LOCALLY CONFORMAL KÄHLER STRUCTURES ON
COMPACT SOLVMANIFOLDS

HIROSHI SAWAI

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Abstract

Let \((M, g, J)\) be a compact Hermitian manifold and \(\Omega\) the fundamental 2-form of \((g, J)\). A Hermitian manifold \((M, g, J)\) is said to be locally conformal Kähler if there exists a closed 1-form \(\omega\) such that \(d\Omega = \omega \wedge \Omega\). The purpose of this paper is to investigate a relation between a locally conformal Kähler structure and the adapted differential operator on compact solvmanifolds.

Introduction

Let \((M, g, J)\) be a 2n-dimensional compact Hermitian manifold. We denote by \(\Omega\) the fundamental 2-form, that is, the 2-form defined by \(\Omega(X, Y) = g(X, JY)\). A Hermitian manifold \((M, g, J)\) is said to be locally conformal Kähler if there exists a closed 1-form \(\omega\) such that \(d\Omega = \omega \wedge \Omega\). The closed 1-form \(\omega\) is called Lee form. In the case \(\omega = df\), \((M, e^{-f} g, J)\) is Kähler. The main non-Kähler examples of locally conformal Kähler manifolds are Hopf manifolds [15], Inoue surfaces [14] and generalized Kodaira–Thurston manifolds [5] (cf. [6]). Note that Inoue surfaces and generalized Kodaira–Thurston manifolds have a structure of a compact solvmanifold. In this paper, we investigate locally conformal Kähler structures on a compact solvmanifold \(\Gamma\backslash G\) with a left-invariant complex structure, where \(G\) is a simply-connected solvable Lie group and \(\Gamma\) is a lattice of \(G\), that is, a discrete co-compact subgroup.

An \(n\)-dimensional complex manifold \(M\) is said to be complex parallelizable if it admits holomorphic vector fields \(\{X_1, \ldots, X_n\}\) which are linearly independent at every point. Wang [16] proved that a compact complex parallelizable manifold \(M\) is biholomorphic to a homogeneous space \(D\backslash G\), where \(G\) is a simply connected complex Lie group and \(D\) is a discrete subgroup of \(G\). Abbena–Grassi [1] proved that a non-toral compact complex parallelizable manifold has no locally conformal Kähler structures. The author in [12] proved that if a compact nilmanifold \(M\) with a left-invariant complex structure has a locally conformal Kähler structure, then \(M\) is biholomorphic to a Kodaira–Thurston manifold or a generalized Kodaira–Thurston manifold.

A locally conformal Kähler manifold \((M, g, J)\) is said to be a generalized Hopf manifold if the Lee form \(\omega\) is parallel with respect to metric \(g\). Vaisman [15] proved
that a generalized Hopf manifold has a structure of a principal $S^1$-bundle over a compact Sasakian manifold. Hopf manifolds and generalized Kodaira–Thurston manifolds are generalized Hopf manifolds. Tricerri [14] constructed a locally conformal Kähler structure on Inoue surfaces with non-parallel Lee form $\omega$. In Section 3, we prove that

**Main Theorem.** Let $(\Gamma \backslash G, g, J)$ be a locally conformal Kähler solvmanifold of $\dim \Gamma \backslash G \geq 4$ with a left-invariant complex structure and $\omega$ the Lee form. We assume that $(\Gamma \backslash G, g, J)$ satisfies the following conditions:

1. There exists a left-invariant closed 1-form $\omega_0$ such that the Lee form $\omega$ is cohomologous to $\omega_0$.
2. The fundamental 2-form $\Omega$ of $(g, J)$ can be written as $\Omega = -\omega \wedge \eta + d\eta$, where $\eta = -\omega \circ J$.

Then $\Gamma \backslash G$ has a left-invariant locally conformal Kähler structure with the parallel Lee form $\omega_0$, in particular, $\Gamma \backslash G$ is a generalized Hopf manifold.

In Section 4, we see that Kodaira–Thurston manifolds and secondary Kodaira–Thurston manifolds are examples of Main Theorem.

Hasegawa [7] proved that a complex structure on a 4-dimensional compact solvmanifold is induced from a left-invariant complex structure on the solvable Lie group and classified 4-dimensional compact solvmanifolds with a complex structure. By this classification, we have a classification of 4-dimensional locally conformal Kähler solvmanifolds in Section 4.

1. **Preliminaries**

Let $(M = \Gamma \backslash G, g, J)$ be a compact locally conformal Kähler solvmanifold such that $J$ is left-invariant. In this section, we see that the locally conformal Kähler metric $g$ induces a left-invariant locally conformal Kähler metric and consider its fundamental 2-form on the Lie algebra $\mathfrak{g}$ of $G$.

