solution \(p = 1 - d, r = c\) when \(a + d = 3\), and solution \(p = -1 - d, r = c\) when \(a + d = -3\).

**Example 1.10.** For the torus bundle \(M_\phi, \phi = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\), \(-1 \notin D(M_\phi)\). Indeed for \(\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\), if \(a + d \pm 2\) has prime decomposition \(p_1^{e_1} \cdots p_n^{e_n}\) such that \(p_i = 4l + 3\) and \(e_i = 2m + 1\) for some \(i\), then \(-1 \notin D(M_\phi)\). Since if the equation \(p^2 + (d - a)pr/c - br^2/c = -1\) has integer solution, \(((a + d)^2 - 4)r^2 - 4d^2/c^2\) should be a square of rational number. That is \(((a + d)^2 - 4)r^2 - 4d^2 = s^2\) for some integer \(s\). Therefore \((a + d + 2)(a + d - 2)r^2\) is a sum of two squares. By a fact in elementary number theory, neither \(a + d + 2\) nor \(a + d - 2\) has \(4k + 3\) type prime factor with odd power (see p. 279, [7]).

**Example 1.11.** Note if \(-1 \in D(M)\), then \(k \in D(M)\) implies \(-k \in D(M)\). For the torus bundle \(M_\phi, \phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\), among the first 20 integers \(0, \) exactly \(1, 4, 5, 9, 11, 16, 19, 20 \in D(M_\phi)\).

1.4. **Organization of the paper.** Theorems 1.6 and 1.7 will be proved in Sections 3 and 4 respectively. To prove these theorems, we need have a careful look of the structures of torus bundle and semi-bundles. This is carried out in Section 2.

We explain more about Section 2. The most convenient and useful reference for us is “Notes on basic 3-manifold topology” by Hatcher [4], which is not formally published, but widely circulated (see http://www.math.cornell.edu/~hatcher/). In particular Chapter 2 of [4] is devoted to the study of torus bundles and semi-bundles. Theorems 2.3 and 2.4 about classifications of torus bundles and semi-bundles are quoted from [4] directly. It seems that the proof of Theorem 2.4 in [4] missed an existed and rather complicated case, so we rewrite a proof for it (most parts still follow that in [4]). Lemma 2.6 studies incompressible surfaces in torus semi-bundle, which relies on the proof of Theorem 2.4. Then Proposition 1.5 is proved by using Theorem 2.4, Lemma 2.6, and Lemma 2.8 which presents the relation between gluing maps of a torus semi-bundles and its torus bundle double covers. Finally, Theorem 2.9 studies lifting of maps between torus semi-bundles to their torus bundle double covers.

2. **Structures of orientable torus bundles and semi-bundles**

2.1. **Some elementary facts.** All facts in this sub-section are known, and one can find them in [6], or more directly in [4].
DEFINITION 2.1. Suppose an oriented 3-manifold $M'$ is a circle bundle with a given section $F$, where $F$ is a compact surface with boundary components $c_1, \ldots, c_n, \ldots, c_{n+b}$ with $n > 0$. On each boundary component of $M'$, orient $c_i$ and the circle fiber $l_i$ so that the product of their orientations match with the induced orientation of $M'$. Now attaching $n$ solid tori $S_i$ to the first $n$ boundary tori of $M'$ so that the meridian of $S_i$ is identified with slope $r_i = a_i c_i + b_i l_i$ with $a_i > 0$. Denote the resulting manifold by $M$ which has the Seifert fiber structure extended from the circle bundle structure of $M'$.

We will denote this Seifert fibering of $M$ by $M(\pm g, b; r_1, \ldots, r_s)$ where $g$ is the genus of the section $F$ of $M$, with the sign $+$ if $F$ is orientable and $-$ if $F$ is nonorientable, here ‘genus’ of nonorientable surfaces means the number of $RP^2$ connected summands. When $b = 0$, call $e(M) = \sum r_i$ the Euler number of the Seifert fibration.

Another view of $N$ described in Fig. 2 (a): $N$ is obtained from $S^1 \times I \times I$ by identifying $S^1 \times I \times \{0\}$ with $S^1 \times I \times \{1\}$ via a diffeomorphism $\rho$ which reflects both the $S^1$ and $I$ factors. Fig. 2 (b) is a schematic picture of $N$ which will be used in the paper.

We list some properties of $N$ as:

Lemma 2.2. (1) $N$ has two types of Seifert fiber structures:
I: $M(0, 1; 1/2, -1/2)$ in which $l_0$ on $\partial N$ is a regular fiber and $l_\infty$ is the boundary of the section defining the Seifert invariant.
II: $M(-1, 1; \ )$ in which $l_\infty$ on $\partial N$ is a regular fiber and $l_0$ is the boundary of the section defining the Seifert invariant.

(2) $N$ has three types of essential (orientable, incompressible, $\partial$-incompressible) surfaces:
I. A torus parallel to $\partial N$.
II. An annulus whose boundary is $l_\infty$ in $\partial N$ (Fig. 3 (a)) which does not separate $N$.
III. An annulus whose boundary is $l_0$ in $\partial N$ (Fig. 3 (b)) which separates $N$.

(3) Suppose $M$ is a torus bundle or semi-bundle and $F$ is a closed incompressible surface in $M$, then $F$ is union of parallel tori.
2.2. Classifications of torus bundles and semi-bundles. Orientable torus bundles and semi-bundles are classified by two theorems below.

**Theorem 2.3** ([3]; [4], Theorem 2.6). An orientable torus bundle $M_\phi$ is diffeomorphic to $M_\psi$ if and only if $\phi$ conjugates to $\psi^{-1}$ in $GL_2(\mathbb{Z})$.

**Theorem 2.4** ([4], Theorem 2.8). The torus semi-bundle $N_\phi$ is diffeomorphic to $N_\psi$ if and only if $\phi = \left( \begin{array}{cc} \pm 1 & 0 \\ 0 & \pm 1 \end{array} \right) \psi^{-1} \left( \begin{array}{cc} \pm 1 & 0 \\ 0 & \pm 1 \end{array} \right)$ in $GL_2(\mathbb{Z})$, with independent choices of signs understood.

Proof. (We start the proof as that in [4].) Suppose $f : N_\phi \rightarrow N_\psi$ is a diffeomorphism and $T$, $T'$ are the torus fibers of $N_\phi$, $N_\psi$ respectively. $N_\psi \setminus T' = N_1 \cup N_2$ where $N_1$, $N_2$ are homeomorphic to $N$.

Since $f$ is a diffeomorphism, two components of $N_\psi \setminus f(T)$ are both homeomorphic to $N$. We can isotope $f$, such that every component of $f(T) \cap N_i$ is an essential surface in $N_i$, $i = 1, 2$. So $f(T) \cap N_i$ is in the three types listed in Lemma 2.2 (2). Thus either $f(T)$ is parallel to $T'$, or $f$ takes $l_0$ or $l_\infty$ to $l_0'$ or $l_\infty'$.

Suppose $f(T)$ is parallel to $T'$. We can assume $f(T) = T'$. Then $\phi$ must be obtained from $\psi$ by composing on the left and right homeomorphisms of $\partial N$ which extend to homeomorphisms of $N$. Such homeomorphisms must preserve both $l_0$ and $l_\infty$ (may reverse the directions), since $l_0$ is the unique slopes of the boundaries of essential separating annulus and $l_\infty$ is the unique slopes of the boundaries of essential non-separating annulus in $N$. Theorem 2.4 is proved in this situation.

Suppose $\psi$ takes $l_0$ or $l_\infty$ to $l_0'$ or $l_\infty'$. Then there are three cases as below:

- **Case (1)** $\psi$ takes $l_\infty$ to $l_0$ (if $\psi$ takes $l_0$ to $l_\infty$, then we consider $\psi^{-1}$).
- **Case (2)** $\psi$ takes $l_\infty$ to $l_\infty$.
- **Case (3)** $\psi$ takes $l_0$ to $l_0$.

(The proof in [4] claims that only Case (3) is possible, while we show below that only Case (2) is impossible.)

Case (1). Now $\psi = \left( \begin{array}{cc} z & 1 \\ 1 & 0 \end{array} \right)$, and $N_\psi = M(-1,0;1/2,-1/2,z)$, and $e(M) = z$. Note:
Fig. 4. Cut $N_i$ through type (II) surfaces.

Fig. 5. Cut $N_i$ through type (III) surfaces.

(i) $f(T) \cap N_1$ are $n$ parallel annuli $A_1, \ldots, A_n$ of type (II) (see Fig. 4), which are located in a cyclic order in $N$. Set $\partial A_i = a_i \cup a'_i$, then $2n$ circles $a_1, \ldots, a_n, a'_1, \ldots, a'_n$ are located in cyclic order in $\partial N_1$.

(ii) $f(T) \cap N_2$ are annuli $B_1, \ldots, B_n$ of type (III) (see Fig. 5), where $B_{i+1}$ is next to $B_i$, $i = 1, \ldots, n-1$ in $N_2$. Set $\partial B_i = b_i \cup b'_i$ then $2n$ circles $b_1, \ldots, b_n, b'_n, \ldots, b'_1$ are located in cyclic order in $\partial N_2$.

If $n = 1$, we can check that $\psi$ pastes $A_1$ and $B_1$ to a Klein bottle, which contradicts the fact that $f(T)$ is torus. When $n > 1$, we can assume $\psi$ pastes $a_1$ to $b_1$ and pastes $a_2$ to $b_2$, after reindexing $A_i$ if necessary. By the orders of sequences of $a_1, \ldots, a_n, a'_1, \ldots, a'_n$ and $b_1, \ldots, b_n, b'_1, \ldots, b'_n$ on $\partial N_1$ and $\partial N_2$, we have $a_i$ is pasted to $b_i$, and $a'_i$ pasted to $b'_{n-i}$, $i = 1, \ldots, n$. So $A_1, A_{n-i}, B_i, B_{n-i}$ are pasted to one component of $f(T)$ in $N_\psi$, and $f(T)$ has $[(n+1)/2]$ components. Since $f(T)$ is connected, we have $n = 2$.

Now $N_1 \setminus f(T)$ can be presented as two $I$-bundles over annulus: $I \times A_1$ and $I \times A_2$, where $f(T) \cap N_1 = A_1 \cup A_2$, as in Fig. 4. $N_2 \setminus f(T)$ can be presented as an $I$-bundle
over annulus \( I \times B \) as in Fig. 6 (a) and two solid tori \( P_1 \) and \( P_2 \) with the core of 
\( P_1 \cap \partial N_2 \) to be the \((2, 1)\) curve of \( \partial P_i \) as in Fig. 6 (b).

If we glue those five pieces along \( \partial N \), we get two components of \( N_\psi \setminus f(T) \) which 
are \( N'_1 = P_1 \cup_{I \cap N} I \times A_1 \cup_{I \cap N} P_2 \) and \( N'_2 = I \times A_2 \cup_{I \cap N} I \times B \) (re-index \( A_i \) if needed), 
each of them is a copy of \( N \). Moreover under the inherited Seifert structure of \( N_\psi \), 
\( N'_1 = M(0, 1; 1/2, -1/2) \) and \( N'_2 = M(-1, 1; .) \).

If we consider that \( M(-1, 0; 1/2, -1/2, z) \) is obtained by identifying \( N'_1 \) and \( N'_2 \) along \( f(T) \), we get a new semi-bundle structure so that \( f(T) \) become a fiber torus. Since 
the Euler number of the Seifert structure is \( z \), the new gluing map must be 
\[
\begin{pmatrix}
z & 1 \\
1 & 0
\end{pmatrix}^{\pm 1}
\]
This reduces us to the situation that \( f(T) \) is parallel to \( T' \).

Case (2). Both \( f(T) \cap N_i \) are type (II) surfaces, for \( i = 1, 2 \) (Fig. 4). Hence 
\( f(T) \cap N_1 \) is exactly as that in Case (1) (i). Similarly, \( f(T) \cap N_2 \) are \( n \) parallel annulus \( B_1, \ldots, B_n \) located in a cyclic order in \( N \). Set \( \partial B_i = b_i \cup b'_i \), then \( 2n \) circles 
\( b_1, \ldots, b_n, b'_1, \ldots, b'_n \) are located in cyclic order in \( \partial N_2 \).

We can assume \( \psi \) paste \( a_1 \) to \( b_1 \) and paste \( a_2 \) to \( b_2 \) (re-index \{\( B_i \)\} if needed). 
Then we have \( a_i \) is pasted to \( b_i \), and \( a'_i \) pasted to \( b'_i \), \( i = 1, \ldots, n \). So \( A_i \) and \( B_i \) are 
pasted to one component of \( f(T) \) in \( N_\psi \). Since \( f(T) \) is connected, \( n = 1 \). But here 
\( f(T) \) does not separate \( N_\psi \), it is impossible.

Case (3). (We copy the proof of [4] for this case.) Now \( \psi = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \), and \( N_\psi = 
M(0, 0; 1/2, 1/2, -1/2, -1/2, z), \ e(N_\psi) = z \). (Both \( f(T) \cap N_i \) are type (III).)