Let $\Omega$ be the fundamental 2-form of $(g, J)$. There exists a closed 1-form $\omega$ such that $d\Omega = \omega \wedge \Omega$. We now assume that the associated Lee form satisfies the condition 1 in Main Theorem. Then there exists a left-invariant closed 1-form $\omega_0$ such that $\omega_0 - \omega = df$. This assumption holds for completely solvmanifolds [9]. Then we define a left-invariant 2-form $\Omega_0$ by

$$\Omega_0(X, Y) = \int_M (e^f \Omega)(X, Y) \, d\mu$$

for left-invariant vector fields $X, Y$, where $d\mu$ is the volume element induced by a bi-invariant volume element on $G$. Belgun [3] proved that $\Omega_0$ is $J$-invariant and it is the fundamental 2-form of a Hermitian structure $(\langle , \rangle, J)$ on the solvable Lie algebra $\mathfrak{g}$ of $G$ such that $d\Omega_0 = \omega_0 \wedge \Omega_0$. 

Definition 1.1. Let \( g \) be a Lie algebra and \( \alpha \) a closed 1-form on \( g \). We define the adapted algebraic complex \( (\wedge g^*, d_\alpha) \) with the differential operator \( d_\alpha \):

\[
d_\alpha \beta := \alpha \wedge \beta + d\beta
\]

for \( \beta \in \wedge^p g^* \). Note that \( d_\alpha^2 = 0 \) because \( \alpha \) is closed. A \( p \)-form \( \beta \) is called \( \alpha \)-closed if \( d_\alpha \beta = 0 \). It is called \( \alpha \)-exact if there exists a \((p-1)\)-form \( \gamma \) such that \( \beta = d_\alpha \gamma \).

On the above Hermitian structure \((\langle \cdot, \cdot \rangle, J)\) on \( g \), since \( \omega_0 \) is a closed 1-form, the fundamental 2-form \( \Omega_0 \) is \(-\omega_0\)-closed. We say that \((\langle \cdot, \cdot \rangle, J)\) on \( g \) is locally conformal Kähler. In Section 4, we consider a relation between a property of Lee form and the adapted differential operator on 4-dimensional solvmanifolds.

We next show that \( \Omega_0 \) is \(-\omega_0\)-exact if the second condition in Main Theorem also holds true.

Proposition 1.2 (cf. [3]). Let \( \Omega_0 \) be the 2-form on \( g \) above. If \( \Omega = -\omega \wedge \eta + d\eta \), then \( \Omega_0 \) is \(-\omega_0\)-exact. In addition, if \( \eta = -\omega \circ J \), then \( \eta_0 = -k \omega_0 \circ J \), where \( k = \int_M e^f d\mu \).

Proof. Since \( \Omega = -\omega \wedge \eta + d\eta \) and \( \omega = \omega_0 - df \), we see that

\[
e^f \Omega = e^f(-\omega \wedge \eta + d\eta) = e^f(-\omega_0 \wedge \eta + df \wedge \eta + d\eta) = -\omega_0 \wedge e^f \eta + d(e^f \eta) \wedge \eta + e^f d\eta = -\omega_0 \wedge e^f \eta + d(e^f \eta).
\]

Then we get

\[
\Omega_0(X, Y) = \int_M (-\omega_0 \wedge e^f \eta + d(e^f \eta))(X, Y) d\mu
\]

\[
= -\omega_0(X) \int_M (e^f \eta)(Y) d\mu + \omega_0(Y) \int_M (e^f \eta)(X) d\mu + \int_M d(e^f \eta)(X, Y) d\mu,
\]

for \( X, Y \in g \).

Now, we define a 1-form \( \eta_0 \) on \( g \) by \( \eta_0(X) = \int_M (e^f \eta)(X) d\mu \) for \( X \in g \). Since \( d\mu \) is the right-invariant volume element, its Lie derivative \( L_X d\mu \) along a left-invariant vector fields \( X \) is zero. Then, for any function \( F \) on \( M \), we see that

\[
(XF) d\mu = (L_X F) d\mu = L_X(F d\mu) = di(X)F d\mu + i(X) d(F d\mu) = di(X)F d\mu,
\]
where \( i(X) \) is the interior product with the vector field \( X \). Therefore, by Stokes’s theorem, we have

\[
\int_M d(e^f \eta)(X, Y) \, d\mu \\
= \int_M X((e^f \eta)(Y)) \, d\mu - \int_M Y((e^f \eta)(X)) \, d\mu - \int_M (e^f \eta)([X, Y]) \, d\mu \\
= \int_M di(X)(e^f \eta(Y)) \, d\mu - \int_M di(Y)(e^f \eta(X)) \, d\mu - \eta_0([X, Y]) \\
= d\eta_0(X, Y).
\]

Thus we see that

\[
\Omega_0(X, Y) = -\omega_0(X) \int_M (e^f \eta)(Y) \, d\mu + \omega_0(Y) \int_M (e^f \eta)(X) \, d\mu + \int_M d(e^f \eta)(X, Y) \, d\mu
\]

\[
= -\omega_0(X)\eta_0(Y) + \omega_0(Y)\eta_0(X) + d\eta_0(X, Y)
\]

\[
= (-\omega_0 \wedge \eta_0 + d\eta_0)(X, Y).
\]

Similarly, if \( \eta = -\omega \circ J \), then we have

\[
\eta_0(X) = \int_M (e^f \eta)(X) \, d\mu = -\int_M (e^f \omega)(JX) \, d\mu = -\int_M (e^f \omega_0 - e^f df)(JX) \, d\mu
\]

\[
= -\omega_0(JX) \int_M e^f \, d\mu + \int_M d(e^f)(JX) \, d\mu
\]

\[
= -\omega_0(JX) \int_M e^f \, d\mu + \int_M (JX)(e^f) \, d\mu
\]

\[
= -\left( \int_M e^f \, d\mu \right) \omega_0(JX).
\]

Thus we see that \( \eta_0 = -k\omega_0 \circ J \), where \( k = \int_M e^f \, d\mu \).

By Proposition 1.2, we see that \( \Omega_0 = -\omega_0 \wedge k\eta_0' + k \, d\eta_0' = k(-\omega_0 \wedge \eta_0' + d\eta_0') = kd_{\omega_0} \eta_0' \), where \( \eta_0' = (1/k)\eta_0 = -\omega_0 \circ J \). Then the locally conformal Kähler structure \((\Omega_0, J)\) induces the locally conformal Kähler structure \((\Omega_0' = d_{\omega_0}(-\omega_0 \circ J), J)\) on the solvable Lie algebra \( \mathfrak{g} \). Therefore, by replacing \( \Omega_0 \) by \( \Omega_0' \), in order to prove Main Theorem, it is enough to show that if \( (\mathfrak{g}, \Omega_0 = d_{\omega_0}(-\omega_0 \circ J), J) \) is a locally conformal Kähler solvable Lie algebra, then the Lee form \( \omega_0 \) is parallel with respect to \( \langle , \rangle \) (see Section 3).

2. The adjoint operators and the inner product

Let \( \langle , , \rangle, J \) be the locally conformal Kähler structure as we consider in Section 1 on solvable Lie algebra \( \mathfrak{g} \), namely \( \Omega_0 \) can be written as \( \Omega_0 = d_{\omega_0}\eta_0 = -\omega_0 \wedge \eta_0 + d\eta_0 \),
where $\omega_0$ is a closed 1-form on $g$ and $\eta_0 = -\omega_0 \circ J$. In this section, we investigate the properties of $(g, \langle , \rangle, J)$.

Let $\gamma$ be the canonical isomorphism from $g^*$ to $g$ induced by the inner product $\langle , \rangle$. Put $A = \gamma(\omega_0)$. By this normalization, we may assume that $\langle A, A \rangle = 1$.

We easily see that an abelian Lie algebra of dimension equal to or more than 4 has no locally conformal Kähler structures. From now on, we assume that $g$ is not abelian.

Since $g$ is solvable, $[g, g]$ is nilpotent. We take the descending central series for $[g, g]$: $n = [g, g] \supset n^{(1)} = [n, n] \supset n^{(2)} = [n, n^{(1)}] \supset \cdots \supset n^{(r)} \supset n^{(r+1)} = 0$, where $n^{(i+1)} = [n, n^{(i)}]$ $(i \geq 1)$ and $n^{(r)} \neq 0$. We easily see that $n^{(r)}$ is contained in the center $Z(n)$ of $n$. Note that, for $X \in g$, $\text{ad}(X)n^{(i)} \subset n^{(i)}$ for each $i$. Then we get

**Lemma 2.1.** $Jn^{(r)} \subset [g, g]^{\perp}$, where $[g, g]^{\perp}$ is the orthogonal component of $[g, g]$.

**Proof.** For $Z \in n^{(r)}$ and $X \in [g, g]$, we see that

$$\langle JZ, X \rangle = \Omega_0(X, Z) = (-\omega_0 \wedge \eta_0 + d\eta_0)(X, Z) = 0,$$

because $\omega_0$ is closed and $Z \in Z(n)$. \hfill $\square$

**Lemma 2.2.** For $Z, Z' \in n^{(r)}$, $[JZ, Z'] = [JZ', Z]$ and $[JZ, JZ'] = 0$.

**Proof.** The Nijenhuis tensor $N_J$ vanishes, because $J$ is integrable. Since $Z, Z' \in n^{(r)}$, we see that


Note that $[JZ, Z'], [Z, JZ'] \in n^{(r)}$. It follows that $J([JZ, Z'] + [Z, JZ']) \in [g, g]^{\perp}$ by Lemma 2.1. Thus we have $[JZ, Z'] + [Z, JZ'] = 0$ and $[JZ, JZ'] = 0$. \hfill $\square$

By Lemmas 2.1 and 2.2, we have

**Proposition 2.3.** For $U, V, W \in n^{(r)}$,

$$\langle \text{ad}(JU)V, W \rangle + \omega_0(JV)(U, W) = \langle V, \text{ad}(JU)W \rangle + \omega_0(JW)(U, V).$$