We may assume that \( f(T) \) has been isotoped to be either vertical or horizontal in 
this Seifert fibering. Since a connected horizontal essential surface is not separating, 
\( f(T) \) must be vertical. Then \( f(T) \) must separate \( M(0, 0; 1/2, 1/2, -1/2, -1/2, z) \) into 
two copies of \( N \) both having the inherited Seifert structure \( M(0, 1; 1/2, -1/2) \). We can 
rechoose the semi-bundle structure so that \( f(T) \) become a fiber torus. Then for the new 
torus semi-bundle structure the gluing map must also be \( \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \).
This reduces us to the situation that \( f(T) \) is parallel to \( T' \).
2.3. Incompressible surfaces.

**Lemma 2.5** ([4], Lemma 2.7). For a torus bundle $M_\phi$, if $\phi$ is not conjugate to $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then any essential closed surface in $M_\phi$ is isotopic to a union of torus fibers.

**Lemma 2.6.** If a torus semi-bundle $N_\phi$ has no torus bundle structure, then any essential closed surface in $N_\phi$ is isotopic to copies of torus fibers of a torus semi-bundle structure on $N_\phi$, which is isomorphic to $N_\phi$.

Proof. Let $F$ be an essential close surface in $N_\phi = N_1 \cup N_2$. By Lemma 2.2 (3), $F$ is a union of parallel tori. For our purpose we may assume that $F$ is a torus. Isotope $F$ so that $F \cap N_i$ is essential in $N_i$. Then each component of $F \cap N_i$ must be in one of the three types listed in Lemma 2.2.

If $F \cap N_i$ is of type (I), then the proof is finished.

There are two cases remaining:

(a) Both $F \cap N_i$ are of type (II) for $i = 1, 2$ (Fig. 4). Then $N_i \setminus F$ are I-bundles over $N_i \setminus F$. Gluing those two I-bundles along $\partial N$ will get an I-bundle over $F$ and $N_\phi$ is obtained from this I-bundle by identifying its top and bottom, which provides a torus bundle structure of $N_\phi$.

(b) Some $F \cap N_i$ is of type (III), say $F \cap N_2$ (Fig. 5). Then $F$ is the same as $f(T)$ either in Case (1) or Case (3) of the proof of Theorem 2.4, depends on $F \cap N_1$ is of type (III) or type (II).

As indicated in the proof of Theorem 2.4, we can rechoose the new torus semi-bundle structure $N_\psi$ so that $F$ become a fiber torus; moreover if choosing suitable coordinates, we can make $\psi$ to be $\phi$. \qed

2.4. Coordinates of torus semi-bundles. Call a map $g: (M, \partial M) \rightarrow (M', \partial M')$ is proper if $g^{-1}(\partial M') \subset \partial M$.

**Lemma 2.7.** If $V = T \times I$ with the two boundaries $T^+$, $T^-$ and $g: (V, T^+, T^-) \rightarrow (N, \partial N)$ is a proper map, then $(g|_{T^+})_* = \tau_s \cdot (g|_{T^-})_*$, where $\tau_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof. Let $p: T \times I \rightarrow N$ be the double covering and $\tau$ be the deck transformation map.

Since $g_\ast(\pi_1(V)) = (g|_{T^+})_\ast(\pi_1(T^+)) \subset \pi_1(\partial N) \subset \pi_1(N)$, thus $g$ can be lifted to a map $\tilde{g}: V \rightarrow T \times I$.

\[
\begin{array}{c}
(T \times I, T \times \{0\}, T \times \{1\}) \xrightarrow{\tau} (T \times I, T \times \{1\}, T \times \{0\}) \\
(V, T^+, T^-) \xrightarrow{g} (N, \partial N)
\end{array}
\]
From the commuted diagram above, we have:

\[
\begin{align*}
\{ g|_{r^-} &= p|_{r\times\{1\}} \circ \tilde{g}|_{r^-}, \\
\{ g|_{r^+} &= p|_{r\times\{1\}} \circ \tau|_{r\times\{0\}} \circ \tilde{g}|_{r^+}.
\end{align*}
\]

We can choose coordinate on \((T \times I', T \times [0], T \times [1])\), such that \(p|_{r\times\{1\}} = id\).

When considering fundamental group, we have \((\tilde{g}|_{r^-})_s = (\tilde{g}|_{r^+})_s\). Thus by the above equation:

\[
(\tilde{g}|_{r^+})_s = \tau_s \cdot (\tilde{g}|_{r^-})_s
\]

where \(\tau_s = (\gamma|_{r\times\{0\}})_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\).

**Lemma 2.8.** A torus semi-bundle \(N_\phi\) is doubly covered by a torus bundle \(M_{\tau\phi\tau^{-1}}\) where \(\tau(x, y) = (x + \pi, -y)\) with suitable choice of coordinate \((x, y)\) on the torus.

**Proof.** Let \(N_\phi = N_1 \cup_\phi N_2\) with \(\partial N_1 = \partial N_2 = T\). Let \(p: M \to N_\phi\) be the double cover, where \(M\) is a torus bundle, \(p^{-1}(N_i) = M_i\) is homeomorphic to \(T \times I\). \(p^{-1}(T) = T_1 \cup T_2\). Cut \(M\) along \(T_1, T_2\), get \(M \setminus T_1 \cup T_2\). The two boundaries of \(M_i\) are denoted by \(T_i'\), \(T_1\) is pasted to \(T_2\) by \(\psi\), \(T_1'\) is pasted to \(T_2'\) by \(\psi'\). Let \(p_i = p|_{M_i}\). All of these are shown in Fig. 7.

We can choose coordinate on \(T_1, T_2\), such that \((p_i|_{T_i})_s = id\). Since \(T_i'\) is parallel to \(T_i\), we can identify \(\pi_1(T_i')\) with \(\pi_1(T_i)\). By Lemma 2.7, we have \((p_i|_{T_i'})_s = \tau_s \cdot (p_i|_{T_i})_s\).
From Fig. 7, we know that
\[
\begin{align*}
(p_2|T_s) \circ \psi &= \phi \circ (p_1|T_s), \\
(p_2|T'_s) \circ \psi' &= \phi \circ (p_1|T'_s).
\end{align*}
\]

Then we get
\[
\begin{align*}
\psi &= \phi, \\
\psi' &= \tau \circ \phi \circ \tau.
\end{align*}
\]

Thus \( M \) has the torus bundle structure \( M_{\psi' \psi^{-1}} = M_{\tau \phi \tau^{-1}} \).

By Theorem 2.4, and the fact that \( \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}^{-1} = \begin{pmatrix} -z & 1 \\ 1 & 0 \end{pmatrix} \), with suitable choice of canonical coordinates of \( \partial N \), we can set \( \phi \) is one of the four matrices: \( \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} \), \( \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \), \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( abcd \neq 0, ad - bc = 1 \).