**Proof.** Since $\Omega_0 = -\omega_0 \wedge \eta_0 + d\eta_0$ and $\omega_0$ is closed 1-form, we see that

$$\langle \text{ad}(JU)V, W \rangle = \Omega_0(J \text{ad}(JU)V, W) = -\Omega_0(\text{ad}(JU)V, JW)$$

$$(2.1) = \langle \omega_0 \wedge \eta_0 - d\eta_0)(\text{ad}(JU)V, JW)$$

$$= -\omega_0(JW)\eta_0(\text{ad}(JU)V) - d\eta_0(\text{ad}(JU)V, JW).$$
Since $\eta_0 = -\omega_0 \circ J$, we get

$$-\omega_0(JW)\eta_0(\text{ad}(JU)V) = \omega_0(JW) \cdot d\eta_0(JU, V)$$

(2.2)

$$= \omega_0(JW)\{\Omega_0(JU, V) + \omega_0 \wedge \eta_0(JU, V)\}$$

$$= \omega_0(JW)(U, V) - \omega_0(JW)\omega_0(JU)\omega_0(JV).$$

From the derivation conditions and Lemma 2.2, we get

$$d\eta_0(\text{ad}(JU)V, JW) = d\eta_0(JU, [V, JW]) - d\eta_0(V, [JU, JW])$$

$$= d\eta_0(JU, [V, JW])$$

$$= d\eta_0(JU, [W, JW])$$

$$= d\eta_0([JU, W], JV) + d\eta_0(W, [JU, JV])$$

$$= d\eta_0([JU, W], JV) = -d\eta_0(JV, [JU, W]).$$

It follows that

$$-d\eta_0(\text{ad}(JU)V, JW)$$

$$= d\eta_0(JV, [JU, W])$$

$$= \Omega_0(JV, [JU, W]) + \omega_0 \wedge \eta_0(JV, [JU, W])$$

(2.3)

$$= \{V, [JU, W]\} - \omega_0(JV) \cdot d\eta_0(JU, W)$$

$$= \{V, [JU, W]\} - \omega_0(JV)\{\Omega_0(JU, W) + \omega_0 \wedge \eta_0(JU, W)\}$$

$$= \{V, [JU, W]\} - \omega_0(JV)(U, W) + \omega_0(JV)\omega_0(JU)\omega_0(JW).$$

From (2.1) and (2.2), (2.3), we have

$$\langle \text{ad}(JU)V, W \rangle = \omega_0(JW)(U, V) - \omega_0(JW)\omega_0(JU)\omega_0(JV)$$

$$+ \langle V, [JU, W]\rangle - \omega_0(JV)(U, W) + \omega_0(JV)\omega_0(JU)\omega_0(JW),$$

$$\langle \text{ad}(JU)V, W \rangle + \omega_0(JV)(U, W) = \langle V, \text{ad}(JU)W \rangle + \omega_0(JW)(U, V).$$

It is well-known that if a solvable Lie group $G$ admits a lattice $\Gamma$, then the solvable Lie algebra $\mathfrak{g}$ of $G$ is unimodular. We define

**Definition 2.4.** A solvable Lie algebra $\mathfrak{g}$ is called strongly unimodular if, for $X \in \mathfrak{g}$, $\text{tr} \ ad(X)|_{\eta^{(i)}} = 0$ for each $i$.

Benson–Gordon [4] proved that if a solvable Lie group $G$ admits a lattice $\Gamma$, then the solvable Lie algebra $\mathfrak{g}$ of $G$ is strongly unimodular.

We take an orthonormal frame $\{Z_1, \ldots, Z_m\}$ of $\eta^{(i)}$ and consider the strongly unimodular conditions of $\text{ad}(JZ_i)$ from $\eta^{(i)}$ to $\eta^{(i)}$ for each $i$. 
For each $i$, let

$$\text{ad}(JZ_i)(Z_1, \ldots, Z_m) = (Z_1, \ldots, Z_m) \begin{pmatrix} a_{11}(i) & \cdots & a_{1m}(i) \\ \vdots & \ddots & \vdots \\ a_{m1}(i) & \cdots & a_{mm}(i) \end{pmatrix}.$$ 

By Proposition 2.3, we see that

$$\langle \text{ad}(JZ_i)Z_j, Z_k \rangle + \omega_0(JZ_j)(Z_i, Z_k) = \langle Z_j, \text{ad}(JZ_i)Z_k \rangle + \omega_0(JZ_k)(Z_i, Z_j),$$

$$a_{kj}(i) + \delta_{ik}\alpha_0(JZ_j) = a_{jk}(i) + \delta_{jk}\alpha_0(JZ_k).$$

It follows that, in the case of $j, k \neq i$,

$$a_{kj}(i) = a_{jk}(i)$$

and, in the case of $j = i, k \neq i$,

$$a_{ki}(i) = a_{ik}(i) + \alpha_0(JZ_k).$$

Then we get

$$\begin{equation}
(2.4) \quad (a_{jk}(i)) = A_i + \begin{pmatrix}
\text{line } i \\
\omega_0(JZ_1) \\
\vdots \\
\omega_0(JZ_m)
\end{pmatrix},
\end{equation}$$

where $A_i$ is an $(m \times m)$-symmetric matrix.