When \( \phi \) is in the first three matrices, \( N_\phi \) is a Seifert manifold with Euler number \( z \). \( N_\phi \) is \( E^3 \) manifold if \( z = 0 \) and is Nil manifold if \( z \neq 0 \). Now suppose \( \phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( abcd \neq 0, ad - bc = 1 \). Then by Lemma 2.8, \( N_\phi \) is double covered by \( M_{\tau \phi \tau^{-1}} \). Since
\[
(\tau \phi \tau^{-1})_s = \tau \cdot \phi \cdot \tau \cdot \phi^{-1} = \begin{pmatrix} ad + bc & -2ab \\ -2cd & ad + bc \end{pmatrix},
\]
we have
\[
|\text{Trace}((\tau \phi \tau^{-1})_s)| = 2|ad + bc| = 2|ad - bc + 2bc| = 2|2bc + 1| > 2.
\]

By Proposition 1.3, \( M_{\tau \phi \tau^{-1}} \) admits Sol geometry, thus \( N_\phi \) admits Sol geometry. The first part of Proposition 1.5 is proved.

If \( N_\phi \) also has torus bundle structure, it must have non-separating essential torus. Recall the proof of Lemma 2.6, an essential torus in \( N_\phi \) can be non-separating only if case (a) is happened, and in this case \( \phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \) under suitable choice of canonical coordinates, and \( N_\phi \) does have torus bundle structure. This finishes the “moreover” part of Proposition 1.5.

2.5. Lifting automorphism from semi-bundle to bundle.

**Theorem 2.9.** Suppose \( f: N_\phi \to N_\psi \) is a non-zero degree map and \( f^{-1}(T') \) is a union of copies of \( T \), where \( T, T' \) are the torus fiber of \( N_\phi, N_\psi \) respectively. Then we
have commute diagram

\[
\begin{array}{ccc}
M & \xrightarrow{j} & M' \\
p & & p' \\
N_\phi & \xrightarrow{f} & N_\psi
\end{array}
\]

where \( M, M' \) are the torus bundle which are double covers of \( N_\phi, N_\psi \) respectively and \( \tilde{f} : M \to M' \) is a lift of \( f \).

Proof. We only have to check \( f_\ast(p_\ast(\pi_1(M))) \subset p'_\ast(\pi_1(M')) \).

Let \( \tilde{T}, \tilde{T}' \) be one of the lifting of \( T, T' \) in \( M, M' \) respectively. In torus bundle \( M \), we have the exact sequence:

\[
1 \to \pi_1(\tilde{T}) \to \pi_1(M) \to \pi_1(S^1) \to 1.
\]

In torus semi-bundle \( N_\phi \), we have another exact sequence:

\[
1 \to \pi_1(T) \to \pi_1(N_\phi) \to \mathbb{Z}_2 \ast \mathbb{Z}_2 \to 1.
\]

Since \( f^{-1}(T') \) is a union of copies of \( T \), we can assume \( f(T) = T' \). Then we have the commuted diagram (every row is exact):

\[
\begin{array}{ccccccccc}
1 & \to & \pi_1(\tilde{T}) & \overset{i_1}{\to} & \pi_1(M) & \overset{j_1}{\to} & \pi_1(S^1) & \to & 1 \\
\langle p \rangle \ast & & \downarrow p_\ast & & \downarrow p_\ast & & \downarrow \tilde{p}_\ast & & \\
1 & \to & \pi_1(T) & \overset{i_1}{\to} & \pi_1(N_\phi) & \overset{j_1}{\to} & \mathbb{Z}_2 \ast \mathbb{Z}_2 & \to & 1 \\
\langle f \rangle \ast & & \downarrow f_\ast & & \downarrow f_\ast & & \downarrow \tilde{f}_\ast & & \\
1 & \to & \pi_1(T') & \overset{i_2}{\to} & \pi_1(N_\psi) & \overset{j_2}{\to} & \mathbb{Z}_2 \ast \mathbb{Z}_2 & \to & 1 \\
\langle p' \rangle \ast & & \downarrow p'_\ast & & \downarrow p'_\ast & & \downarrow \tilde{p}'_\ast & & \\
1 & \to & \pi_1(\tilde{T}') & \overset{i_2}{\to} & \pi_1(M') & \overset{j_2}{\to} & \pi_1(S^1) & \to & 1,
\end{array}
\]

here \( \tilde{p}_\ast, \tilde{p}'_\ast, \tilde{f}_\ast \) are the maps among the fundamental groups of the base spaces of fiber bundles induced by the maps among the fundamental groups of the total spaces.

We present the group \( \mathbb{Z}_2 \ast \mathbb{Z}_2 \) by \( \langle a, b \mid a^2 = b^2 = 1 \rangle \) and choose the generator \( a, b \) such that \( \tilde{p}_\ast(1) = ab, \tilde{p}'_\ast(1) = ab \) (here \( 1 \) is the generator of \( \pi_1(S^1) \)).

Since \( a^2 = b^2 = 1 \), so \( \tilde{f}_\ast(a)^2 = \tilde{f}_\ast(b)^2 = 1 \), then \( \tilde{f}_\ast(a), \tilde{f}_\ast(b) \) must be of the form \( ab \cdots ba \) or \( ba \cdots ab \), and \( \tilde{f}_\ast(ab) = (ab)^k \) or \( (ba)^k = (ab)^{-k} \). So \( \tilde{f}_\ast(\tilde{p}_\ast(\pi_1(S^1))) \subset \tilde{p}'_\ast(\pi_1(S^1)) \).
For any \( \alpha \in \pi_1(M) \), let \( \beta = f_\alpha(p_\alpha(\alpha)) \). Since \( j_2(\beta) = \tilde{f}_\alpha(\tilde{p}_\alpha(\tilde{j}_1(\alpha))) \in \tilde{p}_\alpha(\pi_1(S^1)) \), and there is \( \gamma \in \pi_1(M') \) such that \( \tilde{p}_\alpha(\tilde{j}_2(\gamma)) = j_2(\beta) \), so

\[
j_2(p'_\alpha(\gamma) \cdot \beta^{-1}) = \tilde{p}'_\alpha(\tilde{j}_2(\gamma)) \cdot j_2(\beta^{-1}) = j_2(\beta) \cdot j_2(\beta^{-1}) = 1.
\]

Since \( (p'|)_\alpha \) is an isomorphism, there is \( \delta \in \pi_1(T') \) such that \( i_2((p'|)_\alpha(\delta)) = p'_\alpha(\gamma) \cdot \beta^{-1} \). We have

\[
p'_\alpha(i_2(\delta^{-1}) \cdot \gamma) = i_2((p'|)_\alpha(\delta^{-1})) \cdot p'_\alpha(\gamma) = (p'_\alpha(\gamma) \cdot \beta^{-1})^{-1} \cdot p'_\alpha(\gamma) = \beta.
\]

So \( f_\alpha(p_\alpha(\pi_1(M))) \subseteq p'_\alpha(\pi_1(M')) \), thus \( \tilde{f} \) exists. \( \square \)

3. The degrees of self maps of torus bundles

We are going to prove Theorem 1.6 (ref. Proposition 1.3). There are two cases to consider:

**CASE 1:** \( \phi \) is conjugated to \( \pm \left( \begin{array}{ll} 1 & 0 \\ n & 1 \end{array} \right) \). Now \( M_\phi \) is a Seifert manifold whose Euler number of Seifert fibering \( e(M_\phi) \) is equal to \( n \).

(1.I) If \( n = 0 \), \( M_\phi \) is \( T^3 \) or \( S^1 \times S^1 \times S^1 \). Here \( \phi = \pm \left( \begin{array}{ll} 1 & 0 \\ 0 & 1 \end{array} \right) \), any \( 2 \times 2 \) integer matrix \( A \) commutes with \( \phi \), so \( M_\phi \) admits self maps of any degrees.

(1.II) If \( n \neq 0 \), for a none zero degree map \( f: M_\phi \rightarrow M_\phi \), by [12, Corollary 0.4], \( f \) is homotopic to a covering map \( g: M_\phi \rightarrow M_\phi \). We can choose a suitable Seifert fibering of \( M_\phi \) such that \( g \) is a fiber preserving map. Denote the orbifold of \( M_\phi \) by \( O(M_\phi) \). By [10, Lemma 3.5], we have:

\[
\begin{align*}
\{ e(M_\phi) &= e(M_\phi) \cdot \frac{l}{m}, \\
\deg(g) &= l \cdot m,
\end{align*}
\]

where \( l \) is the covering degree of \( O(M_\phi) \rightarrow O(M_\phi) \) and \( m \) is the fiber degree.

Since \( e(M_\phi) \neq 0 \), from equation (3.1) we get \( l = m \). Thus \( \deg(f) = \deg(g) \) is a square number. Conversely, given a square number \( l^2 \), it is easy to construct a covering map \( f: M_\phi \rightarrow M_\phi \) of degree \( l^2 \).

**CASE 2:** \( \phi = \left( \begin{array}{ll} a & b \\ c & d \end{array} \right) \) is not conjugated to \( \pm \left( \begin{array}{ll} 1 & 0 \\ 0 & 1 \end{array} \right) \).

**Theorem 3.1.** Suppose \( \phi \) is not conjugated to \( \pm \left( \begin{array}{ll} 1 & 0 \\ 0 & 1 \end{array} \right) \) \( M_\phi \) admits a self map of degree \( l \neq 0 \) if and only if there exist a \( 2 \times 2 \) nondegenerate integer matrix \( A \) and a positive integer \( k \) such that \( l = k \cdot e \cdot \det(A) \) and \( A \cdot \phi_\alpha = (\phi_\alpha)^k \cdot A \) where \( e = \pm 1 \).

**Proof.** For a torus fiber \( T \subseteq M_\phi \), \( T \) is incompressible. Suppose \( f: M_\phi \rightarrow M_\phi \) is a self-map of degree \( l \neq 0 \). By [6, Lemma 6.5], \( f \) is homotopic to \( g: M_\phi \rightarrow M_\phi \).
such that \( g^{-1}(T) \) is an incompressible surface of \( M_\phi \). Thus by Lemma 2.5, \( g^{-1}(T) \) is isotopic to a union of torus fibers.

Suppose \( M_\phi \setminus g^{-1}(T) \) has \( k \) components \( V_1, \ldots, V_k \). Each \( V_i \) is a \( T \times I \). Denote two torus boundary components of \( V_i \) by \( T_i^+ \) and \( T_i^- \), and the homeomorphism gluing \( T_i^- \) to \( T_{i+1}^+ \) by \( \psi_i \) see Fig. 8. Then \( M_{\phi_1 \circ \cdots \circ \phi_i} = M_\phi \). By choosing suitable coordinate on the torus fiber, we have \( \psi_k \circ \cdots \circ \psi_0 = \phi^\epsilon, \epsilon = \pm 1 \) according to Theorem 2.3. Below we assume \( \psi_k \circ \cdots \circ \psi_0 = \phi \) (replace \( \phi \) by \( \phi^{-1} \) if needed). Let \( \bar{g} : M_\phi \setminus g^{-1}(T) \to M_\phi \setminus T \) be the map induced by \( g \). We have the following commuted diagram:

\[
\begin{array}{c}
M_\phi \setminus g^{-1}(T) \xrightarrow{\bar{g}} M_\phi \setminus T \\
\cup \psi_i \downarrow \quad \downarrow \phi^\epsilon \\
M_\phi \xrightarrow{\iota} M_\phi.
\end{array}
\]

(3.2)

Denote the restriction of \( \bar{g} \) to \( V_i \) by \( g_i \). From the commuted diagram in Fig. 8, we have:

\[
g_{i+1}|_{T_{i+1}^+} \circ \psi_i = \phi^\epsilon \circ g_i|_{T_i^-},
\]

(3.3)

where \( \epsilon = \pm 1 \), \( i = 1, \ldots, k \) and if \( i = k \) then \( i + 1 \) is 1.
Since $T_i$ is parallel to $T_i^+$, we can identify $\pi_1(T_i^-)$ with $\pi_1(T_i^+)$. Thus $(g_i|_{T_i^-})_s = (\psi_k)_s$ and $(\psi_1)_s \cdots (\psi_1)_s = \phi_s$ on fundamental group. The identity (3.3) deduces that:

\[
(g_1|_{T_i^-})_s \cdot \phi_s = (g_1|_{T_i^-})_s \cdot (\psi_k)_s \cdots (\psi_1)_s
\]

\[
= (g_{k+1}|_{T_{i+1}^-})_s \cdot (\psi_k)_s \cdots (\psi_1)_s
\]

\[
= \phi^e_s \cdot (g_{k+1}|_{T_{i+1}^-})_s \cdot (\psi_k)_s \cdots (\psi_1)_s
\]

\[
= \phi^e_s \cdot (g_{k+1}|_{T_{i+1}^-})_s \cdot (\psi_k)_s \cdots (\psi_1)_s
\]

\[
= \cdots
\]

\[
= (\phi^e_s)^k \cdot (g_1|_{T_i^-})_s.
\]

Set $A = (g_1|_{T_i^-})_s$ and get:

\[
(3.4) \quad A \cdot \phi_s = (\phi^e_s)^k \cdot A.
\]

Clearly $|\deg(g)| = k|\det(A)|$. The sign of $\deg(g)$ is decided by $e$ and the sign of $\det(A)$. Thus $l = \deg(f) = \deg(g) = k \cdot e \cdot \det(A)$.