Moreover, put

$$A_i = \text{row } i \begin{pmatrix}
\cdots \\
\ast \\
\vdots \\
\vdots \\
\ast \\
\ast \\
\ast
\end{pmatrix},$$

for each $i$. Since $[JZ_i, Z_j] = [JZ_j, Z_i]$, we get

$$a_{jj}(i) = \langle \text{ad}(JZ_i)Z_j, Z_j \rangle = \langle \text{ad}(JZ_j)Z_i, Z_j \rangle = a_{ij}^i.$$
for $j \neq i$. Then $A_i$ can be written as

\[
A_i = \text{row } i \begin{pmatrix}
\cdots & \star & \cdots \\
\vdots & \ddots & \vdots \\
\star & \cdots & \star \\
\cdots & \cdots & \cdots \\
\star & \cdots & \star \\
\end{pmatrix}
\]

Thus (2.4) can be expressed as follows:

\[
(2.5) \quad (a_{jk}(i)) = \text{row } i \begin{pmatrix}
\cdots & \star & \cdots \\
\vdots & \ddots & \vdots \\
\star & \cdots & \star \\
\cdots & \cdots & \cdots \\
\star & \cdots & \star \\
\end{pmatrix}
\]

From the strongly unimodular condition of $\text{ad}(J Z_i)|_{\alpha_0}$, we have

**Proposition 2.5.** For each $i$, \((\sum_j a^j_i) + \omega_0(J Z_i) = 0.\)

Then we have

**Corollary 2.6.** \(\sum_i \omega_0(J Z_i)^2 = 1.\)

Proof. For each $i$, we see that

\[
1 = \langle Z_i, Z_i \rangle = \Omega_0(J Z_i, Z_i) = (-\omega_0 \land \eta_0 + d\eta_0)(J Z_i, Z_i) = -\omega_0(J Z_i)\eta_0(Z_i) - \eta_0([J Z_i, Z_i]) = \omega_0(J Z_i)^2 + \omega_0(J [J Z_i, Z_i]).
\]
because $\eta_0 = -\omega_0 \circ J$. From (2.5),
\[1 = \langle Z_i, Z_i \rangle = \omega_0(J Z_i)^2 + \omega_0(J [J Z_i, Z_i]),\]
\[(2.6)\]
\[1 = \omega_0(J Z_i)^2 + \omega_0 \circ J \left( \sum_j (a_j^i + \omega_0(J Z_j)) Z_j \right),\]
\[1 = \omega_0(J Z_i)^2 + \sum_j a_j^i \omega_0(J Z_j) + \sum_j \omega_0(J Z_j)^2.\]

Take the sum of (2.6) for $i$, we get
\[
\sum_i 1 = \sum_i \omega_0(J Z_i)^2 + \sum_i \sum_j a_j^i \omega_0(J Z_j) + \sum_i \omega_0(J Z_j)^2,
\]
\[m = \sum_i \omega_0(J Z_i)^2 + \sum_i \left( \sum_j a_j^i \right) \omega_0(J Z_i) + \sum_i \omega_0(J Z_i)^2.\]

By Proposition 2.5, we have
\[
m = \sum_i \omega_0(J Z_i)^2 + \sum_i (-\omega_0(J Z_i)) \omega_0(J Z_i) + \sum_i \omega_0(J Z_i)^2,
\]
\[
m = \sum_i \omega_0(J Z_i)^2 - \sum_i \omega_0(J Z_i)^2 + \sum_i \omega_0(J Z_i)^2,
\]
\[
m = m \sum_i \omega_0(J Z_i)^2,
\]
which implies that $\sum_i \omega_0(J Z_i)^2 = 1.$

**Corollary 2.7.** $A = \sum_i \omega_0(J Z_i) J Z_i \in J \mathfrak{n}^{(r)}$.

**Proof.** Since $A = \gamma(\omega_0)$, it can be given by
\[A = \sum_i \omega_0(J Z_i) J Z_i + B,
\]
where $B \in (J \mathfrak{n}^{(r)})^\perp \cap [\mathfrak{g}, \mathfrak{g}]^\perp$. From $\langle A, A \rangle = 1$, we have
\[1 = \langle A, A \rangle = \sum_i \omega_0(J Z_i)^2 \langle J Z_i, J Z_i \rangle + \langle B, B \rangle = \sum_i \omega_0(J Z_i)^2 + \langle B, B \rangle.
\]

By Corollary 2.6, we have $\langle B, B \rangle = 0$, which implies that $B = 0.$
3. Structure on the solvable Lie algebra

We use same notation introduced in Section 2. In this section, we prove that $J A$ is in the center $Z(g)$ of $g$ and have Main Theorem.

We see $JA = - \sum_i \omega_0(JZ_i)Z_i \in \mathfrak{n}^{(r)}$ by Corollary 2.7. Put $Z_0 = JA$. Note that $\eta_0(Z_0) = -\omega_0 \circ J(JA) = 1$ and $\eta_0(A) = -\omega_0 \circ J(A) = 0$. We get

Lemma 3.1. For $X \in \mathfrak{g}$, $d\eta_0(A, X) = 0$ and $d\eta_0(Z_0, X) = 0$.

Proof. Since $\eta_0 = -\omega_0 \circ J$, we see that

$$d\eta_0(A, X) = \Omega_0(A, X) + \omega_0 \wedge \eta_0(A, X) = \langle A, JX \rangle + \eta_0(X) = \omega_0(JX) + \eta_0(X)$$

$$= 0.$$ 

Similarly, we get

$$d\eta_0(Z_0, X) = \Omega_0(Z_0, X) + \omega_0 \wedge \eta_0(Z_0, X) = -(X, JZ_0) - \omega_0(X) = \langle X, A \rangle - \omega_0(X)$$

$$= 0. \qed$$

Then we see that

Proposition 3.2. For $U \in \mathfrak{n}^{(r)}$, $\text{ad}(A)U = 0$.