Conversely, we set $\psi_i = \cdots = \psi_{k-1} = id$, $\psi_k = \phi$ and construct the map $\tilde{g}: M_\phi \setminus g^{-1}(T) \to M_\phi \setminus T$ such that $\tilde{g}|_{V_i} = (\phi^e \cdot (g_i)^{-1} \circ A) \times id: T \times I \to T \times I$ for $i = 1, \ldots, k$. This construction fits the commuted diagram (3.2). Thus we get the quotient $g: M_\phi \to M_\phi$ whose degree is equal to $k \cdot e \cdot \det(A)$. \hfill \Box

Suppose $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ where $p, q, r, s \in \mathbb{Z}$. We use equation (3.4) to solve $p, q, r, s$ and then can determine $l$ by Theorem 3.1.

(2.1) If $\phi$ is Anosov which means the absolute value of one eigenvalue of $\phi$ is larger than 1 while the other is less than 1. In this case, the $k$ in the equation (3.4) must be equal to 1. We have:

\[
\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\epsilon \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix}.
\]

Solve this matrix equation and get:

\[
A = \begin{cases} 
\begin{pmatrix} p & br \\ r & cp + (d-a)r \end{pmatrix} & (\epsilon = 1), \\
\begin{pmatrix} p & p(d-a) - br \\ r & cp + (d-a)r \end{pmatrix} & (\epsilon = -1)
\end{cases}
\]

where $br/c, (d-a)r/c, (p(d-a) - br)/c \in \mathbb{Z}$.
By Theorem 3.1, we have:

\[ l = p^2 + \frac{(d-a)}{c} \cdot pr - \frac{b}{c} \cdot r^2. \]

(2.II) If \( \phi \) is periodic, may assume \( \phi \) is either \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \), or \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), or \( \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \).

(A) If \( \phi = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \) (\( \phi \) has order 3), the equation (3.4) means:

\[ A \cdot \phi^k = \begin{cases} A & (k \equiv 0 \mod 3), \\
\phi \cdot A & (k \equiv 1 \mod 3), \\
\phi^2 \cdot A & (k \equiv 2 \mod 3). \end{cases} \]

After solving all the above possible cases, we get:

\[
A = \begin{cases}
\begin{pmatrix} p & q \\ -q & p - q \end{pmatrix} & (k \equiv 1 \mod 3, \epsilon = 1), \\
\begin{pmatrix} p & q \\ q - p & -p \end{pmatrix} & (k \equiv 1 \mod 3, \epsilon = -1), \\
\begin{pmatrix} p & q \\ q - p & -p \end{pmatrix} & (k \equiv 2 \mod 3, \epsilon = 1), \\
\begin{pmatrix} p & q \\ -q & p - q \end{pmatrix} & (k \equiv 2 \mod 3, \epsilon = -1).
\end{cases}
\]

If \( k \equiv 0 \mod 3 \), we have \( A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), which induces degree 0 map.

By Theorem 3.1:

\[ l = \begin{cases}
\begin{pmatrix} p^2 - pq + q^2 \\ k \cdot (p^2 - pq + q^2) \end{pmatrix} & (k \equiv 1 \mod 3), \\
\begin{pmatrix} -p^2 + pq - q^2 \\ k \cdot (-p^2 + pq - q^2) \end{pmatrix} & (k \equiv 2 \mod 3). 
\end{cases} \]

It’s easy to deduce that:

\[ l = (3t + 1)(p^2 - pq + q^2), \quad t, p, q \in \mathbb{Z}. \]

The same method is applied to the other two cases and we get:

(B) If \( \phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), then:

\[ l = (4t + 1)(p^2 + q^2), \quad t, p, q \in \mathbb{Z}. \]
(C) If $\phi = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, then:

$$l = (6t + 1)(p^2 - pq + q^2), \quad t, p, q \in \mathbb{Z}.$$ 

4. The degrees of self maps of torus semi-bundles

We are going to prove Theorem 1.7 (ref. Proposition 1.5). We will assume that torus semi-bundle $N_\phi$ considered in this section has no torus bundle structure, otherwise $D(N_\phi)$ is determined in Section 3.

Suppose the degree of $f : N_\phi \to N_\phi$ is $l \neq 0$ and $T$ is a torus fiber of $N_\phi$. By [6, Lemma 6.5], $f$ is homotopic to $g : N_\phi \to N_\phi$ such that $g^{-1}(T)$ is incompressible in $N_\phi$. Thus by Lemma 2.6 and its proof (also ref. the proof of Theorem 2.4), we have $g^{-1}(T)$ is isotopic to either a union of torus fibers, or a union of torus fibers of another semi-bundle structure which is isomorphic to the original one. Also the later case happen only if $N_\phi$ is a Nil manifold. Note by Theorem 2.9 and the proof in Section 3 (1.II), Nil 3-manifolds admits no orientation reversing homeomorphism.

Suppose now $g^{-1}(T)$ has $k$ connected components, then $N_\phi \setminus g^{-1}(T)$ has two copies of $N$, denoted by $V_0$ and $V_k$, and $k-1$ copies of $T \times I$, denoted by $V_i$, $i = 1, \ldots, k-1$. Denote the boundaries of $V_0$ and $V_k$ by $T_0^-$ and $T_k^+$, the boundaries of $V_i$ by $T_i^+$ and $T_i^-$, $i = 1, \ldots, k-1$, and the gluing map from $T_i^-$ to $T_{i+1}^+$ by $\psi_i$ ($i = 0, \ldots, k-1$) see Fig. 9.

Then $N_{\psi_1 \circ \cdots \circ \psi_0} = N_\phi$, and $\psi_{k-1} \circ \cdots \circ \psi_0 = \phi^\epsilon$, $\epsilon = \pm 1$ by Theorem 2.4 (with a suitable orientation of the canonical coordinate). Below we assume $\psi_{k-1} \circ \cdots \circ \psi_0 = \phi$ (replace $\phi$ by $\phi^{-1}$ if needed). Let $\bar{g} : N_\phi \setminus g^{-1}(T) \to N_\phi \setminus T$ be the map induced by $g$, and we have commuted diagram:

$$\begin{array}{ccc}
N_\phi \setminus g^{-1}(T) & \xrightarrow{\bar{g}} & N_\phi \setminus T \\
\cup \psi_i \downarrow & & \phi^\epsilon \\
N_\phi & \xrightarrow{g} & N_\phi.
\end{array}$$

(4.1)

Since $T_i^+$ is parallel to $T_i^-$, we can identity $\pi_1(T_i^+)$ with $\pi_1(T_i^-)$ ($i = 0, \ldots, k-1$). Thus $(\psi_{k-1})_s \cdots (\psi_0)_s = \phi_s$ on fundamental group. Denote the restriction of $\bar{g}$ on $V_i$ by $g_i$. Then $g : V_i \to N_1$ if $i$ even, and $g : V_i \to N_2$ if $i$ odd.