Proof. By a straightforward computation, we see that

$$\langle \text{ad}(A)U, \text{ad}(A)U \rangle = \Omega_0(J \circ \text{ad}(A)U, \text{ad}(A)U)$$

$$= (-\omega_0 \wedge \eta_0 + d\eta_0)(J \circ \text{ad}(A)U, \text{ad}(A)U)$$

$$= -\omega_0(J \circ \text{ad}(A)U)\eta_0(\text{ad}(A)U) + d\eta_0(J \circ \text{ad}(A)U, \text{ad}(A)U).$$

By Lemma 3.1 and the derivation conditions,

$$\langle \text{ad}(A)U, \text{ad}(A)U \rangle = \omega_0(J \circ \text{ad}(A)U) d\eta_0(A, U) - d\eta_0(\text{ad}(A)U, J \circ \text{ad}(A)U)$$

$$= -d\eta_0(\text{ad}(A)U, J \circ \text{ad}(A)U)$$

$$= -d\eta_0(A, [U, J \circ \text{ad}(A)U]) + d\eta_0(U, [A, J \circ \text{ad}(A)U])$$

$$= d\eta_0(U, [A, J \circ \text{ad}(A)U]).$$

Now, since $JA = Z_0$ and $\text{ad}(A)U \in \mathfrak{n}^{(r)}$, we see that

$$[A, J \circ \text{ad}(A)U] = [-JZ_0, J \circ \text{ad}(A)U] = 0$$

by Lemma 2.2. It follows that $d\eta_0(U, [A, J \circ \text{ad}(A)U]) = 0$, which implies that $\text{ad}(A)U = 0. \qed$
Therefore we have

**Theorem 3.3.** \( Z_0 \in Z(g) \).

**Proof.** Let \( X \in g \). By Lemma 3.1, we see that

\[
\langle \text{ad}(X)Z_0, \text{ad}(X)Z_0 \rangle = \Omega_0(J \circ \text{ad}(X)Z_0) \text{ad}(X)Z_0)
\]

\[
= (-\omega_0 \wedge \eta_0 + d\eta_0)(J \circ \text{ad}(X)Z_0) \text{ad}(X)Z_0)
\]

\[
= -\omega_0(J \circ \text{ad}(X)Z_0)\eta_0(\text{ad}(X)Z_0) + d\eta_0(J \circ \text{ad}(X)Z_0) \text{ad}(X)Z_0)
\]

\[
= \omega_0(J \circ \text{ad}(X)Z_0) d\eta_0(X, Z_0) + d\eta_0(J \circ \text{ad}(X)Z_0) \text{ad}(X)Z_0)
\]

\[
= d\eta_0(J \circ \text{ad}(X)Z_0) \text{ad}(X)Z_0).
\]

Moreover, from the derivation conditions and \( Z_0 \in Z([g, g]) \),

\[
\langle \text{ad}(X)Z_0, \text{ad}(X)Z_0 \rangle = d\eta_0(J \circ \text{ad}(X)Z_0) \text{ad}(X)Z_0) = -d\eta_0(\text{ad}(X)Z_0, J \circ \text{ad}(X)Z_0)
\]

\[
= -d\eta_0(X, [Z_0, J \circ \text{ad}(X)Z_0]) + d\eta_0(Z_0, [X, J \circ \text{ad}(X)Z_0])
\]

\[
= -d\eta_0(X, [Z_0, J \circ \text{ad}(X)Z_0]).
\]

Now, since \( \text{ad}(X)Z_0 \in n^{(r)} \), we see that

\[
[Z_0, J \circ \text{ad}(X)Z_0] = [\text{ad}(X)Z_0, J Z_0] = [A, \text{ad}(X)Z_0] = 0
\]

by Lemma 2.2 and Proposition 3.2. It follows that \( d\eta_0(X, [Z_0, J \text{ad}(X)Z_0]) = 0 \), which implies that \( \text{ad}(X)Z_0 = 0 \) for \( X \in g \).

From Theorem 3.3, we have

**Corollary 3.4.** \( \text{ad}(A) \circ J = J \circ \text{ad}(A) \).

**Proof.** Let \( X \in g \). Since \( J \) is integrable, we see that

\[
\]

\[
\]

From \( Z_0 \in Z(g) \), \( [A, X] + J[A, JX] = 0 \). Then we have our claim.

**Corollary 3.5.** Lee form \( \omega_0 \) is parallel, if and only if \( \Omega_0 \) is \( \text{ad}(A) \)-invariant.
Proof. Let $\nabla$ be the Riemannian connection of $\langle \ , \rangle$. For $X, Y \in \mathfrak{g}$, we see that

$$2(\nabla_X \omega_0)(Y) = -2\omega_0(\nabla_X Y) = -2\langle A, \nabla_X Y \rangle$$

$$= -\langle [A, X], Y \rangle - \langle X, [A, Y] \rangle - \langle A, [X, Y] \rangle$$

$$= -\Omega_0(J [A, X], Y) - \Omega_0(J X, [A, Y]) - \omega_0([X, Y])$$

$$= -\Omega_0(J [A, X], Y) - \Omega_0(J X, [A, Y]),$$

because $\omega_0$ is a closed 1-form. From Corollary 3.4, we get

$$2(\nabla_X \omega_0)(Y) = -\Omega_0([A, J X], Y) - \Omega_0(J X, [A, Y]) = (\text{ad}^*(A) \Omega_0)(J X, Y).$$

Thus we have our claim. \qed

Proof of Main Theorem. By Corollary 3.5, to prove Main Theorem, it is enough to show that $\Omega_0$ is $\text{ad}(A)$-invariant.