**Lemma 4.1.** Under the canonical basis $(l_0, l_\infty)$, $(g_0|_{T_i^-})_s$ is of the form

$$\begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}$$

where $n \neq 0$, $m, n \in \mathbb{Z}$, and so is $(g_k|_{T_i^-})_s$.

**Proof.** Let $g : N \to N$ be a proper map, we argue that under the basis $(l_0, l_\infty)$, $(g|_{\partial N})_s$ is of the form

$$\begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}$$

where $n \neq 0$, $m, n \in \mathbb{Z}$.
Choose a presentation \( \pi_1(N) = \langle a, b \mid a = bab \rangle \) with \( l_0 = a^2 \) and \( l_\infty = b \). Suppose \( g_*(a) = a^{m'b^q}, \ g_*(b) = a^pb^n \). Since \( g_*(a) = g_*(b)g_*(a)g_*(b) \), we get:

\[
(a^{m'b^q} a^{m'b^q} a^pb^n) = (a^{m'+2p} b^{(1)^m+p-n+(-1)^p} q+n).
\]

Thus:

\[
\begin{align*}
m' &= m' + 2p, \\
q &= (-1)^{m'+p} \cdot n + (-1)^p \cdot q + n, \quad \Rightarrow \quad \begin{cases} p = 0, \\
m' \text{ odd, or } p = 0,
\end{cases} \\
&\text{or } n = 0.
\end{align*}
\]

Abandon the case that \( p = n = 0 \) for \( g_0 \) is non-zero degree map and let \( m' = 2m + 1 \), we get: \( g_*(a) = a^{2m+1b^q}, \ g_*(b) = b^n \).

Since \( \pi_1(\partial N) = \langle a^2, b \mid [a^2, b] = 1 \rangle \) and \( g_*(a^2) = a^{2m+1} b^q a^{2m+1} b^q = a^{4m+2} \), we have

\[
(g|_{\partial N})_* = \begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}.
\]

**Theorem 4.2.** If \( N_\phi \) has no torus bundle structure, then \( N_\phi \) admits a self map of degree \( l \neq 0 \) if and only if there exist a positive integer \( k \) and two integer matrices \( A_1, A_2 \) of form \( \begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix} \), \( m, n \in \mathbb{Z}, \ n \neq 0 \), satisfying the following equation:

\[
A_2 \cdot \phi_* = \begin{cases} (\phi_*^{\epsilon} \cdot \tau_* \cdot \phi_*^{\epsilon} \cdot \tau_*)^{-1} \cdot \phi_*^{\epsilon} \cdot \tau_* \cdot \phi_*^{\epsilon} \cdot A_1 & (k = 2s), \\
(\phi_*^{\epsilon} \cdot \tau_* \cdot \phi_*^{\epsilon} \cdot \tau_*)^{k} \cdot \phi_*^{\epsilon} \cdot A_1 & (k = 2s + 1),
\end{cases}
\]

such that \( l = k \cdot \epsilon \cdot \det(A_1) \) where \( \epsilon = \pm 1 \).
Proof. From Fig. 9, we know that:

\begin{equation}
(4.2) \quad g_{i+1}|T_{i+1}^+ \circ \psi_i = \begin{cases} 
\phi^\epsilon \circ g_i|T_i^- & (i \equiv 0 \text{ mod } 2), \\
\phi^{-\epsilon} \circ g_i|T_i^- & (i \equiv 1 \text{ mod } 2), 
\end{cases}
\end{equation}

where \( \epsilon = \pm 1, i = 0, \ldots, k - 1. \)

Thus if \( k = 2s \) is even, then:

\begin{equation}
(4.3) \quad (g_k|T^-_s)_s \cdot \phi_s = (g_k|T^-_s)_s \cdot (\psi_{k-1})_s \cdots (\psi_0)_s \quad \text{by Fig. 9}
\end{equation}

\begin{equation}
= \phi_s^{-\epsilon} 
\quad \text{by (4.2)}
\end{equation}

\begin{equation}
\quad = \phi_s^\epsilon \cdot \tau_s \cdot (g_k|T^-_{k-1})_s \cdot (\psi_{k-2})_s \cdots (\psi_0)_s 
\end{equation}

\begin{equation}
\quad = \cdots
\end{equation}

\begin{equation}
\quad = (\phi_s^\epsilon \cdot \tau_s \cdot \phi_s^{-\epsilon} \cdot \tau_s)^{s-1} \cdot \phi_s^\epsilon \cdot \tau_s \cdot (g_0|T^-_0)_s.
\end{equation}

If \( k = 2s + 1 \) is odd, then:

\begin{equation}
(4.4) \quad (g_k|T^-_s)_s \cdot \phi_s = (g_k|T^-_s)_s \cdot (\psi_{k-1})_s \cdots (\psi_0)_s
\end{equation}

\begin{equation}
= \phi_s^\epsilon \cdot (g_{k-1}|T^-_{k-1})_s \cdot (\psi_{k-2})_s \cdots (\psi_0)_s
\end{equation}

\begin{equation}
= \phi_s^\epsilon \cdot \tau_s \cdot (g_{k-1}|T^-_{k-1})_s \cdot (\psi_{k-2})_s \cdots (\psi_0)_s
\end{equation}

\begin{equation}
= \cdots
\end{equation}

\begin{equation}
= (\phi_s^\epsilon \cdot \tau_s \cdot \phi_s^{-\epsilon} \cdot \tau_s)^{s-1} \cdot \phi_s^\epsilon \cdot \tau_s \cdot (g_0|T^-_0)_s.
\end{equation}

It is easy to see that \(|\deg(g)| = k|\det(g_0|T^-_0)_s||. The sign of \( \deg(g) \) is decided by both \( \epsilon \) and the sign of \( \det(g_0|T^-_0)_s \). Thus \( i = \deg(f) = \deg(g) = k \cdot \epsilon \cdot \det(g_0|T^-_0)_s \).

Finally by applying Lemma 4.1, we finish the proof of one direction of Theorem 4.2.

Conversely, if given \( k, A_1, A_2 \), then we can easily construct the maps \( g_0, g_k: N \to N \) such that \( (g_0|T^-_0)_s = A_1, (g_k|T^-_s)_s = A_2 \). Set \( \psi_0 = \cdots = \psi_{k-2} = id, \psi_{k-1} = \phi \) and \( g_i: T \times I \to N \) (\( i = 1, \ldots, k - 1 \)) is a map such that:

\begin{equation}
g_i|T_i^+ = \begin{cases} 
\phi^\epsilon \circ g_i|T_{i-1}^- & (i \equiv 1 \text{ mod } 2), \\
\phi^{-\epsilon} \circ g_i|T_{i-1}^- & (i \equiv 0 \text{ mod } 2).
\end{cases}
\end{equation}

Then \( \tilde{g} = \bigcup g_i \) fits the commutative diagram (4.1). Thus we get the quotient map \( g: N_\phi \to N_\phi \) of degree \( k \cdot \epsilon \cdot \det(A_1). \)

Given \( \phi_s = \left( \begin{array}{cc} a & b \\
& c & d \end{array} \right) \in GL_2(\mathbb{Z}) \) and suppose \( (g_0|T^-_0)_s = \left( \begin{array}{cc} 2m + 1 & 0 \\
0 & n \end{array} \right), (g_k|T^-_s)_s = \left( \begin{array}{cc} 2m' + 1 & 0 \\
0 & n' \end{array} \right) \) where \( m, n, m', n' \in \mathbb{Z}. \)
CASE 1: \( abcd \neq 0, ad - bc = 1 \). (It should be noted that \( (\tau \phi \tau \phi^{-1})_s \) is Anosov.) Since \( g: N_{\phi} \to N_{\phi} \) satisfies \( g^{-1}(T) \) is copies of torus fiber, by Theorem 2.9 \( g \) can be lift to \( g': M_{\phi \tau \phi^{-1}} \to M_{\tau \phi \tau \phi^{-1}} \). By the argument of Anosov monodromy case in Section 3, the degree of \( g' \) in the \( S^1 \) direction is 1. So we have \( k = 1 \).

By equation (4.4), we have:

\[
(g_1|_{T_1^+})_s \cdot \phi_s = \phi_s^\epsilon \cdot (g_0|_{T_0^-})_s.
\]

If \( \epsilon = 1 \), then:

\[
\begin{pmatrix}
2m' + 1 & 0 \\
0 & n'
\end{pmatrix}
\cdot
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= 
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\cdot
\begin{pmatrix}
2m + 1 & 0 \\
0 & n
\end{pmatrix}.
\]

Solving this matrix equation we have:

\[
\begin{cases}
2m' + 1 = 2m + 1, \\
0 = 0,
\end{cases}
\]

\[
\begin{cases}
n = 2m + 1, \\
m' = m,
\end{cases}
\]

Thus \( (g_0|_{T_0^-})_s = 
\begin{pmatrix}
2m + 1 & 0 \\
0 & 2m + 1
\end{pmatrix} \) which means:

\[
\deg(g) = k \cdot \epsilon \cdot \det((g_0|_{T_0^-})_s) = (2m + 1)^2.
\]

If \( \epsilon = -1 \), then:

\[
\begin{pmatrix}
2m' + 1 & 0 \\
0 & n'
\end{pmatrix}
\cdot
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= 
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^{-1}
\cdot
\begin{pmatrix}
2m + 1 & 0 \\
0 & n
\end{pmatrix}.
\]

Solving this matrix equation we have:

\[
\begin{cases}
2m' + 1 = -(2m' + 1), \\
0 = -(2m + 1) \cdot a,
\end{cases}
\]

\[
\begin{cases}
n = -(2m' + 1), \\
(2m' + 1) \cdot a = (2m + 1) \cdot d,
\end{cases}
\]

\[
\begin{cases}
n' = -(2m + 1).
\end{cases}
\]

Suppose \( (2m + 1) = u \cdot a / \gcd(a, d) \), then both \( u \) and \( a / \gcd(a, d) \) must be odd. Similarly, since \( n = 2m' + 1 = -u \cdot d / \gcd(a, d) \) is odd, then \( d / \gcd(a, d) \) is odd also.

Thus \( (g_0|_{T_0^-})_s = 
\begin{pmatrix}
u \cdot a / \gcd(a, d) & 0 \\
0 & -u \cdot d / \gcd(a, d)
\end{pmatrix} \) which means:

\[
\deg(g) = k \cdot \epsilon \cdot \det((g_0|_{T_0^-})_s) = u^2 \cdot \frac{ad}{\gcd(a, d)^2}.
\]

This degree can be realized here if and only if \( ad / \gcd(a, d)^2 \) is odd.
CASE 2: $abcd = 0$. Then there are three subcases.

(2.I) $\phi_s = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$.
In this case $N_{\phi}$ is a torus bundle which has been discussed in Section 3.

(2.II) $\phi_s = \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$, or equivalently $\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}$.
When $z \neq 0$, we discuss the following four possible cases:

(A) If $\epsilon = 1$ and $k = 2s$ is even, then by equation (4.3), we have the following equation:

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} 1 & zk \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$ 

This equation has no solution.

(B) If $\epsilon = -1$ and $k = 2s$ is even, then by equation (4.3):

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} 1 & 0 \\ zk & -1 \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$ 

This equation has no solution either.

(C) If $\epsilon = 1$ and $k = 2s + 1$ is odd, then by equation (4.4):

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} 0 & 1 \\ 1 & kz \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$ 

Solving this matrix equation:

$$\begin{cases} n = (-1)^s(2m' + 1), \\ n' = (-1)^s(2m + 1), \\ n' = (-1)^skn. \end{cases}$$

So $2m + 1 = kn$, thus $k$ is odd, if $k$ exists.

Then $(g_k|_{T_s^+})_s = \begin{pmatrix} 2m' + 1 & 0 \\ 0 & k(2m' + 1) \end{pmatrix}$ which means:

$$\deg(g) = k \cdot \epsilon \cdot \det((g_0|_{T_s^+})_s) = k \cdot \epsilon \cdot \det((g_k|_{T_s^+})_s) = k^2 \cdot (2m' + 1)^2.$$ 

This degree is an odd square number. In another hand, when $k = 1$, all odd square number can be realized as a degree: $(g_k|_{T_s^+})_s = \begin{pmatrix} 2m' + 1 & 0 \\ 0 & 2m' + 1 \end{pmatrix}$.

(D) If $\epsilon = -1$ and $k = 2s + 1$ is odd, then by equation (4.4):

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} -zk & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$ 

This equation has no solution.
When $z = 0$, the same method will show that $\text{deg}(g)$ is odd, and all odd numbers can be realized.

\[(2.\text{III}) \quad \phi_3 = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.
\]

In this case, $\text{deg}(g)$ can be determined as in case (2.II).

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