Let $X, Y \in \mathfrak{g}$. By a straightforward computation, we see that

$$(\text{ad}^*(A) \Omega_0)(X, Y) = -\Omega_0([A, X], Y) - \Omega_0(X, [A, Y])$$

$$= (\omega_0 \wedge \eta_0 - d\eta_0)([A, X], Y) + (\omega_0 \wedge \eta_0 - d\eta_0)(X, [A, Y])$$

$$= -\omega_0(Y) \eta_0([A, X]) - d\eta_0([A, X], Y)$$

$$+ \omega_0(X) \eta_0([A, X]) - d\eta_0(X, [A, Y]).$$

By Lemma 3.1, we get $\eta_0([A, X]) = -d\eta_0(A, X) = 0$ and $\eta_0([A, Y]) = 0$. Moreover, from the derivation conditions,

$$d\eta_0([A, X], Y) + d\eta_0(X, [A, Y]) = d\eta_0(A, [X, Y]) = 0,$$

which implies that $\Omega_0$ is $\text{ad}(A)$-invariant. This completes the proof of Main Theorem. \qed

4. Examples

In this section, we give examples of Main Theorem and consider locally conformal Kähler structures on 4-dimensional compact solvmanifolds. Note that a compact Kähler solvmanifold is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus ([2], [8]).

Hasegawa [7] classified a 4-dimensional compact solvmanifold and proved that any complex structure on such solvmanifold is induced from a left-invariant complex structure on Lie group. By this classification, we see that a 4-dimensional locally conformal Kähler solvmanifold is biholomorphic to Kodaira–Thurston manifold, Secondary Kodaira–Thurston manifold or Inoue surfaces.
We see that Kodaira–Thurston manifolds and secondary Kodaira–Thurston manifolds are examples of Main Theorem:

EXAMPLE 4.1 (Kodaira–Thurston manifold [5]). Let \( G \) be a 3-dimensional nilpotent Lie group given by

\[
G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.
\]

The Lie group \( G \) admits a lattice \( \Gamma \). Let \( \mathfrak{g} \) be the nilpotent Lie algebra corresponding to \( G \). \( \mathfrak{g} \times \mathbb{R} \) is given by

\[
\mathfrak{g} \times \mathbb{R} = \text{span}\{X, Y, Z, A : [X, Y] = Z\}.
\]

Let \( \{x, y, z, \omega\} \) be the dual base of \( \{X, Y, Z, A\} \):

\[
dx = dy = d\omega = 0, \quad dz = -x \wedge y.
\]

We define a left-invariant metric \( \langle \cdot, \cdot \rangle \) on \( \Gamma \setminus G \times S^1 \) such that \( \{X, Y, Z, A\} \) is an orthonormal frame and a left-invariant complex structure \( J \) by \( JA = Z, JX = Y \). Then \( (\Gamma \setminus G \times S^1, \langle \cdot, \cdot \rangle, J) \) is a locally conformal Kähler manifold with the fundamental 2-form given by

\[
\Omega_0 = -\omega \wedge z - x \wedge y = d_{-\omega} z = d_{-\omega} (-\omega \circ J).
\]

Hence the fundamental 2-form \( \Omega_0 \) is \( -\omega \)-exact. We easily see that Lee form \( \omega \) is parallel.

EXAMPLE 4.2 (Secondary Kodaira–Thurston manifold (cf. [15])). Let \( G \) be a 4-dimensional solvable Lie group given by

\[
G = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}(x \sin t + y \cos t) & \frac{1}{2}(x \cos t - y \sin t) & z \\ 0 & \cos t & \sin t & x \\ 0 & -\sin t & \cos t & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x, y, z \in \mathbb{R} \right\}.
\]

The Lie group \( G \) admits a lattice \( \Gamma \):

\[
\Gamma = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}(u \sin 2\pi s + v \cos 2\pi s) & \frac{1}{2}(u \cos 2\pi s - v \sin 2\pi s) & w \\ 0 & \cos 2\pi s & \sin 2\pi s & u \\ 0 & -\sin 2\pi s & \cos 2\pi s & v \\ 0 & 0 & 0 & 1 \end{pmatrix} : s, u, v, w \in \mathbb{Z} \right\}.
\]
Let $g$ be the solvable Lie algebra corresponding to $G$:


Let $\{\omega, x, y, z\}$ be the dual base of $\{A, X, Y, Z\}$:

$$d\omega = 0, \quad dx = -\omega \wedge y, \quad dy = \omega \wedge x, \quad dz = -x \wedge y.$$ 

We define a left-invariant metric $\langle , \rangle$ on $\Gamma \backslash G$ such that $\{A, X, Y, Z\}$ is an orthonormal frame and a left-invariant complex structure $J$ by $JA = Z, JX = Y$. Then $(\Gamma \backslash G, \langle , \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge z - x \wedge y = d\omega \wedge \bar{z} = d\omega(-\omega \circ J).$$

Hence the fundamental 2-form $\Omega_0$ is $-\omega$-exact. We easily see that Lee form $\omega$ is parallel.

A locally conformal Kähler structure on Inoue surfaces are different from one on solvmanifolds as Main Theorem:

**Example 4.3 (Inoue surface $S^0$ [11], [14]).** Let $B \in \text{SL}(3, \mathbb{Z})$ be a unimodular matrix with eigenvalues $\alpha, \beta, \bar{\beta}$ such that $\beta \neq \bar{\beta}$, and eigenvectors $(a_1, a_2, a_3), (b_1, b_2, b_3)$ of $\alpha, \beta$, respectively. Then we define a structure of the group on $\mathbb{H} \times \mathbb{C} = \{(x + \sqrt{-1}\alpha', z): x, t \in \mathbb{R}, z \in \mathbb{C}\}$ as follows:

$$(x + \sqrt{-1}\alpha', z) \cdot (x' + \sqrt{-1}\alpha', z') = (\alpha' x' + x + \sqrt{-1}\alpha' \bar{\alpha}', \beta' z' + z).$$

It can be expressed by

$$G = \left\{ \begin{pmatrix} \alpha' & 0 & 0 & x \\ 0 & \beta' & 0 & z \\ 0 & 0 & \bar{\beta}' & \bar{z} \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x \in \mathbb{R}, z \in \mathbb{C} \right\}.$$ 

Thus we easily see that the Lie group $G$ is solvable and it admits a lattice $\Gamma$:

$$\Gamma = \left\{ \begin{pmatrix} \alpha' & 0 & 0 & x \\ 0 & \beta' & 0 & w_1 \\ 0 & 0 & \bar{\beta}' & w_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} : x, u, w_1, w_2 \in \mathbb{Z} \right\},$$

where $P = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \in \text{GL}(3, \mathbb{C})$. Let $g$ be the solvable Lie algebra corresponding to $G$:

$$g = \text{span}\{A, X, Y_1, Y_2: [A, X] = -2rX, [A, Y_1] = rY_1 + \theta Y_2, [A, Y_2] = rY_2 - \theta Y_1\},$$
where $\alpha = e^{-2r}$ and $\beta = e^{r + \sqrt{-1} \theta}$. Let $\{\omega, x, y_1, y_2\}$ be the dual base of $\{A, X, Y_1, Y_2\}$:

$$d\omega = 0, \quad dx = 2r\omega \wedge x, \quad dy_1 = -r\omega \wedge y_1 + \theta\omega \wedge y_2, \quad dy_2 = -r\omega \wedge y_2 - \theta\omega \wedge y_1.$$ 

We define a left-invariant metric $\langle , \rangle$ on $\Gamma \backslash G$ such that $\{A, X, Y_1, Y_2\}$ is an orthonormal frame and a left-invariant complex structure $J$ by $JA = X$, $JY_1 = Y_2$. Then $(\Gamma \backslash G, \langle , \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega = -\omega \wedge x - y_1 \wedge y_2.$$ 

Note that the fundamental 2-form $\Omega$ is not $-2r\omega$-exact. We see that the Lee form $\omega$ is not parallel: $(\nabla X \omega) X \neq 0$.

**Example 4.4** (Inoue surface $S^+ \ [5, 11, 14]$). Let $G$ be a 4-dimensional solvable Lie group given by

$$G = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}y & \frac{1}{2}x & z \\ 0 & e^t & 0 & x \\ 0 & 0 & e^{-t} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x, y, z \in \mathbb{R} \right\}.$$ 

We can construct a lattice $\Gamma$ on $G$ (cf. [13]). Let $\mathfrak{g}$ be the solvable Lie algebra corresponding to $G$:


Let $\{\omega, x, y, z\}$ be the dual base of $\{A, X, Y, Z\}$:

$$d\omega = 0, \quad dx = -\omega \wedge x, \quad dy = \omega \wedge y, \quad dz = -x \wedge y.$$ 

We define a left-invariant metric $\langle , \rangle$ on $\Gamma \backslash G$ such that $\{A, X, Y, Z\}$ is an orthonormal frame and a left-invariant complex structure $J$ by $JA = X$, $JZ = X$. Then $(\Gamma \backslash G, \langle , \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge y - z \wedge x.$$ 

Note that the fundamental 2-form $\Omega$ is not $-\omega$-exact. We see that the Lee form $\omega$ is not parallel: $(\nabla Y \omega) Y \neq 0$.

We mention that an Inoue surface $S^-$ is not of the form $G \backslash G$, but it is a double covering space of an Inoue surface $S^+$ (cf. [10]). Then an Inoue surface $S^-$ has a locally conformal Kähler structure.
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Numazu National College of Technology
3600 Ooka, Numazu
Japan

e-mail: sawai@numazu-ct.ac.jp